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Comparison of pure and combined search strategies for single and multiple targets⋆

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Abstract. We address the generic problem of random search for a point-like target on a line. Using the measures of search reliability and efficiency to quantify the random search quality, we compare Brownian search with Lévy search based on long-tailed jump length distributions. We then compare these results with a search process combined of two different long-tailed jump length distributions. Moreover, we study the case of multiple targets located by a Lévy searcher.

1 Introduction

Searching for randomly located targets is a central problem in many branches of the sciences comprising all scales from the smallest to the largest: examples include chemical reactions, in which a molecule has to find a reactive target such as the search of transcription factor proteins for a specific binding spot on a DNA chain [1,2], the question of molecular signal detection [3–6], white blood cells trying to locate intruding pathogens [7], spider monkeys searching for food in a tropical forest [8], human rescue operations [9], the hunt for submarines [10] and, more mathematically, algorithms for finding the minima in a complex search space [11].

In society the development of search strategies like the search for land mines, castaways or victims of avalanches belongs to the realm of *operations research* [12,13]. In ecology and biology understanding the foraging of biological organisms forms part of the new discipline of *movement ecology* [14,15]. Prominent examples for the latter are wandering albatrosses searching for food [16–18], marine predators diving for prey [19,20], and bees collecting nectar [21]. Within this context the *Lévy Flight Hypothesis* (LFH) attracted considerable attention [22]: it predicts that under certain mathematical conditions scale-free jump processes called *Lévy flights* (LFs) [23] minimise the search time [16,17,24]. The LFH implies that, for instance, for a bumblebee searching for rare flowers the flight lengths should be distributed according to a power law. This prediction is completely different from the paradigm put forward by Pearson more than a century ago who proposed to model the movements of biological organisms by simple random walks [25]. Pearson's theory entails that the movement lengths are distributed according to a Gaussian distribution, contrasting the Lévy stable form underlying the LFH. Both Lévy and Gaussian dynamics represent fundamentally different, *pure* classes of stochastic processes.

However, in complex biological reality animals, or humans, may not search according to a simple, pure stochastic process, as other factors may come into play. For example, they often have a limited perception while moving with high speed. In this case a more promising search strategy is to switch between a slow recognition mode during which targets can be found and fast relocations [26–28]. These *intermittent search strategies* can *combine* different types of motion such as Brownian, ballistic motion, or LFs [29–34]. They may also include various distributions for the switching times from one phase to another [35]. The optimal search strategy then depends on the specific types of motion and the dimension of the search space [28,35–37].

For real world problems it is furthermore crucial that a searcher does not only eventually find a target but also that the search is successful within a limited time span, for instance, if the search is a rescue operation or if a starving animal searches for food to survive. This means that the search needs to be *efficient*. However, on top of this it often must also be *reliable* in that a given target is not missed out but found with sufficiently high probability [38,39].
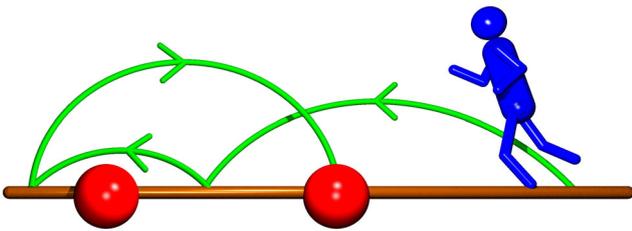
Fig. 1. Lévy searcher multiply hopping across (overshooting) the two targets before eventually hitting the right target.

From a mathematical point of view, finding a single target when hitting it (in the attempt of sweeping past it) defines a *first passage problem* [40,41], which corresponds to the situation where a searcher searches uninterruptedly while continuously moving. On the other hand, landing on a single target after a relocation process has been completed can be formulated as a *first arrival problem* [38,39,42]. This distinction relates to the situation described above, when a searcher does not perceive targets while moving but only if it comes at rest exactly on the target, or sufficiently close to it (see Fig. 1). Calculating first passage and first arrival times for stochastic processes are well-defined mathematical problems, that, in certain cases such as for Brownian motion and in one dimension, can be solved exactly. Remarkable recent progress has been reported for first passage problems, such as the universality of certain classes of mean and global mean first passage times [30,43,44]. Moreover, it has been discussed that mean search times are not always meaningful, as they may not be representative [45,46], or that they are vastly different from the most probably first passage time obtained from the full distribution of first passage times [47,48]. However, these studies only address the problem of finding *single* targets. For solving the problem of how to find *multiple* targets other techniques are needed. The extreme case of the time needed to find all targets in a given domain with certainty is called the *covering time* [49]. Recent work has demonstrated that for a broad range of stochastic processes on networks there holds a certain universality for the distributions of cover times [50]. Clearly, first passage as well as arrival problems for finding single targets and cover times for finding all of them define extreme cases of search problems. We here consider the case of a finite number of targets and explicitly calculate the splitting probabilities to locate one of the targets. This setting is distinct from the previously studied case of equally spaced targets in a system with periodic boundary conditions [35,51].

The structure of our paper is as follows: in Section 2 we briefly review two important applications of search theory to biology. The first one is the problem to understand the foraging of biological organisms. Within this context we also elaborate on the role of *Lévy walks*, which, apart from subdiffusive processes and Lévy flights, define a special case of *continuous time random walks*, that in turn are the central topic of this Special Issue. The second one is the problem of search along DNA chains by DNA binding proteins. Section 3 starts by introducing

two basic quantities for judging the quality of a mathematical search problem, namely, the search reliability and the search efficiency. In Section 4 these two quantities are calculated explicitly for the single-target first arrival problem of the two fundamental stochastic processes of pure Brownian motion and a pure LF search process, the latter both with and without a bias. Here we review known results in the literature. Section 5 generalises this theory to single-target search by a combination of two Lévy stable processes, which yields a new result. In Section 6 we address the problem of search for more than one target by a pure stochastic process. As a specific example, we consider Brownian and Lévy search of two targets by calculating the first arrival density, the search reliability, and the search efficiency. We conclude with a brief outlook in Section 7. In Appendix A we collect a number of technical results.

2 Search research: two examples from biology

2.1 Search for food by biological organisms

The advantage of random search based on random walks with long-tailed, scale-free jump length distributions was postulated by Shlesinger and Klafter already in 1986 [22]. The groundbreaking moment for the popularisation of this concept came with the 1996 article by Viswanathan and colleagues: In this study the flight times of albatrosses were recorded during their foraging excursions in the South Atlantic [16]. It was found that the distribution of flight times obeyed an asymptotic power law $\simeq t^{-2}$. Assuming that the birds move with a constant average speed one can associate these flight times with the respective power law distribution $\simeq |x|^{-2}$ of flight lengths. This suggests that the albatrosses were searching for food by performing *Lévy flights*. For more than a decade albatrosses were thus considered to be the most prominent case study of animal foraging by LFs. This work spawned a large number of related studies suggesting that many other animals like goats and deer [52,53], bumblebees [17], spider monkeys [8], marine predators [19,20], and microzooplankton [54] also perform Lévy search [24]. Heavytailed distributions were also found to be characteristic for human movement dynamics [55,56]. The discussion of the LF nature of the flight of the albatross recently saw an interesting twist. While a re-analysis of the albatross flights showed that they generally are *not* LFs [57], strong evidence was presented according to which LFs are indeed a search pattern for individual albatrosses [58].

The mathematical underpinning for the relevance of long-tailed probability laws was provided in the works starting with Lévy [59] as well as Gnedenko and Kolmogorov [60]. Their work showed that specific types of power laws, the *Lévy alpha stable distributions* [59,61–63], obey a generalised central limit theorem. Their result thus generalises the conventional central limit theorem for Gaussian distributions, which explains why Brownian motion with a Gaussian probability distribution is universally observed in a huge variety of physical phenomena.

But, Gaussian tails decay faster than power laws, which implies that for Lévy-distributed flight lengths there is a larger probability to yield long flights than for flight lengths obeying Gaussian statistics. The generalised central limit theorem then guarantees that for sufficiently many steps in the Lévy flight a well defined limit distribution for the probability distribution emerges. The only difference is that this Lévy stable law is not universal as the Gaussian, but characterised by a specific alpha stable index [59–63]. Intuitively, Lévy flights with their chance of long, decorrelating jumps should be better suited to scan a large space for randomly distributed targets than Brownian motion. In turn Brownian search should outperform Lévy motion when the targets are dense. This is the basic idea underlying the LFH mentioned above [17]. The motivating question is: *What is the best statistical strategy to adapt in order to search efficiently for randomly located, sparse objects?* The LFH stipulates that *Lévy motion provides an optimal search strategy for sparse, randomly distributed, immobile, revisitable targets in unbounded domains* [17,24].

To be precise we note that there exist two formulations of continuous time random walk processes with long-tailed, scale-free relocation distributions. One pertains to Lévy flights, these are fully Markovian processes in which the jumps occur instantaneously, separated with a well defined sojourn time. In exchange the mean squared displacement of Lévy flights diverges [61–64]. This divergence is remedied in the *Lévy walk* model, in which a spatiotemporal coupling between relocation lengths and waiting times exists, such that long jumps are penalised by long waiting times [63–67]. In fact a specific Lévy walk model was investigated by computer simulations in the analysis leading to the LFH [17]. Remarkably, in the limit of sparse food both Lévy walk and flight dynamics lead to the same optimal Lévy stable exponent $\alpha = 1$ for the distribution of relocation lengths [17,34]. We also note that a rigorous mathematical proof of the LFH to date remains elusive, while empirical tests are debated in literature [18,68–72].

2.2 Search along DNA chains

To activate or downregulate individual genes on the genome, in biological cells specific DNA-binding proteins need to locate and then bind to designated binding sites on the DNA chain. For long it had been assumed that a good estimate for the associated binding rate is the celebrated Smoluchowski result for molecular aggregation [73]. In vitro experiments for the search rate of the Lac repressor protein remarkably showed a rate that was larger by around two orders of magnitude [74]. Building on earlier work of Adam and Delbrück [75] and Richter and Eigen [76], the so-called facilitated diffusion model was developed by Berg and von Hippel, and coworkers [77–79]. The main idea of the facilitated diffusion model is the possibility that the searching protein may not only diffuse in the bulk volume of the reaction container, but it may also intermittently associate with the DNA chain and perform a random sliding motion on it. Hereby the linear topology of the DNA leads to a transient dimensional reduction of the random search, effecting a similar advantage as the intermittent search model discussed above. Namely, the one-dimensional search makes sure that the target, if close-by, will be located with high probability. Significant oversampling due to the recurrent motion of one-dimensional diffusion is avoided by the intermittent volume excursions, that decorrelate the position of the protein before it rebinds to the DNA. Indeed, this approach to good approximation explains the observed speedup compared to the Smoluchowski limit, see the recent review [80].

In single molecule measurements the various search modes can be verified directly or indirectly, for instance, the existence of the one-dimensional sliding motion [81–83], association-dissociation events leading to the change between sliding and bulk diffusion [84], intersegmental transfer between different segments of DNA [85], and the role of the three-dimensional DNA conformations [86]. A number of theoretical studies highlight the role of the intermittency of the search for the efficiency of the process. Thus, Halford and Marko [87], Coppey et al. [88], Erskine et al. [89], Givaty and Levy [90], and Klenin et al. [91] considered the competition of one- and three-dimensional diffusion. Slutsky and Mirny [92] argue that the one-dimensional search needs to consist of search and recognition modes such that the protein can slide sufficiently fast while retaining its binding selectivity. Including intersegmental transfers or jumps of the protein between chemically remote but physically close segments of the DNA chain further improves the search efficiency, especially when the three-dimensional search mode is repressed, for instance, at certain salt conditions [34,93,94]. Interestingly, in the limit of long DNA chains and sufficiently fast reorganisation of the DNA conformation, intersegmental jumps effect an LF search by the binding protein [34]. In fact, this may be the only example for an LF, which is not hampered by a diverging second moment: as the jumps are long-tailed in terms of the chemical distance measured along the backbone of the DNA molecule, but are local in the real, embedding space, this divergence is physically meaningful.

Intersegmental jumps may even assist in avoiding "roadblocks" in the form of other non-specifically DNA-bound proteins [95,96]. A concise overview over the various facilitated diffusion search modes is given in [97]. More recent progress includes the formulation of facilitated diffusion for the in vivo case of living bacteria cells [2] and the inclusion of effects of the DNA sequence [98–100]. Finally, effects of the crowded cytoplasm of living cells were considered by different approaches in [101,102]. A path integral formulation of the downregulation of one gene by the product of a steering gene, including the stochasticity of the regulation process [103] was given in [104]. A noteworthy result of that study is that the efficiency of the protein search for its target binding site crucially depends on the initial distance from this target [104], a result that is consistent with the so-called rapid search hypothesis based on bioinformatics research [1].

We note that while there exists full experimental evidence for the intermittent search of DNA binding proteins, the showcase examples of Lac repressor proteins or EcoRV restriction enzymes are in fact not universal, and many proteins simply occur at sufficiently high concentrations and no significant improvement over pure three-dimensional diffusion to locate their binding site would be expected [105,106]. Yet for those proteins whose number per cell is small, facilitated diffusion is essential [107].

3 Defining search reliability and search efficiency

As mentioned above, the key quantities to characterise the success of a search strategy are the reliability and efficiency. The former quantifies the probability whether the search process is ever successful, the latter is a measure for how long the search takes. We define the search reliability as the cumulative probability P of the first arrival to reach the target. In terms of the survival probability $\mathscr{S}(t)$ (of not hitting the target up to time t), we thus write $P = 1 - \mathscr{S}(\infty)$ [40,41]. Expressing the survival probability in terms of the first arrival time density $\wp_{\mathrm{fa}}(t)$, we thus find the relation

$$P = 1 - \mathscr{S}(\infty) = \int_0^\infty \wp_{\mathrm{fa}}(t)dt. \qquad (1)$$

Using the Laplace transform, defined through

$$\wp_{\mathrm{fa}}(s) = \int_0^\infty \wp_{\mathrm{fa}}(t)e^{-st}dt, \qquad (2)$$

we find the relation [38]

$$P = \lim_{s \to 0} \wp_{\mathrm{fa}}(s). \qquad (3)$$

The search reliability depends on the exact type of the random search process as well as the geometrical details (dimension, distance from the starting position to the target, etc.). The arrival time density $\wp_{\mathrm{fa}}(t)$ can be determined from the (fractional) Fokker-Planck equation of the search process, equipped with a sink term [38,39,42].

For search in one dimension by LFs without a bias the search reliability is unity if $\alpha > 1$ and zero otherwise [38,39,42], which is consistent with previous results [108]. For search in the presence of a bias (water stream for marine searchers, winds for airborne foragers, etc.) the search reliability can vary between zero and unity [38,39], which is similarly true also for Brownian motion [40]: when the bias pushes a searcher away from the target the search reliability is exponentially suppressed by a Boltzmann-like factor [40]. A search reliability of unity does not necessarily imply recurrence of the motion. For instance, LFs with $\alpha = 1$ in one dimension and Brownian motion in two dimensions are recurrent but their search reliability is zero.

The second quantity of interest is the search efficiency. Most of the theoretical studies consider a probabilistic searcher with a limited radius of perception. Motivated

by [109], in this case two basic definitions of the search efficiency are considered to be either

$$\mathrm{Efficiency}_1 = \frac{\text{visited number of targets}}{\text{number of steps}}, \qquad (4)$$

or

$$\mathrm{Efficiency}_2 = \frac{\text{visited number of targets}}{\text{distance travelled}}. \qquad (5)$$

The first definition applies especially to *saltatory search*, where a searcher moves in a jump-like fashion and is able to detect the target only around the landing point after a jump. The second definition is adapted to *cruise motion*, during which the searcher keeps exploring the search space continuously during the whole search process. An example for the former scenario is given by a regulatory protein that moves in three-dimensional space and occasionally binds to the DNA of a biological cell until it finds its binding site. The latter scenario would correspond to an eagle or vulture whose excellent eyesight permits them to scan their environment for food during their entire flight. For LFs, equation (4) presents a natural choice while equation (5) is better suited for processes like Brownian motion and finite-velocity Lévy walks.

In what follows we focus on the limit of sparse targets. Concretely, we consider a single or a finite number of targets. For a single target and saltatory motion we argued that the efficiency should be defined from equation (4) with proper averaging [38,39]. In our continuous time model the number of steps is naturally substituted by the time of the process. We choose the following averaging [38,39]

$$\mathcal{E} = \left\langle \frac{1}{t} \right\rangle = \int_0^\infty \wp_{\mathrm{fa}}(s)ds \qquad (6)$$

over the inverse search times. This choice appears more meaningful than taking an average of the form $1/\langle t \rangle$, as the latter would produce a zero efficiency when the mean search time diverges. Our definition (6) instead pronounces short and intermediate search times.

4 Search of a single target by a single Lévy flight searcher

Below we use the search reliability and efficiency to characterise search strategies of the motion governed by two Lévy stable processes and search by a single Lévy stable process for more than one target. Before, we recall the main properties of the search of a single Lévy searcher in an environment without and with an external bias, as well as the limit of a Brownian searcher.

The properties of an LF search process can be calculated from a space-fractional Fokker-Planck diffusion equation [64] for the non-normalised density function $f(x,t)$ [42],

$$\frac{\partial f(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha f(x,t)}{\partial |x|^\alpha} - \wp_{\mathrm{fa}}(t)\delta(x - x_1), \qquad (7)$$

where the target, represented as a δ-sink, is located at $x = x_1$. The generalised diffusion coefficient has physical

dimensions $[K_\alpha] = \mathrm{cm}^\alpha/\mathrm{s}$. We assume that at $t = 0$ the searcher is placed at $x = x_0$, that is, $f(x, 0) = \delta(x - x_0)$. The δ-sink effects the condition $f(x_1, t) = 0$ [34,42]. In equation (7) the fractional derivative $\partial^\alpha/\partial|x|^\alpha$ on an infinite axis is conveniently represented in terms of its Fourier transform [64]

$$\int_{-\infty}^{\infty} e^{ikx} \frac{\partial^\alpha}{\partial|x|^\alpha} f(x,t)dx = -|k|^\alpha f(k,t). \tag{8}$$

Integrating over the coordinate x in equation (7) yields the survival probability $\mathscr{S}(t)$. Its negative time derivative then delivers the probability density of first arrival [42],

$$\wp_{\mathrm{fa}}(t) = -\frac{d}{dt} \int_{-\infty}^{\infty} f(x,t)dx. \tag{9}$$

The density function $f(x, t)$ can be determined from equation (7) by application of combined Laplace and Fourier transforms, defined in terms of

$$f(k,s) = \int_0^\infty dt\, e^{-st} \int_{-\infty}^{\infty} dx\, e^{ikx} f(x,t). \tag{10}$$

The solution reads [42]

$$f(k,s) = \frac{e^{ikx_0} - \wp_{\mathrm{fa}}(s)e^{ikx_1}}{s + K_\alpha|k|^\alpha}. \tag{11}$$

Integrating this result over k yields

$$\int_{-\infty}^{\infty} f(k,s)dk = f(x=0,s) = 0, \tag{12}$$

and thus

$$W(x_1 - x_0, s) - W(0,s)\wp_{\mathrm{fa}} = 0, \tag{13}$$

where $W(x, t)$ is the solution (Green's function) of equation (7) *without* the sink term and reads

$$W(x,s) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{s + K_\alpha|k|^\alpha}dk \tag{14}$$

in Laplace space. Hence the probability of first arrival becomes

$$\wp_{\mathrm{fa}}(s) = \frac{\displaystyle\int_{-\infty}^{\infty} dk\, \frac{e^{ik(x_1 - x_0)}}{s + K_\alpha|k|^\alpha}}{\displaystyle\int_{-\infty}^{\infty} dk\, \frac{1}{s + K_\alpha|k|^\alpha}}. \tag{15}$$

We now use this result together with our definitions equations (3) and (6) to assess the random search dynamics by a pure Brownian and LF searchers for a single target.

4.1 Brownian search

If the search is performed by a Brownian searcher in equation (7) we take $\alpha = 2$ and the first arrival density can be computed analytically. In Laplace space it reads

$$\wp_{\mathrm{fa}}(s) = \exp\left(-|x_1 - x_0|\sqrt{\frac{s}{K_2}}\right). \tag{16}$$
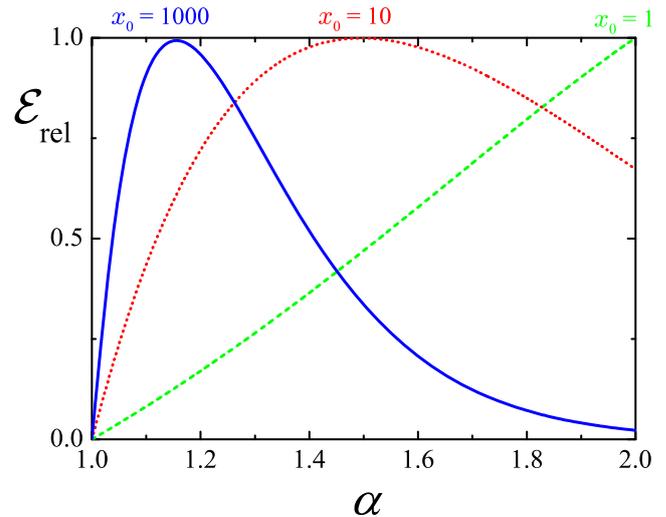
Fig. 2. Relative search efficiency for LF search for a single point-like target as a function of the stable index α according to equation (19), displayed for the initial searcher-target separations $x_0 = 1$ (green dashed curve), $x_0 = 10$ (red dotted curve), and $x_0 = 1000$ (blue continuous curve). We take $K_\alpha = 1$.

Back-transformed, we find in real time that

$$\wp_{\mathrm{fa}}(t) = \frac{|x_1 - x_0|}{\sqrt{4\pi K_2 t^3}} \exp\left(-\frac{(x_1 - x_0)^2}{4K_2 t}\right). \tag{17}$$

This is the well known Lévy-Smirnov density [41]. The search reliability (3) in this case is $P = 1$ and the efficiency reads

$$\mathcal{E}_2 = \frac{2K_2}{(x_1 - x_0)^2}. \tag{18}$$

4.2 First arrival for Lévy searcher

The first arrival density $\wp_{\mathrm{fa}}(s)$ for an LF searcher in Laplace space can be computed in terms of Fox' H-functions [39]. From the small s limit of this function one can see that for $\alpha \le 1$ the search reliability is $P = 0$, that is, the search is unsuccessful with probability one, due to the diverging first absolute moment $\langle|x|\rangle$ of this process. For $\alpha > 1$, $\langle|x|\rangle$ is finite and the reliability is $P = 1$. By integration of the corresponding H-function expression in Laplace space one gets a simple equation for the search efficiency [39]

$$\mathcal{E}_\alpha = \frac{\alpha K_\alpha}{|x_1 - x_0|^\alpha} \left|\cos\left(\frac{\pi\alpha}{2}\right)\right| \Gamma(\alpha), \tag{19}$$

for $1 < \alpha < 2$. The exact shape of the relative efficiency $\mathcal{E}_{\mathrm{rel}} = \mathcal{E}_\alpha/\mathcal{E}_{\mathrm{opt}}$, where $\mathcal{E}_{\mathrm{opt}}$ is the maximal (optimal) value of $\mathcal{E}_\alpha$ for a given value of the index α, is displayed in Figure 2. In particular, for short initial distance x_0 to the target, the Brownian strategy is superior.
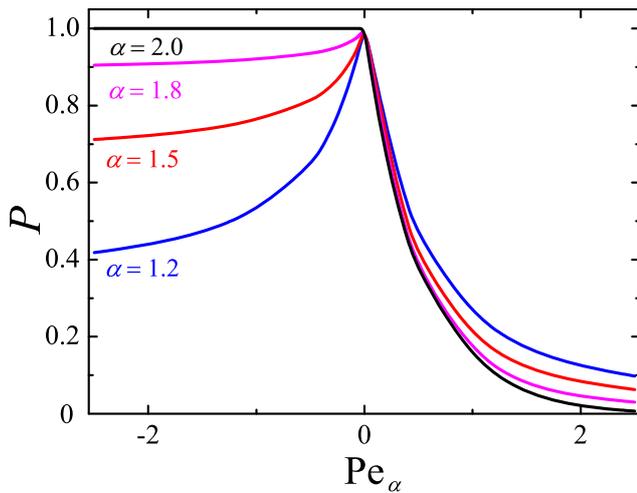
Fig. 3. Dependence of the search reliability P for a biased LF on the generalised Péclet number Pe_α, for the indicated values of the stable index α. Positive values of Pe_α correspond to uphill search.

4.3 Lévy search in the presence of a bias

The dynamic equation (7) can be generalised for search in the presence of different external potentials. Even a simple bias, stemming from, for instance, an underwater current for marine predators or wind in the case of airborne foragers, changes the search performance significantly [38,39]. The space-fractional Fokker-Planck equation then reads

$$\frac{\partial f(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha f(x,t)}{\partial |x|^\alpha} - v \frac{\partial f(x,t)}{\partial x} - \wp_{\text{fa}}(t)\delta(x - x_1). \tag{20}$$

Here v denotes the external, constant bias of dimension $[v] = \text{cm s}^{-1}$, and the rest of the terms are the same as in equation (7). The search reliability in this case depends solely on a single parameter, the generalised Peclét number [38,39]

$$Pe_\alpha = \frac{v|x_1 - x_0|^{\alpha-1}}{2K_\alpha}. \tag{21}$$

In Figure 3 the search reliability is shown as a function of Pe_α for various values of α. The target can be either in an uphill or downhill location relative to the starting point of the searcher. Positive values of Pe_α correspond to the uphill scenario, in which the searcher has to fight the bias in order to reach the target. In this scenario the reliability increases with decreasing stable index α (as long as $\alpha > 1$). In contrast, for the downhill scenario LFs are less reliable than Brownian motion, because LFs allow overshoots or leapovers [110] and, hence, an LF searcher may be eventually lost [39]. More details about the search properties by LFs in the presence of an external bias can be found in References [38,39].

The exact nature of the external potential landscape creating the bias field influences the search properties. In reference [111] the fractional Fokker-Planck equation for search processes was considered for different point sink strengths for free diffusion, diffusion with a constant bias,

and for an harmonic external potential. A finite strength of the sink describes a finite probability of absorption. The results for the arrival time density in reference [111] are consistent with our results in [38,39].

5 Search by a combination of two Lévy processes

What happens when we combine two search strategies? This question was analysed previously in terms of a fractional Fokker-Planck equation for the DNA search on a long DNA chain in reference [34], combining Brownian and LF search. In the language of search processes used here this process was further studied in reference [112]. In this section we analyse the intermittent motion with two different LF search strategies governed by the dynamic equation

$$\frac{\partial f(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha f(x,t)}{\partial |x|^\alpha} + K_\mu \frac{\partial^\mu f(x,t)}{\partial |x|^\mu} - \wp_{\text{fa}}(t)\delta(x), \tag{22}$$

which can be directly derived from the corresponding continuous time random walk model. Here, we introduced the two stable indices α and μ, and the target position is fixed at $x = 0$. The two diffusion coefficients K_α and K_μ measure the relative frequency with which jump lengths are drawn from the corresponding stable laws with indices α and μ. The remaining variables have the same meaning as before. The first arrival density takes on the form, similar to (15),

$$\wp_{\text{fa}}(s) = \frac{\int_0^\infty \dfrac{\cos kx_0}{s + K_\alpha k^\alpha + K_\mu k^\mu} dk}{\int_0^\infty \dfrac{1}{s + K_\alpha k^\alpha + K_\mu k^\mu} dk}$$

$$= \frac{\int_0^\infty \dfrac{\cos k}{st_\mu + pk^\alpha + k^\mu} dk}{\int_0^\infty \dfrac{1}{st_\mu + pk^\alpha + k^\mu} dk}, \tag{23}$$

where $t_\mu = |x_0|^\mu/K_\mu$ is a time scale of LFs with the index μ, and we define

$$p = \frac{K_\alpha}{K_\mu} x_0^{\mu-\alpha}. \tag{24}$$

Analogously to the case of a single LF searcher, if both $\alpha > 1$ and $\mu > 1$ the search reliability is unity, $P = 1$ (the motion is recurrent). Vice versa, if both values are less than or equal to unity, then $P = 0$. Thus, only the case $\alpha < 1$ and $\mu > 1$ is of interest. The search reliability then reads (see Appendix A):

$$P = \frac{\sin\left(\frac{\pi(1-\alpha)}{\mu-\alpha}\right)}{2\sqrt{\pi}}$$

$$\times H_{31}^{12}\left[\frac{2}{p^{\frac{1}{\mu-\alpha}}}\middle|\begin{matrix}(1,\frac{1}{2})\left(\frac{1-\alpha}{\mu-\alpha},\frac{1}{\mu-\alpha}\right)(\frac{1}{2},\frac{1}{2})\\(\frac{1-\alpha}{\mu-\alpha},\frac{1}{\mu-\alpha})\end{matrix}\right], \tag{25}$$
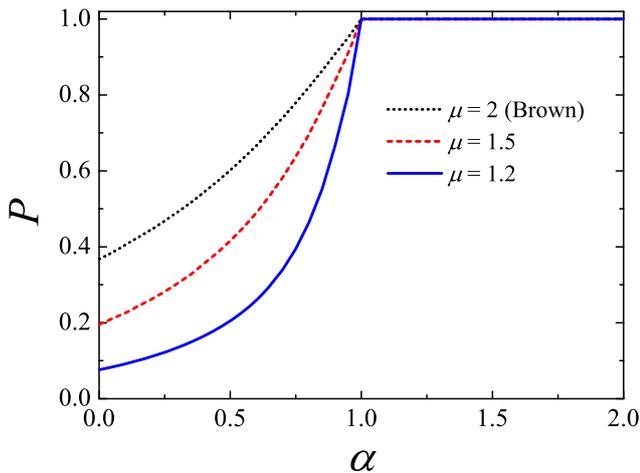
Fig. 4. Search reliability P for two combined LF searchers with stable indices α and μ as function of α, for $p = 1$ and different values of μ.

Fig. 5. Search efficiency $\mathcal{E}_{\alpha,\mu}$, normalised versus the single-LF efficiency $\mathcal{E}_{\mu}$ as function of the parameter p for the search by two LFs. Continuous lines show the numerical results. Dashed lines represent the corresponding asymptotics derived in Appendix B.

which is a generalisation of equation (16) in reference [112] for the search reliability of combined Lévy and Brownian motion.

Figure 4 illustrates the influence of the competing stable index μ on the search reliability P, shown as function of α. While the qualitative behaviour for $\mu < 2$ stays the same as for Brownian search ($\mu = 2$) [112], the cumulative probability of finding the target decreases with μ once $\alpha < 1$. If α is kept constant while μ is varied between 1 and 2 then the search reliability increases from 0 to some value lower than 1 (not shown here). This result can be rephrased in terms of the classical problem whether a process is recurrent or transient [62]: a recurrent motion revisits the points in the domain of interest, in our notations it corresponds to $P = 1$. We can see that the combination of recurrent motion ($P = 1$ for LFs with $\mu > 1$) with transient LFs with $\alpha \leq 1$ ($P = 0$) leads to search reliabilities between 0 and 1, that is, the combined motion is transient with $P > 0$. This is one of the central results of this paper, and it is consistent with our findings for combined Lévy-Brownian search [112], compare also the discussion in reference [34].

Following the parametrisation in result (23) we plot the search efficiency $\mathcal{E}_{\alpha,\mu}$ (see Appendix B) for the dual LF search as function of the dimensionless parameter p in Figure 5, for the case $\alpha < \mu < 2$. For $\alpha > 1$, $\mathcal{E}_{\alpha,\mu}$ converges to the efficiency $\mathcal{E}_{\alpha}$ of a single LF searcher with index α in the limit $p \to \infty$, that is, $\mathcal{E}_{\alpha>1,\mu} \sim p$. For $\alpha = 1$ we find $\mathcal{E}_{1,\mu} \sim p/\ln p$. These two asymptotics are the same as for the combined Lévy-Brownian search [112]. However, for $\alpha < 1$ the asymptotic power law changes to the expression $\mathcal{E}_{\alpha<1,\mu} \sim p^{(\mu-1)/(\mu-\alpha)}$. The derivation of these power laws can be found in Appendix B.

The comparison of the strategies for different α values and $\mu = 1.5$ for different values of the initial distance x_0 is shown in Figure 6. We see that for small x_0 the optimal strategy is Brownian ($\alpha = 2$), while for larger x_0 it changes to values smaller than 2. This behaviour
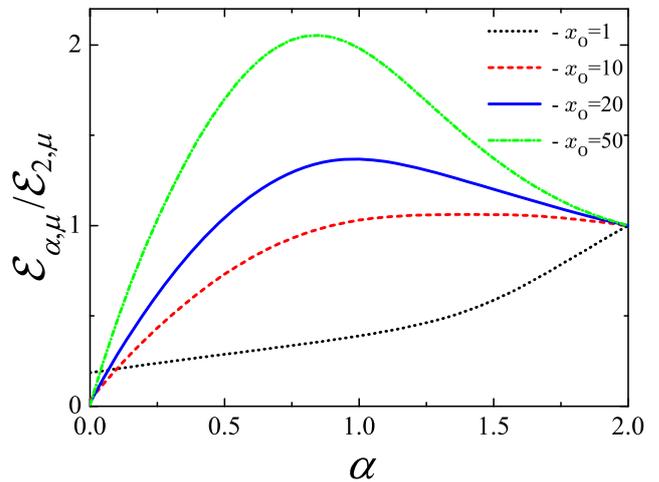
Fig. 6. Search efficiency $\mathcal{E}_{\alpha,\mu}$, normalised versus the efficiency $\mathcal{E}_{2,\mu}$, of two Lévy processes with indices α and μ as function of α for fixed x_0 and $\mu = 1.5$. We choose $K_\alpha = 1$ and $K_\mu = 1$.

is analogous to the combination of Brownian and Lévy strategies in reference [112]. The search for a nearby target should be more local in comparison to the search for far away targets.

6 Search for two and multiple targets by a single Lévy searcher

In this section we return to the case of a single Lévy searcher but consider the situation with multiple, point-like targets. We first consider two targets, placed at x_1 and x_2. Then the dynamic equation for the process becomes (7)

$$\frac{\partial f(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha f(x,t)}{\partial |x|^\alpha} - \wp_{\text{fa}1}(t)\delta(x - x_1)$$
$$- \wp_{\text{fa}2}(t)\delta(x - x_2). \quad (26)$$

Integrating over the position x it follows, analogously to the above, that

$$\wp_{\mathrm{fa}1}(t) + \wp_{\mathrm{fa}2}(t) = -\frac{d}{dt}\int_{-\infty}^{\infty} f(x,t)dx. \qquad (27)$$

The decrease of the survival probability is thus due to the flux into either of the two targets. Equation (26) can be solved in Fourier-Laplace space, producing

$$f(k,s) = \frac{e^{ikx_0} - \wp_{\mathrm{fa}1}(s,x_0)e^{ikx_1} - \wp_{\mathrm{fa}2}(s,x_0)e^{ikx_2}}{s}. \qquad (28)$$

The inverse Fourier transform can now be taken in the same way as above. As we here have two conditions of the form $f(x = x_1, s) = f(x = x_2, s) = 0$, this inversion is related to position x_1 and x_2 of the sinks, and we find the set of linear equations

$$W(x_1 - x_0, s) - \wp_{\mathrm{fa}2}W(x_1 - x_2, s) - \wp_{\mathrm{fa}1}W(0,s) = 0,$$
$$W(x_2 - x_0, s) - \wp_{\mathrm{fa}1}W(x_2 - x_1, s) - \wp_{\mathrm{fa}2}W(0,s) = 0. \qquad (29)$$

The density of first arrival is the sum of fluxes to both targets,

$$\wp_{\mathrm{fa}}(s) = \wp_{\mathrm{fa}1}(s) + \wp_{\mathrm{fa}2}(s). \qquad (30)$$

Let us use the simplified notation

$$W(x_i - x_j, s) = W_{ij} = \int_{-\infty}^{\infty} dk \frac{e^{ik(x_j - x_i)}}{s + K_\alpha |k|^\alpha}, \qquad (31)$$

with $W(0,s) = W_0$. Then the first arrival density becomes

$$\wp_{\mathrm{fa}}(s) = \frac{W_{10} + W_{20}}{W_{12} + W_0}. \qquad (32)$$

Similarly, the splitting first arrival densities are

$$\wp_{\mathrm{fa}1}(s) = \frac{W_{10}W_0 - W_{20}W_{12}}{W_0^2 - W_{12}^2} \qquad (33)$$

and

$$\wp_{\mathrm{fa}2}(s) = \frac{W_{20}W_0 - W_{10}W_{12}}{W_0^2 - W_{12}^2}. \qquad (34)$$

These expressions can be generalised to the case of multiple targets. The corresponding dynamic equation reads

$$\frac{\partial f(x,t)}{\partial t} = K_\alpha \frac{\partial^\alpha f(x,t)}{\partial |x|^\alpha} - \sum_i \wp_{\mathrm{fa}i}(t)\delta(x - x_i), \qquad (35)$$

where $\wp_{\mathrm{fa}i}(t)$ is the splitting first arrival density to target i, and x_i is the position of target i. The formal solution in Fourier-Laplace space reads

$$f(k,s) = \frac{e^{ikx_0} - \sum_i \wp_{\mathrm{fa}i}(s,x_0)e^{ikx_i}}{s + K\alpha|k|^\alpha}. \qquad (36)$$

Similarly to the case with two targets this leads to the system of linear equations

$$\wp_{\mathrm{fa}j}W_0 + \sum_{i \neq j} W_{ij}\wp_{\mathrm{fa}i} = W_{j0}. \qquad (37)$$

This system of n equations with n unknowns has a unique solution which allows one to define all the splitting first arrival densities $\wp_{\mathrm{fa}j}(t)$ as well as the first arrival density $\wp_{\mathrm{fa}}(t) = \sum_j \wp_{\mathrm{fa}j}(t)$. The matrix of coefficients in this system of equations is symmetric. Interestingly, if the targets form an equidistant set, $x_i - x_j = (i-j)\Delta$ with the constant spacing Δ, the matrix of coefficients is the Toeplitz matrix [113].

6.1 Brownian search for two targets

Let us start with the splitting probabilities for Brownian search. In the corresponding case $\alpha = 2$ [39]

$$W_{ij} = \int_{-\infty}^{\infty} dk \frac{e^{ik(x_i - x_j)}}{s + K_2 k^2}$$
$$= \frac{\pi}{\sqrt{sK_2}} \exp\left(-\sqrt{s/K_2}|x_i - x_j|\right). \qquad (38)$$

Hence,

$$\wp_{\mathrm{fa}}(s) = \frac{e^{-|x_1 - x_0|\sqrt{s/K_2}} + e^{-|x_2 - x_0|\sqrt{s/K_2}}}{e^{-|x_1 - x_2|\sqrt{s/K_2}} + 1}. \qquad (39)$$

There exist two different cases. In the first case both targets are on the same side of the starting point ($x_0 < x_1 < x_2$ or $x_0 > x_2 > x_1$). In the second case the starting point is located between the targets ($x_1 < x_0 < x_2$).

Let us consider the first case for $x_0 < x_1 < x_2$. Then,

$$\wp_{\mathrm{fa}}(s) = \frac{e^{-(x_1 - x_0)\sqrt{s/K_2}} + e^{-(x_2 - x_0)\sqrt{s/K_2}}}{e^{-(x_1 - x_2)\sqrt{s/K_2}} + 1}$$
$$= e^{-(x_1 - x_0)\sqrt{s/K_2}}, \qquad (40)$$

or, after Laplace back transformation,

$$\wp_{\mathrm{pa}}(t) = \frac{x_1 - x_0}{2\sqrt{\pi K_2 t^3}} \exp\left(-\frac{(x_1 - x_0)^2}{4K_2 t}\right). \qquad (41)$$

We see that the coordinate x_2 of the second target disappears from the expression of the first arrival probability density and we arrive at the result for a Brownian particle on a semi-infinite axis, as it should, see equations (16) and (17): for the Brownian walker first arrival and first passage are identical, the walker cannot pass the closer target to reach the second target.

In the second case $x_1 < x_0 < x_2$ from equation (39) it follows that

$$\wp_{\mathrm{fa}}(s) = \frac{e^{-(x_0 - x_1)\sqrt{s/K_2}} + e^{-(x_2 - x_0)\sqrt{s/K_2}}}{e^{-(x_2 - x_1)\sqrt{s/K_2}} + 1}. \qquad (42)$$

To compare this solution with the expression for the fluxes in [40] (Eqs. (2.2.10) therein), we note that the latter

can be rewritten in our notation as

$$\wp_{\text{fa}}(s) = \wp_{\text{fa}1}(s) + \wp_{\text{fa}2}(s)$$

$$= \frac{\sinh\left((x_2 - x_0)\sqrt{\frac{s}{K_2}}\right) + \sinh\left((x_0 - x_1)\sqrt{\frac{s}{K_2}}\right)}{\sinh\left((x_2 - x_1)\sqrt{\frac{s}{K_2}}\right)}$$

$$= 2\frac{\sinh\left((x_2 - x_0)\sqrt{\frac{s}{K_2}}\right) + \sinh\left((x_0 - x_1)\sqrt{\frac{s}{K_2}}\right)}{(1 + e^{-(x_2 - x_1)\sqrt{s/K_2}})(e^{(x_2 - x_1)\sqrt{s/K_2}} - 1)}$$

$$= 4\frac{\sinh\left(\frac{(x_2 - x_1)\sqrt{\frac{s}{K_2}}}{2}\right)\cosh\left(\frac{(x_1 + x_2 - 2x_0)\sqrt{\frac{s}{K_2}}}{2}\right)}{(1 + e^{-(x_2 - x_1)\sqrt{s/K_2}})(e^{(x_2 - x_1)\sqrt{s/K_2}} - 1)}$$

$$= \frac{e^{\left(\frac{x_1 + x_2}{2} - x_0\right)\sqrt{s/K_2}} + e^{-\left(\frac{x_1 + x_2}{2} - x_0\right)\sqrt{s/K_2}}}{e^{0.5(x_2 - x_1)\sqrt{s/K_2}}(1 + e^{-(x_2 - x_1)\sqrt{s/K_2}})}. \quad (43)$$

The latter expression is equivalent to equation (42) after division of both denominator and numerator by $\exp(0.5(x_2 - x_1)\sqrt{s/K_2})$. For the Brownian case our results thus coincide with those from literature.

6.2 Long time asymptotics and splitting search reliabilities of an LF for two targets

We now derive the splitting probabilities and splitting search reliabilities for the case of LF search for two targets. The values of the search reliabilities can be found from the asymptotics of $\wp_{\text{fa}}(s)$ in the limit $s \to 0$ (or $t \to \infty$), for the derivation see Appendix C:

$$\wp_{\text{fa}}(s) \approx 1 - \frac{\Lambda(\alpha)(s/K_\alpha)^{1 - \frac{1}{\alpha}}}{2}$$
$$\times \left(|x_1 - x_0|^{\alpha - 1} + |x_2 - x_0|^{\alpha - 1} - |x_2 - x_1|^{\alpha - 1}\right), \quad (44)$$

where

$$\Lambda(\alpha) = \frac{\alpha\Gamma(2 - \alpha)}{\pi(\alpha - 1)}\sin\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi}{\alpha}\right). \quad (45)$$

Due to Minkowski's inequality the combination of absolute values in the brackets of equation (44) is always non-negative. From that expression one can see that the search reliability $P = \wp_{\text{fa}}(s \to 0) = 1$. Now, let us consider the splitting densities. For the first arrival to the first target (see Appendix C), we find

$$\wp_{\text{fa}1}(s) \approx \Big[-|x_1 - x_0|^{\alpha - 1} + |x_2 - x_0|^{\alpha - 1} + |x_2 - x_1|^{\alpha - 1}$$
$$+ \Lambda(\alpha)|x_2 - x_0|^{\alpha - 1}|x_2 - x_1|^{\alpha - 1}(s/K_\alpha)^{1 - 1/\alpha}\Big]/C \quad (46)$$

where we use the abbreviation

$$C = 2|x_2 - x_1|^{\alpha - 1} + \Lambda(\alpha)|x_2 - x_1|^{2\alpha - 2}(s/K_\alpha)^{1 - 1/\alpha}. \quad (47)$$
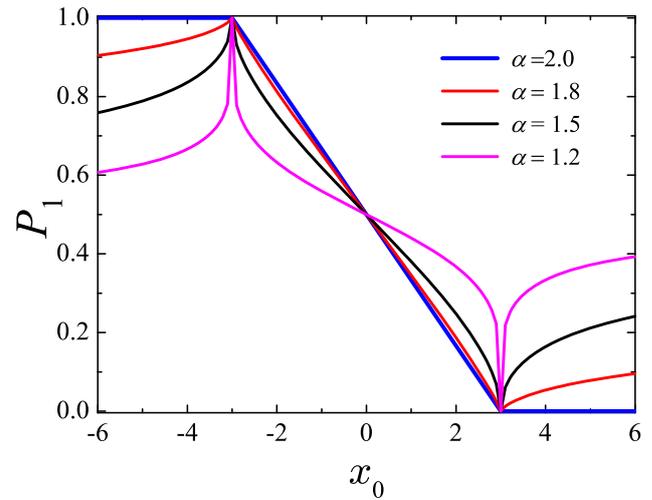
Fig. 7. Search reliability P_1 of hitting target x_1 as a function of the initial release position x_0. The two targets are located at $x_1 = -3$ and $x_2 = 3$, respectively.

Then the splitting search reliability to find the first target becomes

$$P_1 = \frac{-|x_1 - x_0|^{\alpha - 1} + |x_2 - x_0|^{\alpha - 1} + |x_2 - x_1|^{\alpha - 1}}{2|x_2 - x_1|^{\alpha - 1}}$$
$$= \frac{1}{2} + \frac{|x_2 - x_0|^{\alpha - 1} - |x_1 - x_0|^{\alpha - 1}}{2|x_2 - x_1|^{\alpha - 1}}. \quad (48)$$

Similarly, for the second target

$$P_2 = \frac{-|x_2 - x_0|^{\alpha - 1} + |x_1 - x_0|^{\alpha - 1} + |x_2 - x_1|^{\alpha - 1}}{2|x_2 - x_1|^{\alpha - 1}}$$
$$= \frac{1}{2} - \frac{|x_2 - x_0|^{\alpha - 1} - |x_1 - x_0|^{\alpha - 1}}{2|x_2 - x_1|^{\alpha - 1}}. \quad (49)$$

We see that $P_1 + P_2 = 1$ as it should be. Assuming $x_0 < x_1 < x_2$ we see that for the Brownian case, $\alpha = 2$, the probability P_2 equals zero whereas for $\alpha < 2$, $P_2 \neq 0$. It can also be shown that for $x_0 < x_1 < x_2$ one always has $P_2 < P_1$. Thus, expressions (48) and (49) are a consistent generalisation of the classical result for the splitting probabilities on an interval (Eq. (2.2.11) in Ref. [40]).

Figure 7 shows the dependence of the search reliability for the first target P_1 as function of the starting position. For the Brownian case the probability to find the first target is unity if the starting position is to the left of both targets, zero if the start is to the right, and it changes linearly with in-between starting points. This behaviour naturally changes for LF searchers due to the possibility to overshoot the target. The LF searcher may miss the closest target and hit the one beyond. In the limit of faraway searcher release, $x_0 \to \pm\infty$, the searcher can hit either target with a likelihood of $\frac{1}{2}$, which can be proven by taking the corresponding limit in equations (48) and (49). If α decreases from two to unity, that is, when the jump lengths get increasingly longer tailed while the motion is still recurrent, the probabilities P_1 and P_2 approach $\frac{1}{2}$.
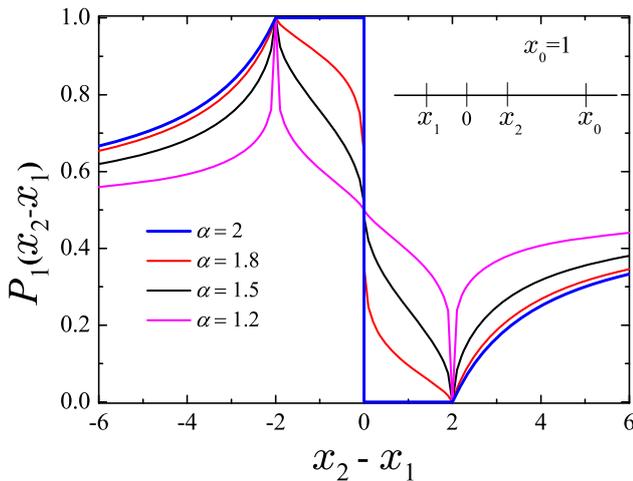
Fig. 8. Splitting search reliability P_1 for hitting target x_1 as function of $x_2 - x_1$. The starting position is fixed at $x_0 = 1$. The targets at x_1 and x_2 are placed symmetrically around 0.

We also see that the search reliability for hitting the closest target drops with decreasing α, while the chance to hit the further target increases.

In Figure 8 we fix the starting position x_0 and change the distance between the targets to show the behaviour of the search reliability P_1 (see the sketch in the inset of the figure). Negative distances correspond to the inverted order of targets, that is, target 1 is located to the right, while target 2 is the left one. Values of $x_2 - x_1 = \pm 2$ correspond either to the situation when x_2 coincides with x_0 or x_1 with x_0, hence, $P_1(-2) = 1$, $P_1(2) = 0$ for any α. For the Brownian case $\alpha = 2$ the splitting search reliability to find the target x_1 is unity when $x_0 \leq x_1 < x_2$ or $x_0 \geq x_1 > x_2$. In the cases $x_0 \leq x_2 < x_1$ and $x_0 \geq x_2 > x_1$ the search reliability to find the target x_1 is zero. When the starting position is located between targets x_1 and x_2, the splitting search reliability for the furthermost target increases. In the limit $x_2 - x_1 \to \infty$ the probability to hit both targets is the same and tends to $\frac{1}{2}$. Similarly to the situation in Figure 7 we see that LF search always provides the possibility to hit both targets, and the likelihood for this to happen approaches $\frac{1}{2}$ for $\alpha \to 1$.

6.3 Search efficiency

In Figures 9 and 10 the target search efficiency is plotted as a function of the starting position x_0. The efficiency exhibits a symmetry with respect to the midpoint between the targets at positions x_1 and x_2. When the starting position is moved towards one of the targets, the search efficiency rapidly increases. If the starting position is between the targets, the search efficiency of the Brownian search exceeds that of LFs. However, once the targets are further apart (Fig. 10) the midpoint between the targets is comparatively far from both targets and the LF strategy becomes advantageous again. In both cases if the searcher starts far to the right or far to the left from both targets
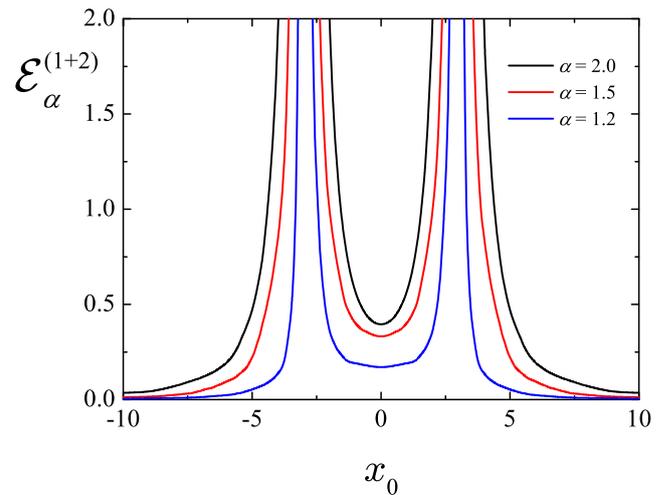
Fig. 9. Search efficiency as function of the starting position x_0. The targets are located at $x_1 = -3$ and $x_2 = 3$, respectively.
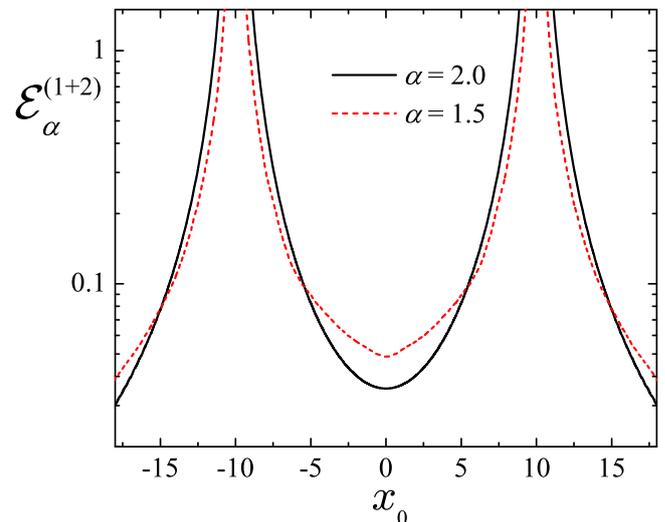
Fig. 10. Search efficiency as function of the starting position x_0. The targets are located at $x_1 = -10$ and $x_2 = 10$, respectively.

then LF search is more efficient than Brownian. The splitting search efficiency of hitting a single target, for instance, the second target (Fig. 11), behaves similarly to the search efficiency to hit either target if the searcher starts to the right of the first target. Naturally, if it starts from the position of the first target, the efficiency of reaching the second one is 0. We note again that if the searcher needs to overshoot one target in order to reach the second target (the region to the left of the first target in Fig. 11), the splitting search efficiency of Brownian motion is 0 because of the absence of overshoots. In contrast LFs are able to do this, and the splitting search efficiency increases with decreasing α.

Search strategies with different α are compared for fixed initial and target positions in Figure 12. As in reference [38] one can see that for small distances from at least on of the targets Brownian search is more efficient than
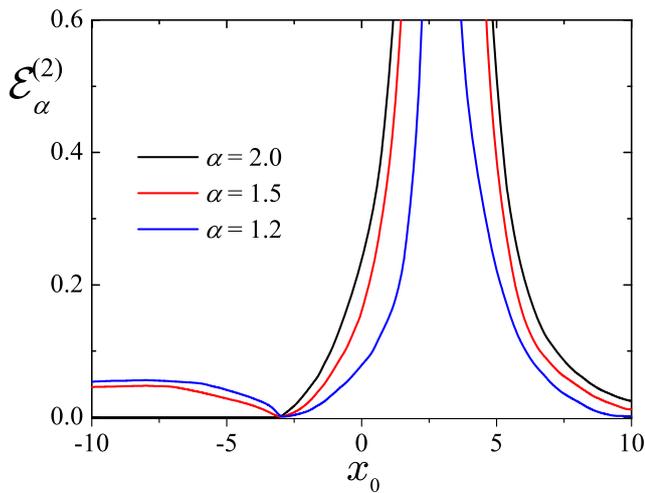
Fig. 11. Splitting search efficiency of hitting target x_2 as function of the starting position x_0. The targets are located at $x_1 = -3$ and $x_2 = 3$, respectively.
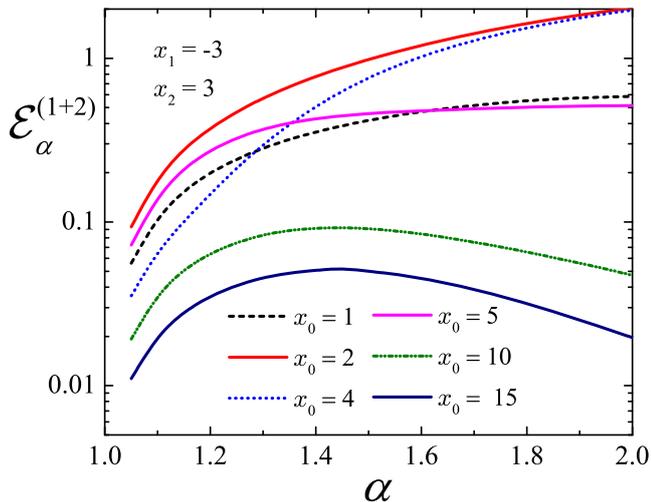
Fig. 12. Search efficiency as function of α, $x_1 = -3$, $x_2 = 3$. $x_0 = 1$ and 2 correspond to the starting position between the targets. The rest of the x_0 value are to the right from the right one.

any type of LFs. However, once the searcher starts further away the longer jumps are a more efficient option. For the same distance $|x_2 - x_0|$ the search efficiency differs, depending on whether one starts on the same half-line with the other target or not. Surprisingly for some α values the efficiency of target search for $x_1 < x_0 < x_2$ can be smaller than for $x_1 < x_2 < x_0$ for the same $x_2 - x_0$ (curves for $x_0 = 1$ and $x_0 = 5$ in Fig. 12).

7 Conclusions

Here we summarised the basic features of LF search for rare targets as well as developed further generalisations of the LF search model. Thus we considered search by a combination of two LF search processes and the search for

multiple targets. Some common patterns can be deduced from all these processes. Namely, in those cases when the target search is easy (for instance, the searcher starts close to the target or the targets, or the target is in a down-hill position relative to the starting point of the searcher) the pure Brownian search strategy beats any LF process, or a combined strategy, with respect to both search reliability and efficiency. However, once the targets are far away, or the search should go in an upstream direction, LF search with its substantial probability of long jumps becomes more successful.

Interestingly, the search properties of combined search strategies cannot be devised from the features of constituent processes. This fact can be clearly demonstrated if one Lévy process has a power-law exponent $\alpha \leq 1$, while the second process has an exponent $\mu > 1$. The first process alone is not capable of locating the target independently of how long the search is, that is, both the search reliability and efficiency are zero. The second process alone eventually locates the target with certainty (in the absence of a potential, which could drive a searcher infinitely away from the target), that is, the second process is recurrent. Non-trivially, the combined process will have a finite search reliability. Our results also show that this effect as well as other search properties stay qualitatively the same in the limiting case when one of the two search processes is Brownian.

As discussed above the generalisation of our approach for the case of many targets leads to similar conclusions regarding successful search strategies. However, some new questions appear in this case. One of the most important is the classical question of splitting probabilities [40]. In the case of two targets we generalise a well-known expression for splitting probabilities of Brownian motion for the case of LF search. The generalised expression reflects the important difference between the continuous exploration of Brownian motion and the jump-like behaviour of LFs. The latter leads to leapovers across a target and, thus, allows the searcher to hit either target from any starting position.

LF search is an idealised process. From a physical point of view the divergence of the second moment of LFs can be avoided by the spatiotemporal coupling offered by Lévy walks – at the expense of the relatively straightforward analytical accessibility. From a biological point of view the assumption that the jump length statistic is not affected by an external bias may be questionable, and some penalty in terms of a thermodynamic efficiency concept should be introduced. However, many of the insights obtained for the simple LF model will, to some extent, also be present in the Lévy walk case, for instance, the question of optimal search for extremely rare targets leads to the same optimal value for the stable index. In a similar way, we believe that other properties such as the splitting behaviour will carry over to more complex search processes. We hope that the results presented here will indeed inspire such research.

Author contribution statement

All authors were involved in the preparation of the manuscript. All authors have read and approved the final manuscript.

Appendix A: Analytical solution for search reliability of two combined LF processes with stable indices $\alpha \leq 1$ and $\mu > 1$

We start with the solution (23) of equation (22). If both stable indices are larger than one, then $P = 1$. If both are smaller or equal to one, then $P = 0$. Hence we concentrate here on the case when $\alpha < 1$ and $\mu > 1$. The search reliability for this case reads

$$P = \wp_{\mathrm{fa}}(s = 0) = \frac{\int_0^\infty \frac{\cos k}{pk^\alpha + k^\mu} dk}{\int_0^\infty \frac{1}{pk^\alpha + k^\mu} dk} = \frac{I_1}{I_2}, \quad (A.1)$$

where

$$I_2 = \int_0^\infty \frac{k^{-\alpha} dk}{p + k^{\mu-\alpha}} = \frac{1}{\mu - \alpha} p^{-\frac{1}{\mu-\alpha}} \frac{\pi}{\sin\left(\pi \frac{1-\alpha}{\mu-\alpha}\right)} \quad (A.2)$$

and

$$I_1 = \frac{1}{p(\mu - \alpha)} \int_0^\infty k^{-\alpha} \cos k H_{11}^{11}\left[\frac{k}{p^{\frac{1}{\mu-\alpha}}} \middle| \begin{matrix} \left(0, \frac{1}{\mu-\alpha}\right) \\ \left(0, \frac{1}{\mu-\alpha}\right) \end{matrix}\right] dk$$

$$= \frac{\sqrt{\pi} 2^{-\alpha}}{p(\mu - \alpha)} H_{31}^{12}\left[\frac{2}{p^{\frac{1}{\mu-\alpha}}} \middle| \begin{matrix} \left(\frac{1+\alpha}{2}, \frac{1}{2}\right), \left(0, \frac{1}{\mu-\alpha}\right), \left(\frac{\alpha}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{\mu-\alpha}\right) \end{matrix}\right]. \quad (A.3)$$

Hence

$$P = \frac{\sin\left(\frac{\pi(1-\alpha)}{\mu-\alpha}\right)}{2\sqrt{\pi}}$$

$$\times H_{31}^{12}\left[\frac{2}{p^{\frac{1}{\mu-\alpha}}} \middle| \begin{matrix} \left(1, \frac{1}{2}\right), \left(\frac{1-\alpha}{\mu-\alpha}, \frac{1}{\mu-\alpha}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \\ \left(\frac{1-\alpha}{\mu-\alpha}, \frac{1}{\mu-\alpha}\right) \end{matrix}\right]. \quad (A.4)$$

Appendix B: Asymptotic behaviour of the search efficiency for $\mu > 1$ and $p \to \infty$

The search efficiency can be expressed through dimensionless units and the timescale of the process with stable index μ as follows

$$\mathcal{E}_{\alpha,\mu} = \frac{1}{t_\mu} \int_0^\infty \wp_{\mathrm{fa}}(p, s) d(st_\mu), \quad (B.1)$$

where $t_\mu = x_0^\mu / K_\mu$ is the time scale of LFs with stable index μ.

B.1 $\alpha > 1$

For $\alpha > 1$ and $\mu > 1$ both LFs have a finite search reliability. In the limit $p \to \infty$ LFs with exponent α dominate the search process, hence, the efficiency should take the form of expression (19).

B.2 $\alpha < 1$

In this case the convergence of expression (23) is due to the term k^μ. Hence the latter cannot be neglected as for $\alpha > 1$. Let us change the variables in equation (B.1) as $st_\mu = p^\nu u$ and $k = p^\gamma \kappa$, where ν and γ will be specified below. Then from equation (B.1) we get

$$\mathcal{E}_{\alpha,\mu} t_\mu = p^\nu \int_0^\infty du \frac{\int_0^\infty \frac{\cos(p^\gamma \kappa)}{p^\nu u + p^{1+\alpha\gamma}\kappa^\alpha + p^{\mu\gamma}\kappa^\mu} p^\gamma d\kappa}{\int_0^\infty \frac{1}{p^\nu u + p^{1+\alpha\gamma}\kappa^\alpha + p^{\mu\gamma}\kappa^\mu} p^\gamma d\kappa}. \quad (B.2)$$

We choose ν and γ such that

$$\nu = 1 + \alpha\gamma = \mu\gamma, \quad (B.3)$$

that is,

$$\gamma = \frac{1}{\mu - \alpha}, \qquad \nu = \frac{\mu}{\mu - \alpha}. \quad (B.4)$$

Then equation (B.2) takes the form

$$\mathcal{E}_{\alpha,\mu} t_\mu = p^\nu \int_0^\infty du \frac{\int_0^\infty \frac{\cos(p^\gamma \kappa)}{u + \kappa^\alpha + \kappa^\mu} d\kappa}{\int_0^\infty \frac{1}{u + \kappa^\alpha + \kappa^\mu} d\kappa}. \quad (B.5)$$

The integral in the denominator converges for all positive u, does not depend on p, and has an upper bound at $u = 0$,

$$f(u) = \int_0^\infty \frac{1}{u + \kappa^\alpha + \kappa^\mu} d\kappa \leq f(0)$$

$$= \int_0^\infty \frac{1}{\kappa^\alpha + \kappa^\mu} d\kappa. \quad (B.6)$$

As for the integral in the numerator, since $p \gg 1$ the main contribution comes from small κ. We thus neglect κ^μ in comparison with κ^α and use the approach from Appendix B in reference [112]. Hence, for the search efficiency we get

$$\mathcal{E}_{\alpha,\mu} t_\mu \approx p^\nu \int_0^\infty \frac{du}{f(u)} \int_0^\infty \frac{\cos(p^\gamma \kappa)}{u + \kappa^\alpha} d\kappa$$

$$\sim p^\nu \int_0^\infty \frac{du}{f(u)} \int_0^\infty d\tau e^{-u\tau} \frac{1}{\tau^{1/\alpha}} l_\alpha\left(\frac{p^\gamma}{\tau^{1/\alpha}}\right), \quad (B.7)$$

where

$$y = p^\gamma / \tau^{1/\alpha},$$
$$\tau = p^{\gamma\alpha} / y^\alpha,$$
$$d\tau = -\frac{1}{\alpha} \frac{p^{\gamma\alpha}}{y^{\alpha+1}} dy. \quad (B.8)$$

Thus

$$
\mathcal{E}_{\alpha,\mu} t_\mu \sim p^{\nu-\gamma+\gamma\alpha} \int_0^\infty \frac{du}{f(u)} \int_0^\infty dy \exp\left(-\frac{up^{\gamma\alpha}}{y^{\gamma\alpha}}\right) y^{-\alpha} l_\alpha(y)
$$

$$
= p^{\nu-\gamma+\gamma\alpha} \int_0^\infty dy\, l_\alpha(y)\, y^{-\alpha} \int_0^\infty \frac{du}{f(u)} \exp\left(-\frac{up^{\gamma\alpha}}{y^{\gamma\alpha}}\right)
$$

$$
= p^{\nu-\gamma} \int_0^\infty dy\, l_\alpha(y) \int_0^\infty \frac{e^{-t} dt}{f\left(\dfrac{y^\alpha t}{p^{\gamma\alpha}}\right)}
$$

$$
\sim p^{\nu-\gamma} \sim p^{(\mu-1)/(\mu-\alpha)}, \quad \text{for } p \to \infty, \tag{B.9}
$$

due to relation (B.6), that is, for $\alpha < 1$

$$
\mathcal{E}_{\alpha,\mu} \sim p^{\frac{\mu-1}{\mu-\alpha}}. \tag{B.10}
$$

B.3 $\alpha = 1$

For $\alpha = 1$ we can use equation (B.5) with $\nu = \frac{\mu}{\mu-1}$ and $\gamma = \frac{1}{\mu-1}$, such that

$$
\mathcal{E}_{\alpha,\mu} t_\mu = p^\nu \int_0^\infty du \frac{\displaystyle\int_0^\infty \frac{\cos(p^\gamma\kappa)}{u+\kappa+\kappa^\mu} d\kappa}{\displaystyle\int_0^\infty \frac{1}{u+\kappa+\kappa^\mu} d\kappa} = p^\nu \int_0^\infty du \frac{f(u)}{f_1(u)}. \tag{B.11}
$$

For the integral in the numerator, similar to the case $\alpha < 1$ (Appendix B.2) the main contribution comes from small κ values due to $p \gg 1$. Hence we can neglect κ^μ in comparison with κ. Thus

$$
f(u) = \int_0^\infty \frac{\cos(p^\gamma\kappa)}{u+\kappa+\kappa^\mu} d\kappa \simeq \int_0^\infty \frac{\cos(p^\gamma\kappa)}{u+\kappa} d\kappa = g(p^\gamma u), \tag{B.12}
$$

where $g(z)$ can be expressed through sine and cosine integrals $\mathrm{Si}(z)$ and $\mathrm{Ci}(z)$ as

$$
g(z) = -\mathrm{Ci}(z)\cos(z) - (\mathrm{Si}(z) - \pi/2)\sin(z), \tag{B.13}
$$

$$
\mathrm{Si}(z) = \int_0^z \frac{\sin y}{y} dy, \tag{B.14}
$$

$$
\mathrm{Ci}(z) = \int_z^\infty \frac{\cos y}{y} dy. \tag{B.15}
$$

The function in the denominator for small arguments u depends on u as $f_1(u) \sim -\ln u$. Equation (B.11) can be rewritten in the form

$$
\mathcal{E}_{\alpha,\mu} t_\mu = p^{\frac{\mu}{\mu-1}} \int_0^1 \frac{du}{f_1(u)} g\left(p^{\frac{1}{\mu-1}} u\right)
$$

$$
+ p^{\frac{\mu}{\mu-1}} \int_1^\infty \frac{du}{f_1(u)} g\left(p^{\frac{1}{\mu-1}} u\right). \tag{B.16}
$$

For the second term in the latter expression one can use the asymptotic of $g(z) \sim 1/z^2$ for $pu \gg 1$ since $p \gg 1$ (see

Eq. (5.2.35) in Ref. [114]). This implies that the contribution from the second term decreases with increasing p at large p values.

The first term can be rewritten as

$$
p^{\frac{\mu}{\mu-1}} \int_0^1 \frac{du}{f_1(u)} g(p^{\frac{1}{\mu-1}} u) = p \int_0^{p^{\frac{1}{\mu-1}}} \frac{dy}{f_1\left(y/p^{\frac{1}{\mu-1}}\right)} g(y). \tag{B.17}
$$

The upper bound of this term is given by ($f_1(y)$ is a monotonously decreasing function of y)

$$
p \int_0^{p^{\frac{1}{\mu-1}}} \frac{dy}{f_1\left(y/p^{\frac{1}{\mu-1}}\right)} g(y) < \frac{p}{f_1(1)} \int_0^\infty dy\, g(y), \tag{B.18}
$$

as $g(y)$ is integrable on $[0, \infty)$ and we can replace the upper limit p of the integral with ∞ at $p \gg 1$. Thus, the first term in equation (B.16) does not grow faster than p. To get a lower bound for the growth limit of large p we use the first mean value theorem [115] and a small argument asymptotic $f_1(u) \sim -\ln u$, yielding

$$
p \int_0^{p^{\frac{1}{\mu-1}}} \frac{dy}{f_1\left(y/p^{\frac{1}{\mu-1}}\right)} g(y) = \frac{p}{f_1\left(y^*/p^{\frac{1}{\mu-1}}\right)} \int_0^{p^{\frac{1}{\mu-1}}} dy\, g(y)
$$

$$
\sim \frac{p}{-\ln\left(y^*/p^{\frac{1}{\mu-1}}\right)} \int_0^\infty dy\, g(y), \tag{B.19}
$$

where $0 < y^* < p^{\frac{1}{\mu-1}}$. Hence

$$
\mathcal{E}_{\alpha,\mu} t_\mu \sim \frac{p}{\ln p}, \tag{B.20}
$$

which is confirmed by numerical simulations.

Appendix C: Derivation of long time asymptotics for two targets

The first arrival density reads

$$
\wp_{\mathrm{fa}}(s) = \frac{\frac{W_{10}}{W_0} + \frac{W_{20}}{W_0}}{\frac{W_{12}}{W_0} + 1}. \tag{C.1}
$$

For the ratio W_{ij}/W_0 the limit of small s was calculated in Appendix A of reference [112]:

$$
\frac{W_{ij}(s)}{W_0(s)} \approx 1 - \Lambda(\alpha) s^{1-\frac{1}{\alpha}} |x_j - x_i|^{\alpha-1} \tag{C.2}
$$

Correspondingly,

$$
\wp_{\mathrm{fa}}(s) \approx \frac{2 - \Lambda(\alpha)(s/K_\alpha)^{1-1/\alpha}\left(|x_1-x_0|^{\alpha-1} + |x_2-x_0|^{\alpha-1}\right)}{2 - \Lambda(\alpha)(s/K_\alpha)^{1-1/\alpha}|x_2-x_1|^{\alpha-1}}
$$

$$
\approx 1 - \frac{\Lambda(\alpha)}{2}(s/K_\alpha)^{1-1/\alpha}
$$

$$
\times \left(|x_1-x_0|^{\alpha-1} + |x_2-x_0|^{\alpha-1} - |x_2-x_1|^{\alpha-1}\right), \tag{C.3}
$$

$$
\begin{aligned}
\wp_{\mathrm{fa1}}(s) &= \frac{W_{20}W_{12} - W_{10}W_0}{W_{12}^2 - W_0^2} = \frac{\left(\frac{W_{20}}{W_0}\right)^2 - \frac{W_{10}}{W_0}}{\left(\frac{W_{12}}{W_0}\right)^2 - 1} \\
&\approx \frac{\left(1 - \Lambda(\alpha)(s/K_\alpha)^{1-1/\alpha}|x_2 - x_0|^{\alpha-1}\right)\left(1 - \Lambda(\alpha)(s/K_\alpha)^{1-1/\alpha}|x_2 - x_1|^{\alpha-1}\right) - \left(1 - \Lambda(\alpha)(s/K_\alpha)^{1-1/\alpha}|x_1 - x_0|^{\alpha-1}\right)}{(1 - \Lambda(\alpha)(s/K_\alpha)^{1-1/\alpha}|x_2 - x_1|^{\alpha-1})^2 - 1} \\
&\approx \frac{-|x_1 - x_0|^{\alpha-1} + |x_2 - x_0|^{\alpha-1} + |x_2 - x_1|^{\alpha-1} + \Lambda(\alpha)|x_2 - x_0|^{\alpha-1}|x_2 - x_1|^{\alpha-1}(s/K_\alpha)^{1-1/\alpha}}{2|x_2 - x_1|^{\alpha-1} + \Lambda(\alpha)|x_2 - x_1|^{2\alpha-2}(s/K_\alpha)^{1-1/\alpha}}.
\end{aligned}
\tag{C.5}
$$

where

$$
\Lambda(\alpha) = \frac{\alpha \Gamma(2-\alpha)}{\pi(\alpha-1)} \sin\left(\frac{\pi\alpha}{2}\right) \sin\left(\frac{\pi}{\alpha}\right).
\tag{C.4}
$$

Due to Minkowski's inequality the combination of the absolute values in the brackets is always non-negative. From that expression one can see that the search reliability is $P = 1$. Now, let us consider the splitting densities. The probability to hit the first target can be written as

See equation (C.5) above.

The second splitting probability density $\wp_{\mathrm{fa2}}$ can be computed in exactly the same way and produces a result, which can be written by swapping x_1 and x_2 in the last expression (C.5).

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