Stochastic processes: Assignment sheet 1

(1) Distributions

(i) Determine the derivative of the function sign(x).

(ii) Evaluate numerically the Fourier transform‡ of the following limiting representations of the δ-function:

$$\delta_\varepsilon(x) = \frac{1}{\sqrt{\pi\varepsilon}} \exp\left(-\frac{x^2}{\varepsilon}\right) \quad \text{and} \quad \delta_k(x)\frac{1}{\pi}\frac{\sin kx}{x}. \tag{1}$$

In these representations $\delta(x)$ corresponds to the limits $\varepsilon \to 0$ and $k \to \infty$. Use three different values for ε and k.

(iii) Show that $\delta(x) = \delta(-x)$ is symmetric; the scaling property $\delta(ax) = |a|^{-1}\delta(x)$ ($a \neq 0$), and $f(x)\delta(x) = f(0)\delta(x)$. Use the concept of the test function space.

(2) Kelvin model

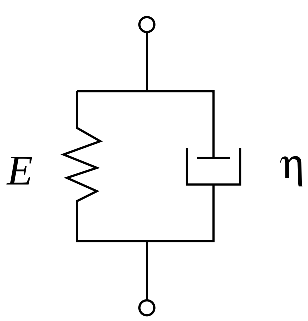

Figure 1: Kelvin model consisting of a Hookean spring with elasticity E and a parallel dashpot with viscosity η.

Determine the time-dependent extension $\varepsilon(t)$ of the Kelvin model shown in figure 1 as function of the strain σ when a load σ_0 is applied at $t = 0$. Start with the stress-strain relation (why?)

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t). \tag{2}$$

‡ The Fourier transform of a function $f(x)$ is

$$f(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx.$$

(3) Diffusion equation

Solve the diffusion equation

$$\frac{\partial}{\partial t} P(x,t) = K \frac{\partial^2}{\partial x^2} P(x,t) \tag{3}$$

by separation of variables for the initial condition $P_0(x) = \delta(x)$ and the two following boundary conditions:

(i) natural boundary conditions $\lim_{|x|\to\infty} P(x,t) = 0$;

(ii) a "box" of length $2a$ with reflecting boundaries defined by $P'(x = \pm a, t) = 0$;

(iii) a "box" of length $2a$ with absorbing boundaries defined by $P(\pm a, t) = 0$.

Provide plots for different times in all three cases.