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Search reliability and search efficiency of combined Lévy–Brownian motion: long relocations mingled with thorough local exploration

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Abstract

A combined dynamics consisting of Brownian motion and Lévy flights is exhibited by a variety of biological systems performing search processes. Assessing the search reliability of ever locating the target and the search efficiency of doing so economically of such dynamics thus poses an important problem. Here we model this dynamics by a one-dimensional fractional Fokker–Planck equation combining unbiased Brownian motion and Lévy flights. By solving this equation both analytically and numerically we show that the superposition of recurrent Brownian motion and Lévy flights with stable exponent $\alpha < 1$, by itself implying zero probability of hitting a point on a line, leads to transient motion with finite probability of hitting any point on the line. We present results for the exact dependence of the values of both the search reliability and the search efficiency on the distance between the starting and target positions as well as the choice of the scaling exponent α of the Lévy flight component.

Keywords: random search process, first passage, first arrival, Lévy flights, Brownian motion

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the most fundamental questions about any type of motion is whether a moving particle starting from a point A is able to reach some pre-selected point B [1]. After the concept of random walks was brought to wide attention by Karl Pearson in 1905 [2], for simple random walks on a lattice George Pólya gave an answer to this question by proving that in one and two dimensions random walks are recurrent, which implies that a walker will visit any lattice site eventually. In three and higher dimensions this motion becomes transient [3]. For random walks where the jump lengths ℓ are drawn from Lévy α-stable distributions, $p(\ell) \sim |\ell|^{-1-\alpha}$ with $0 < \alpha < 2$ [4, 5], the motion is recurrent for $\alpha \geqslant 1$, otherwise it is transient [6]. However, even if the motion is transient there might be a non-zero probability of visiting a certain point. For instance the Brownian motion is transient in three dimensions, however, there exists a finite probability of return[8]. The probability of hitting a given point in space can be directly calculated from the density $\wp_{\mathrm{fa}}(t)$ of the *first arrival time*, or hitting time t to a specific point, which conveniently characterises the hitting process. Integrating $\wp_{\mathrm{fa}}(t)$ over time produces the cumulative probability P of reaching this point, called the *search reliability* [7, 8].

The search reliability is a useful quantity of a search strategy if one is not interested in how long the search might take. However it does not provide any information about the efficiency of a search process in terms of average times for target location. Such information is crucial to assess real world search scenarios [9] occurring over a wide range of spatio-temporal scales, from the search of transcription factor proteins for a specific place on a DNA chain [10–15] through food search by animals [16–18] to rescue operations [19] or algorithms for finding the minima in a complex search space [20]. Any good measure of search efficiency must take into account the specific nature of a particular search process, which a searcher may seek to optimise.

Brownian motion was considered to be the default for a successful random search strategy in most cases until the 1980s. In 1986 Shlesinger and Klafter challenged this dogma, suggesting that Lévy flights (LFs) represent a better strategy if a searcher looks for sparsely distributed targets [21][9]. Due to the divergence of the mean squared average of the jump length ℓ the trajectory of LFs has a different fractal dimension than the trajectory of Brownian motion [1, 23]. This allows one to avoid oversampling, i.e. revisiting the same point several times, which is typical for recurrent Brownian motion in one and two dimensions [17, 24]. In the 1990's Lévy motion was put forward as an optimal foraging strategy for animals searching for sparse food [17, 25, 26] and the approach was recently extended to patchy environments [27]. This so-called LF hypothesis triggered a vivid debate [28–30] in the field of movement ecology [17, 18, 31]. It was argued that Lévy-like motion has been observed for many animals such as albatrosses [32], marine predators [33, 34], terrestrial animals like goats and deer [35, 36] and even microzooplankton [37]. Heavy-tailed distributions were also reported to characterise human movement patterns [38, 39].

However, for certain animals the visual perception of their environment becomes more limited when moving with higher velocity. Observations show that, to remedy this, in these cases the search process alternates between a slow recognition mode during which a target can be found, and fast relocation events where the searcher is insensitive to any target search [16, 40, 41], see figure 1. This poses the need for theoretical modelling to combine saltatory, jump-like, with cruise motion yielding *intermittent strategies*, which feature combinations of

[8] On a simple cubic lattice in three dimensions the returning probability is $\approx$0.34 [1].
[9] The opposite case of a predator hunting a relatively dense herd of prey animals is considered, for instance, in [22].

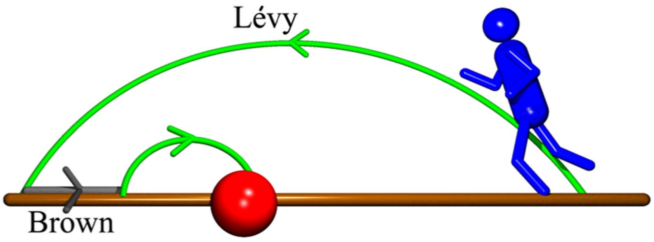

Figure 1. Sketch of intermittent search in one dimension: a searcher (blue) proceeds by a combination of Lévy jumps and Brownian steps until it finds the target (red). Physically, Lévy jumps decorrelate the motion, leading the searcher to sites not previously visited. Brownian motion, instead, provides a thorough local search at the price of oversampling, see text.

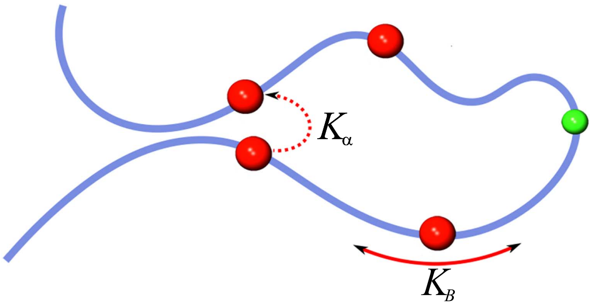

Figure 2. Target search of a protein, or enzyme, along a fast-folding DNA chain allowing a dimensional reduction modelled by intermittent motion: K_α and K_{B} denote the switching rates for performing LFs representing intersegmental transfers through the bulk due to unbinding and rebinding (arrows), respectively Brownian motion mimicking 1D Brownian sliding along the polymer chain until the moving particle (red) finds the target (green) [51].

at least two different types of motion, e.g. Brownian and ballistic motion, or Brownian motion and LFs [16, 42, 43]. A related type of intermittent dynamics is that of composite Brownian motion [29, 30], which was proposed to model the search of a forager or particle in patchy environments. Here inter and intra patch movements are defined by a combination of Brownian modes with different mean step lengths, however, the searcher can detect targets in both modes. Recently it was argued that this type of dynamics was observed in the movements of mussels [44, 45]. Composite Brownian motion can be generalised to an adaptive Lévy walk, where the Brownian inter-patch movement is replaced by Lévy motion [46]. Intermittent dynamics consisting of Brownian and Lévy motion has indeed been observed for a number of biological organisms, such as microzooplankton depending on the density of the prey [37], coastal jellyfish [47], mussels moving in dense environments [48] and a variety of marine predators hunting in different environments [49, 50]. Further generalisations of such models use switching rates [51], or sample the switching times from one mode to another from different distributions [52]. The latter type of modelling was motivated by studying the target search of proteins on fast-folding polymer chains, see figure 2. Formally, these models form special cases of distributed order fractional diffusion equations [53] or diffusion equations whose Laplacian is augmented with a space-fractional term [51]. They have also been derived as long-time approximations for correlated Lévy walks [54]. An optimal strategy

—in the sense of maximising a chosen efficiency—thus depends at least on the type of motion, the switching distributions and the dimension of the search space [16, 52]. On a molecular scale the search patterns of regulatory proteins for their target binding site on finite DNA chains is an LF with a cutoff for the jump length distribution, however, the advantage of the combined search modes through the bulk and along the DNA significantly improve the search rate [55–57].

In previous works the search reliability and efficiency for LFs and Brownian motion were studied separately and compared with each other [7, 8]. Motivated by the examples of intermittent motion referred to above, we here examine a combined process consisting of both Brownian and Lévy components. Our paper is structured as follows: after defining the quantities of interest (the search efficiency and reliability) in section 2, section 3 recalls the results for pure Brownian and LF search. Sections 4 & 5, respectively, then present our results for the search reliability and search efficiency. In section 6 we provide a discussion of our results. Details for the analytical calculations are provided in the appendix.

2. Quantities of interest

We characterise a search strategy by two different quantities. The first one is the search reliability, which is the cumulative probability P of ever reaching the target. It can be expressed through the Laplace image of the first arrival time density $\wp_{\mathrm{fa}}(t)$ as [7]

$$P = \lim_{s\to 0} \wp_{\mathrm{fa}}(s), \tag{1}$$

where the Laplace transform of a function $f(t)$ is defined via $f(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t$. The search reliability depends on the type of random walk as well as geometrical details (dimension, distance from the starting position to the target etc.). Thus $P = 1 - \mathscr{S}$, where $\mathscr{S}$ is the survival probability [59, 60]. The latter quantity can be tackled by solving a (fractional) Fokker–Planck equation with a sink term [7, 8, 58]. For search in one dimension by LFs without a bias the search reliability is unity if $\alpha > 1$ and zero otherwise [58], which is consistent with previous results [6]. For search in the presence of a bias the search reliability can vary between zero and unity [7, 8], which is true even for Brownian motion [59], where for the case of the bias pushing a searcher from the target the search reliability is described by an exponential (Boltzmann) factor [59]. A search reliability of unity does not necessarily imply recurrence of the motion. For instance, LFs with $\alpha = 1$ in one dimension and Brownian motion in two dimensions are recurrent but the search reliability is 0.

The second quantity of interest is the search efficiency. Most of the theoretical studies consider a probabilistic searcher with a limited radius of perception. Motivated by [61], in this case two basic definitions of the search efficiency are considered to be either

$$\text{Efficiency}_1 = \frac{\text{visited number of targets}}{\text{number of steps}}, \tag{2}$$

or

$$\text{Efficiency}_2 = \frac{\text{visited number of targets}}{\text{distance travelled}}. \tag{3}$$

The first definition applies especially to *saltatory search*, where a searcher moves in a jump-like fashion and is able to detect the target only around the landing point after a jump. The second formula is adapted to *cruise motion*, where the searcher keeps exploring the search space continuously during the whole search process. An example for the former scenario is given by a regulatory protein that moves in three dimensional space and occasionally binds to the DNA of a biological cell until it finds its binding [11–15]. The latter scenario would correspond to an eagle or vulture whose excellent eyesight permits them to scan their environment for food during their entire flight. For LFs equation (2) presents a natural choice while equation (3) is better suited for processes like Brownian motion and Lévy walks [63].

In this paper we focus on the limit of a sparse target density, which is approximated by the situation when only one target can be found. For a single target and saltatory motion we argued that the efficiency should be defined from equation (2) with a proper averaging [8]. In our continuous time model the number of steps from (2) is naturally substituted by the time of the process. Since we have one target, the number of targets found on average can be less than one. Obviously, a time averaging is needed, and, hence, we choose

$$\mathcal{E} = \left\langle \frac{1}{t} \right\rangle = \int_0^\infty \wp_{\mathrm{fa}}(s)\mathrm{d}s. \tag{4}$$

Below we use the search reliability & efficiency in the sense of equations (1) & (4) to characterise search strategy of combined Lévy–Brownian motion.

3. First arrival density from a fractional Fokker–Planck equation

The search properties of a process combining LFs and Brownian motion can be effectively calculated from a space-fractional Fokker–Planck diffusion equation similar to the one considered in [51, 58] for the non-normalised probability density function (PDF) $f(x, t)$,

$$\frac{\partial f(x, t)}{\partial t} = K_\alpha \frac{\partial^\alpha f(x, t)}{\partial |x|^\alpha} + K_{\mathrm{B}} \frac{\partial^2 f(x, t)}{\partial x^2} - \wp_{\mathrm{fa}}(t)\delta(x), \tag{5}$$

where without losing generality the target is located at $x = 0$. We assume that at $t = 0$ the searcher is placed at $x = x_0$, i.e., $f(x, 0) = \delta(x - x_0)$. The consequence of the δ-sink at $x = 0$ is the condition $f(0, t) = 0$ [51, 58]. The fractional derivative $\partial^\alpha/\partial|x|^\alpha$ can be introduced in terms of its Fourier transform,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\left[\frac{\partial^\alpha}{\partial|x|^\alpha} f(x, t)\right]\mathrm{d}x = -|k|^\alpha f(k, t). \tag{6}$$

We should note here that in [51] a similar but more specific equation was used for the description of the problem of protein diffusion on a polymer chain. In comparison with our equation (5) it contained additionally two terms, which described the contributions from an adsorption and desorption of particles modelling the exchange of particles with the ambient bulk solvent, and was solved for different initial conditions. For this specific problem the optimal search minimised the mean first arrival time, which is always finite for that case. In our case this quantity can become infinitely large, hence the analysis in [51] is not applicable to the physical situation considered here.

Integration over the position coordinate of equation (5) yields

$$\wp_{\mathrm{fa}}(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} f(x, t)\mathrm{d}x. \tag{7}$$

Hence $\wp_{\mathrm{fa}}(t)$ is the negative time derivative of the survival probability, i.e. $\wp_{\mathrm{fa}}(t)$ is indeed the probability of first arrival: as soon as a walker gets to the sink it is absorbed[10].

[10] We remind the reader that for Brownian motion first arrival and first passage lead to identical results, whereas the definitions are conceptually different for LFs [58, 62].

Analogously to pure search by LFs [58] it is easy to find a solution $f(k, s)$ of equation (5) in Fourier–Laplace space

$$f(k, s) = \frac{\mathrm{e}^{\mathrm{i}kx_0} - \wp_{\mathrm{fa}}(s, x_0)}{s + K_\alpha |k|^\alpha + K_\mathrm{B} k^2}, \tag{8}$$

see also [7]. Integration of equation (8) over k yields:

$$\int_{-\infty}^{\infty} f(k, s)\mathrm{d}k = f(x = 0, s) = 0 = W(-x_0, s) - W(0, s)\wp_{\mathrm{fa}}(s, x_0), \tag{9}$$

where $W(x, s)$ is a solution of equation (5) without the sink term. Hence the probability of first arrival becomes

$$\wp_{\mathrm{fa}}(s) = \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{\mathrm{i}kx_0}}{s + K_\alpha |k|^\alpha + K_\mathrm{B} k^2}}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}k \frac{1}{s + K_\alpha |k|^\alpha + K_\mathrm{B} k^2}}. \tag{10}$$

Equation (10) can be expressed as a function of dimensionless variables as:

$$\wp_{\mathrm{fa}}(s) = \frac{\displaystyle\int_0^{\infty} \frac{\cos k}{st_\mathrm{B} + pk^\alpha + k^2}\mathrm{d}k}{\displaystyle\int_0^{\infty} \frac{1}{st_\mathrm{B} + pk^\alpha + k^2}\mathrm{d}k}, \tag{11}$$

where $t_\mathrm{B} = x_0^2/K_\mathrm{B}$ is the time scale set by Brownian motion over the length x_0 and

$$p = x_0^{2-\alpha} K_\alpha / K_\mathrm{B}. \tag{12}$$

According to equations (1) and (4), in order to get the reliability and efficiency of the combined search, one has to compute, respectively, the limit $s \to 0$ of equation (11) and the integral of equation (11) over s from zero to infinity. Before going into this analysis and its consequences, we recall the results for search by Brownian motion and LFs strategies separately.

3.1. Brownian search

If the search process proceeds only with Brownian moves, i.e. $K_\alpha = 0$, the first arrival density can be computed analytically and in Laplace space reads

$$\wp_{\mathrm{fa}}(s) = \exp\left(-x_0\sqrt{\frac{s}{K_\mathrm{B}}}\right) \tag{13}$$

or, back-transformed to time,

$$\wp_{\mathrm{fa}}(t) = \frac{x_0}{\sqrt{4\pi K_\mathrm{B} t^3}} \exp\left(-\frac{x_0^2}{4K_\mathrm{B} t}\right). \tag{14}$$

This is well known Lévy–Smirnov density [60]. Obviously in this case the search reliability (1) is $P = 1$, and the efficiency $\mathcal{E}_\mathrm{B} = \frac{2K_\mathrm{B}}{x_0^2}$.

3.2. First arrival for pure Lévy search

The expression for $\wp_{\mathrm{fa}}(s)$ for pure Lévy search can be computed in terms of Fox H-functions [8]. In the limit of small s corresponding to the long-time limit, $\wp_{\mathrm{fa}}(s)$ can be computed in terms of elementary functions (see appendix A). For $\alpha \leqslant 1$ the search reliability is $P = 0$, for $\alpha > 1$ the reliability is $P = 1$. By integration of the corresponding H-function expression in Laplace space one gets the simple equation for the search efficiency [8] (see also a derivation without use of H-functions in appendix B),

$$\mathcal{E}_\alpha = \left\langle \frac{1}{t} \right\rangle = \frac{\alpha K_\alpha}{x_0^\alpha} \cos\left(\pi\left(1 - \frac{\alpha}{2}\right)\right)\Gamma(\alpha), \qquad 1 < \alpha < 2. \tag{15}$$

4. Search reliability for combined Lévy–Brownian search

In order to compute the search reliability one has to take the limit $s \to 0$ of expression (11) as pointed out in equation (1). If $\alpha \geqslant 1$, both integrals in (11) diverge at $s = 0$. The divergence occurs at $k \to 0$, and only the values of the integral very close to $k = 0$ (where $\cos k \approx 1$) make a contribution to the integral. Hence $P = 1$, which is understandable intuitively, because if one combines two processes with search reliability $P = 1$, then the combined process should also have this property. Interestingly the search reliability for a combination of LFs with $\alpha = 1$ and Brownian motion also has $P = 1$, while pure LF search with $\alpha = 1$ is absolutely unreliable ($P = 0$) [7].

The case $\alpha < 1$ is less trivial. Even if $s = 0$, both integrals in equation (11) are convergent and the search reliability can be computed in terms of H-functions (appendix C), resulting in

$$P = \frac{\sin\left(\frac{\pi}{2-\alpha}\right)}{2\sqrt{\pi}} H_{31}^{12}\left[\frac{2}{p^{1/(2-\alpha)}} \left| \begin{matrix} \left(1, \frac{1}{2}\right)\left(\frac{1-\alpha}{2-\alpha}, \frac{1}{2-\alpha}\right)\left(\frac{1}{2}, \frac{1}{2}\right) \\ \left(\frac{1-\alpha}{2-\alpha}, \frac{1}{2-\alpha}\right) \end{matrix} \right.\right], \tag{16}$$

where the parameter p is defined in equation (12). Result (16) is naturally independent of the time t_B, because rescaling of the time should not change the search reliability. In the case $\alpha = 0$ the corresponding fractional derivative in equation (5) is of zeroth order, which corresponds to a Fokker–Planck equation with decay term $-f(x, t)$. The result (16) simplifies to $P(\alpha = 0) = \exp(-\sqrt{p}) = \exp(-\sqrt{x_0^{2-\alpha} K_\alpha / K_B})$, in complete agreement with the solution of the diffusion equation with decay term.

The standard expansion for H-functions [66] can be used to find the leading behaviour of equation (16) in the limit of small p,

$$P(p \ll 1) \approx 1 - C_1(\alpha) p^{\frac{1}{2-\alpha}} + \frac{1}{2} p^{\frac{2}{2-\alpha}} - C_2(\alpha) p^{\frac{3-\alpha}{2-\alpha}}, \tag{17}$$

where the coefficients are defined as[11]

$$\begin{aligned} C_1(\alpha) &= \frac{2-\alpha}{2} \sin\left(\frac{\pi}{2-\alpha}\right), \\ C_2(\alpha) &= \frac{(2-\alpha)\sin\left(\frac{\pi}{2-\alpha}\right)}{2^{4-\alpha}\sqrt{\pi}} \frac{\Gamma\left(-\frac{3}{2} + \frac{\alpha}{2}\right)}{\Gamma\left(2 - \frac{\alpha}{2}\right)}. \end{aligned} \tag{18}$$

[11] In the limit of $\alpha \to 1$ the coefficient $C_1(\alpha)$ vanishes. However P does not exceed the value unity, as in this limit we should include the third term in the expansion, as it has the same power. Then C_3 exactly cancels with C_2 and no contradiction appears.

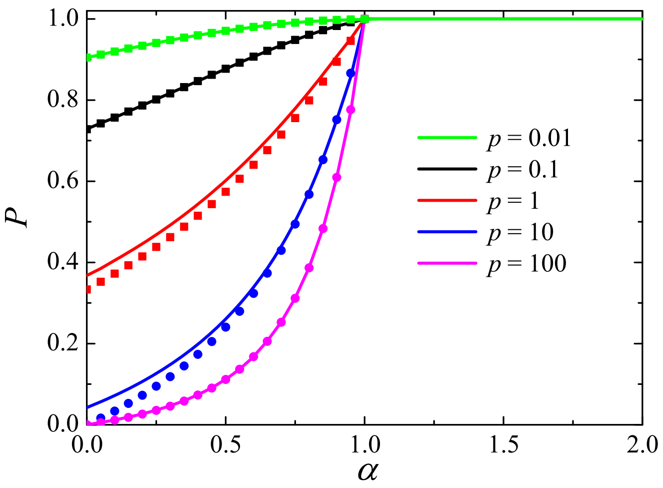

Figure 3. Search reliability P as function of the Lévy stable exponent α for various values of p, equation (12). The continuous curves are obtained by numerical computation of equation (11). The squares for the cases $p = 0.01$ (green), $p = 0.1$ (black) and $p = 1$ (red) are obtained from the approximation (17) for small p. The circles of corresponding colour are plotted from the approximate expression (19) for large p.

From equation (11) one can also find an expansion for P in the limit of large p, which reads (for the derivation see appendix D),

$$P(p \gg 1) \approx \frac{1}{\pi}\Gamma(1-\alpha)\sin\left(\frac{\pi\alpha}{2}\right)(2-\alpha)\sin\left(\frac{\pi}{2-\alpha}\right)p^{(\alpha-1)/(2-\alpha)}. \tag{19}$$

Note that the limiting value of the latter expression for $\alpha \to 1-$ is 1, i.e. the divergence of $\Gamma(1-\alpha)$ is exactly compensated by the convergence of $\sin(\pi/(2-\alpha))$ to zero.

In figure 3 the search reliability P is plotted as a function of the stable index α. As discussed above, for $\alpha \geqslant 1$ the value of P is unity, i.e. the combined Lévy–Brownian search is absolutely reliable. However, when $\alpha < 1$ the hitting probability is less than unity and decreases with α until it reaches the values for the diffusion equation with decay term. The curves were obtained from the numerical computation of the integral ratio in equation (11). The expansion of equation (16) in the limit of small p, i.e. Equation (17) gives a very good approximation for $p \lesssim 1$ (green, red and black squares). For $p \gtrsim 10$ the expansion (19) for the limit $p \gg 1$ works quite well even for small α. For α close to unity, it approximates the numerical solution nicely even for $p \gtrsim 1$.

Obviously P should depend on the distance between the starting position and the target through the parameter p defined in equation (12). If in figures 3 and 4 we kept the diffusion coefficients fixed, for instance, $K_\alpha = 1$ cm$^\alpha$ s^{-1} and $K_B = 1$ cm^2 s^{-1}, the dependence on p would essentially become a dependence on the initial distance between the searcher and the target to the power $(2-\alpha)$. In figure 3 the curves reflect that higher values of p lead to lower amplitudes of the curves in the plot. In figure 4 this dependence is shown explicitly. The larger the initial separation x_0 between the searcher and the target the smaller becomes the value of P. The decrease in α, which corresponds to the higher fraction of long jumps, leads to a drop in the search reliability. Circles show the large p approximation (19), while squares correspond to the small p formula (17). One can see that for $\alpha = 0.75$ (red symbols/curve)

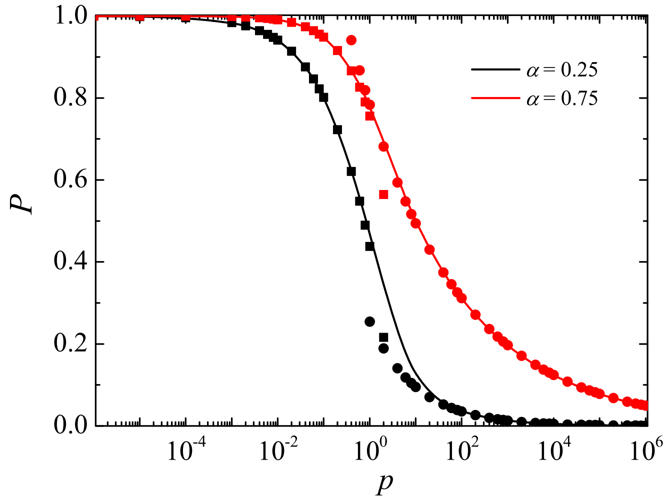

Figure 4. Search reliability as a function of the parameter $p = x_0^{2-\alpha} K_\alpha / K_B$ for various stable indices α. Continuous lines were obtained from numerical solution of equation (11). The squares show the analytical approximation (17) for small p. The circles are plotted from the approximate expression (19) for large p.

these two limiting expressions describe the whole curve quite well. Even for the quite low value $\alpha = 0.25$ (black symbols/curve) the quality of the correspondence of the asymptotic formulas is still very good.

5. Search efficiency for combined Lévy–Brownian search

One can rewrite the definition of the search efficiency (4) as follows,

$$\mathcal{E}(p) = \frac{\mathcal{E}_B}{2} \int_0^\infty \wp_{fa}(p, s) \mathrm{d}(st_B), \tag{20}$$

where $\mathcal{E}_B$ is the Brownian efficiency $\mathcal{E}_B = 2K_B / x_0^2$ of the search process with $p = 0$ and $\wp_{fa}(p, s)$ is determined by equation (11). If $p = 0$, then $\int_0^\infty \wp_{fa}(p, s) \mathrm{d}(st_B) = 2$, i.e. $\mathcal{E}(p = 0) = \mathcal{E}_B$, as it should be. The value of the integral in equation (20) does not depend on the time t_B but only on the value of the parameter p. Hence in figure 5 we plot the ratio of the efficiency of combined Brown–LF search normalised by the efficiency $\mathcal{E}_B$ of pure Brownian search as a function of the parameter p. The continuous curves represent numerical results, and the dashed lines are asymptotes in the limit of large p (for the derivation see appendix F). First, we see that in all cases the efficiency increases monotonically with p. If p is constant then the decrease of α leads to a monotonic decrease of the efficiency. Secondly, there is a qualitative difference for the Brownian–Lévy search with $\alpha > 1$ and $\alpha < 1$. In the former case (for which, as we know, LFs without Brownian motion find the target anyway) the efficiency in the limit of large p is proportional to p. A careful calculation shows that in this case $\mathcal{E} = \mathcal{E}_\alpha$ (appendix F), i.e. the search efficiency is determined only by LFs. This is reasonable, because if both processes lead to the location of the target, only the one with a very large noise strength should matter. In the latter case for which pure LFs would not succeed ($\mathcal{E}_\alpha = 0$) the asymptotic dependence is of power law form $p^{1/(2-\alpha)}$, i.e. even the

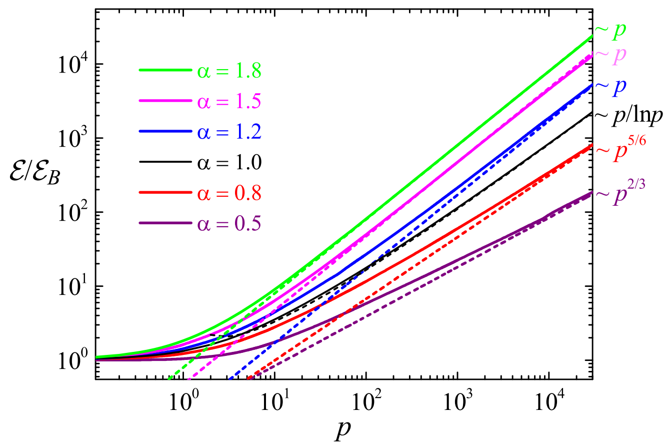

Figure 5. Search efficiency as function of the parameter $p = x_0^{2-\alpha}K_\alpha/K_\mathrm{B}$. Continuous curves correspond to numerical results for the efficiency of the Brown–LF strategy for various values of the stable index α. Dashed lines are asymptotes for the limit $p \to \infty$, which are derived in appendix F.

smallest fraction of Brownian motion makes the location of the target possible. This observation is consistent with the fact that the search reliability does not fall off to zero at $\alpha \leqslant 1$ (cf figure 3) The limiting case $\alpha = 1$ shows a logarithmic correction. An interesting point here is that for $\alpha = 1$ pure LF search is absolutely unreliable, $P = 0$, but the smallest contribution of Brownian motion makes the search absolutely reliable, $P = 1$.

Although figure 5 shows the complete dependence of the efficiency on all original parameters K_α, K_B, x_0, and α, which are combined into a single parameter p, the plot should be interpreted carefully. At first glance it seems that an increase of α leads to an increase of the search efficiency if $p = const$, i.e. Brownian motion will be the most efficient search strategy. However, any change of α implicitly affects the parameter p, compare equation (12). Even if one assumes that K_B is constant the fractional diffusion coefficient K_α will change its dimension with change of α and in order to keep a fixed value of p one needs to change the distance from the target.

In order to fix the starting position and compare the strategies in this practically important case we plot the search efficiency as a function of α for fixed $x_0 = 1$, 10, 20, 50 cm and $K_\alpha = 1\mathrm{cm}^\alpha\mathrm{s}^{-1}$, $K_2 = 1\mathrm{cm}^2\mathrm{s}^{-1}$ in figure 6. The curve for $x_0 = 1$ cm monotonically increases with α, i.e. the Brownian strategy is optimal for finding a nearby target. However, for larger values of x_0 a maximum appears, which shows that the combination of Brownian motion with LFs may perform better for larger initial separation from the target similar to the case of pure LF search as discussed in [7].

6. Discussion and conclusions

To summarise, we found that for complex motion which combines LFs for $\alpha < 1$ with Brownian motion, the search reliability can have intermediate values between zero and unity even if no bias is present. If the Lévy stable exponent α is larger or equal than unity, then $P = 1$. For a process which combines two Lévy motions the qualitative behaviour is the same if one of the exponents is larger than unity. If both of them are smaller than unity, then the process is transient, $P = 0$, and a point-like searcher is unable to find a point-like target. It is

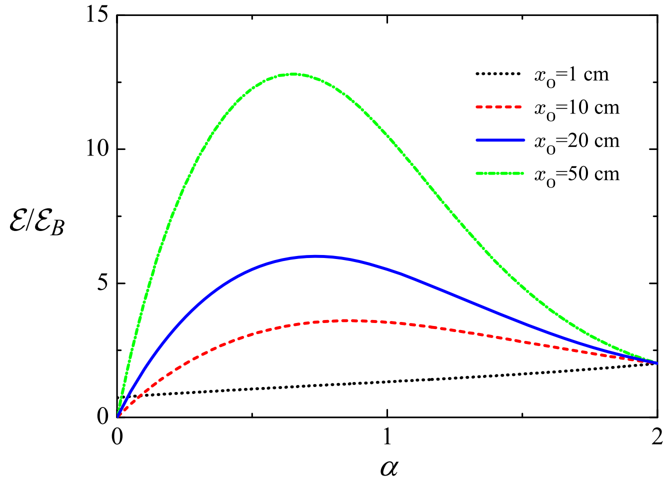

Figure 6. Search efficiency as function of the stable index α for fixed x_0, $K_\alpha = 1\ \mathrm{cm}^\alpha\ \mathrm{s}^{-1}$, $K_2 = 1\ \mathrm{cm}^2\ \mathrm{s}^{-1}$.

interesting to compare our findings with those for the pure Lévy search for a delocalised target —with power-law absorption $a(x) = 1/(|x|^\beta + 1)$, i.e. the target is discovered with a power-law decaying probability—in [64]. In that case, if the stable index $\alpha < 1$, the search still can be absolutely reliable ($P = 1$), if the scaling exponent characterising an absorption probability $\beta \leqslant \alpha$, i.e. the target localisation or absorption probability is delocalised stronger than the LF process.

The search efficiency has a universal behaviour as function of the dimensionless parameter p, which describes the ratio of the noise intensities of the different modes. If the characteristic exponent of the LFs is larger than unity—which includes also Brownian motion —then for large p the efficiency is linear in p, reflecting the fact that only one of the modes defines the properties of the trajectories. Once α becomes unity a logarithmic correction factor appears and the efficiency grows sublinear with p. If the Lévy exponent α is less than unity, the efficiency grows with p as a sublinear power law. The latter case shows that even for high intensity of LF search with $\alpha < 1$ the local search provided by Brownian motion matters and quantitatively affects the search efficiency.

The optimisation of a combined search strategy of LFs with stable index α and Brownian motion shows that for targets in the close vicinity of the starting point one should use more local search strategies, i.e. α should be close to 2, while for distant targets a larger fraction of long jumps increases the search efficiency. This result is consistent with the one-mode case in which LFs are confronted with Brownian motion [7].

It would be interesting to apply our theory to better understand the combined dynamics of biological organisms such as microzooplankton [37], coastal jellyfish [47], moving mussels [48] and marine predators hunting in different environments [49, 50].

Acknowledgments

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Appendix A. Long-time limit of the first arrival density and the search reliability for pure LF search

The probability density of the first arrival for pure LF search is

$$\wp_{\mathrm{fa}}(s) = \frac{\displaystyle\int_0^\infty \frac{\cos kx_0}{s + K_\alpha k^\alpha}\mathrm{d}k}{\displaystyle\int_0^\infty \frac{1}{s + K_\alpha k^\alpha}\mathrm{d}k}. \tag{A.1}$$

In the case $\alpha < 1$ for $s = 0$ the numerator is finite while the denominator diverges for any s, i.e. the search reliability becomes $P = 0$. For $\alpha = 1$ the numerator is finite for $s \neq 0$, whereas the denominator still diverges for all s, thus we have $P = 0$ again. In the case $\alpha > 1$ both the numerator and denominator converge at finite s. Thus, the search reliability is non-zero. Let us consider the case of small s corresponding to long times. The expression (A.1) can be transformed as

$$\begin{aligned}\wp_{\mathrm{fa}}(s) &= \frac{\displaystyle\int_0^\infty \frac{1}{s + K_\alpha k^\alpha}\mathrm{d}k - \int_0^\infty \frac{1 - \cos(kx_0)}{s + K_\alpha k^\alpha}\mathrm{d}k}{\displaystyle\int_0^\infty \frac{1}{s + K_\alpha k^\alpha}\mathrm{d}k}\\ &= 1 - \frac{\displaystyle\int_0^\infty \frac{1 - \cos(kx_0)}{s + K_\alpha k^\alpha}\mathrm{d}k}{\displaystyle\int_0^\infty \frac{1}{s + K_\alpha k^\alpha}\mathrm{d}k}.\end{aligned} \tag{A.2}$$

The integral in the denominator can be computed easily ([66], (2.2.3.5)),

$$\int_0^\infty \frac{1}{s + K_\alpha k^\alpha}\mathrm{d}k = \frac{1}{K_\alpha^{1/\alpha}}\int_0^\infty \frac{1}{s + y^\alpha}\mathrm{d}y = \frac{\pi}{\alpha\sin(\pi/\alpha)}\frac{s^{1/\alpha - 1}}{K_\alpha^{1/\alpha}}. \tag{A.3}$$

Since the integral in the numerator converges at $s \to 0$, we can simply put $s = 0$ while looking at small s (long-time) behaviour. Thus we have

$$\begin{aligned}\int_0^\infty \frac{1 - \cos(kx_0)}{s + K_\alpha k^\alpha}\mathrm{d}k &= \frac{x_0^{\alpha - 1}}{(\alpha - 1)K_\alpha}\int_0^\infty \mathrm{d}y\frac{\sin y}{y^{\alpha - 1}}\\ &= \frac{\Gamma(2 - \alpha)}{\alpha - 1}\sin\left(\frac{\pi\alpha}{2}\right)\frac{x_0^{\alpha - 1}}{K_\alpha}.\end{aligned} \tag{A.4}$$

Hence we get:

$$\wp_{\mathrm{fa}}(s) \approx 1 - \Lambda(\alpha)x_0^{\alpha - 1}K_\alpha^{1/\alpha - 1}s^{1 - 1/\alpha}, \tag{A.5}$$

where

$$\Lambda(\alpha) = \frac{\alpha\Gamma(2 - \alpha)}{\pi(\alpha - 1)}\sin\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi}{\alpha}\right). \tag{A.6}$$

We see that in this case $P = \lim_{s\to 0}\wp_{\mathrm{fa}}(s) = 1$. In order to get the long-time limit with a small-s expansion we note that

$$-\wp_{\mathrm{fa}}'(s) \approx \Lambda(\alpha)x_0^{\alpha - 1}K_\alpha^{1/\alpha - 1}\left(1 - \frac{1}{\alpha}\right)s^{-1/\alpha}. \tag{A.7}$$

The latter expression is the Laplace transform of $t\wp_{\mathrm{fa}}(t)$. Hence, according to Tauberian theorems

$$t\wp_{\mathrm{fa}}(t) \approx \Lambda(\alpha) x_0^{\alpha-1} K_\alpha^{1/\alpha-1}\left(1 - \frac{1}{\alpha}\right)\frac{t^{-1+1/\alpha}}{\Gamma(1/\alpha)}. \tag{A.8}$$

Thus,

$$\wp_{\mathrm{fa}}(t) \approx C(\alpha) x_0^{\alpha-1} K_\alpha^{1/\alpha-1} t^{-2+1/\alpha}, \tag{A.9}$$

where

$$C(\alpha) = \frac{\Gamma(2-\alpha)}{\pi\Gamma(1/\alpha)}\sin\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi}{\alpha}\right). \tag{A.10}$$

Appendix B. Efficiency of pure LF search

The search efficiency in this case becomes

$$\mathcal{E} = \int_0^\infty \mathrm{d}s \wp_{\mathrm{fa}}(s) = K_\alpha \int_0^\infty \mathrm{d}s \frac{\int_0^\infty \frac{\cos kx_0}{s + k^\alpha}\mathrm{d}k}{\int_0^\infty \frac{1}{s + k^\alpha}\mathrm{d}k}. \tag{B.1}$$

The denominator was computed in expression (A.4). We rewrite the expression for the efficiency using the notation for α-stable density as $l_\alpha(x)$ as well as new variables,

$$\begin{aligned}
\mathcal{E} &= \frac{K_\alpha \alpha \sin(\pi/\alpha)}{\pi}\int_0^\infty \mathrm{d}s s^{1-1/\alpha}\int_0^\infty \frac{\cos kx_0}{s + k^\alpha}\mathrm{d}k\\
&= \frac{K_\alpha \alpha \sin(\pi/\alpha)}{\pi}\int_0^\infty \mathrm{d}s s^{1-1/\alpha}\int_0^\infty \cos(kx_0)\mathrm{d}k\int_0^\infty \mathrm{d}\tau e^{-(s+k^\alpha)\tau}\\
&= K_\alpha \alpha \sin(\pi/\alpha)\int_0^\infty \mathrm{d}s s^{1-1/\alpha}\int_0^\infty \mathrm{d}\tau e^{-s\tau}\int_0^\infty \frac{\cos(kx_0)}{\pi}e^{-k^\alpha\tau}\mathrm{d}k\\
&= K_\alpha \alpha \sin(\pi/\alpha)\int_0^\infty \mathrm{d}\tau \tau^{-1/\alpha} l_\alpha\left(\frac{x_0}{\tau^{1/\alpha}}\right)\int_0^\infty \mathrm{d}s s^{1-1/\alpha} e^{-s\tau}\\
&= K_\alpha \alpha \sin(\pi/\alpha)\Gamma\left(2 - \frac{1}{\alpha}\right)\int_0^\infty \mathrm{d}\tau \tau^{-2} l_\alpha\left(\frac{x_0}{\tau^{1/\alpha}}\right)\\
&= |y = x_0/\tau^{1/\alpha}|\\
&= -K_\alpha \alpha^2 \sin(\pi/\alpha)\Gamma\left(2 - \frac{1}{\alpha}\right)x_0^{-\alpha}\int_0^\infty \mathrm{d}y y^{\alpha-1} l_\alpha(y)\\
&= \frac{1}{2}K_\alpha \alpha^2 \sin(-\pi/\alpha)\Gamma\left(2 - \frac{1}{\alpha}\right)x_0^{-\alpha}\langle |y|^{\alpha-1}\rangle,
\end{aligned} \tag{B.2}$$

where $\langle |y|^{\alpha-1}\rangle$ is the $(\alpha-1)$-th moment of a standard α-stable distribution, which is given by [65]

$$\langle |y|^q \rangle = \frac{2}{\pi q} \sin\left(\frac{\pi q}{2}\right)\Gamma(1+q)\Gamma\left(1-\frac{q}{\alpha}\right), \; 0 < q < \alpha. \tag{B.3}$$

After plugging the latter expression into (B.2) we get the final result

$$\mathcal{E} = \frac{\alpha K_\alpha}{x_0^\alpha} \left| \cos\left(\frac{\pi\alpha}{2}\right) \right| \Gamma(\alpha), \qquad 1 < \alpha < 2. \tag{B.4}$$

Appendix C. Derivation of the Fox function solution for $\alpha < 1$ at $s = 0$

The search reliability is

$$P = \wp_{\mathrm{fa}}(s=0) = \frac{\int_0^\infty \frac{\cos k}{pk^\alpha + k^2}\mathrm{d}k}{\int_0^\infty \frac{1}{pk^\alpha + k^2}\mathrm{d}k}, \tag{C.1}$$

and we have [66]

$$\int_0^\infty \frac{1}{pk^\alpha + k^2}\mathrm{d}k = \frac{\pi}{2-\alpha} \frac{p^{-\frac{1}{2-\alpha}}}{\sin\left(\frac{\pi}{2-\alpha}\right)}. \tag{C.2}$$

Since

$$\frac{1}{p + k^{2-\alpha}} = \frac{1}{p}\frac{1}{2-\alpha}H_{11}^{11}\left[\frac{k}{p^{\frac{1}{2-\alpha}}} \left| \begin{matrix} \left(0, \frac{1}{2-\alpha}\right) \\ \left(0, \frac{1}{2-\alpha}\right) \end{matrix} \right.\right], \tag{C.3}$$

we can compute the search reliability in terms of a Fox H-function

$$\begin{aligned} &\frac{1}{p(2-\alpha)} \int_0^\infty k^{-\alpha} \cos k \times H_{11}^{11}\left[\frac{k}{p^{\frac{1}{2-\alpha}}} \left| \begin{matrix} \left(0, \frac{1}{2-\alpha}\right) \\ \left(0, \frac{1}{2-\alpha}\right) \end{matrix} \right.\right]\mathrm{d}k \\ &= \frac{\sqrt{\pi}2^{-\alpha}}{p(2-\alpha)} H_{31}^{12}\left[\frac{2}{p^{\frac{1}{2-\alpha}}} \left| \begin{matrix} \left(\frac{1+\alpha}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2-\alpha}\right), \left(\frac{\alpha}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{2-\alpha}\right) \end{matrix} \right.\right]. \end{aligned} \tag{C.4}$$

Thus

$$\begin{aligned} P &= \frac{2^{-\alpha}\sin\left(\frac{\pi(1-\alpha)}{2-\alpha}\right)}{\sqrt{\pi}p^{\frac{1-\alpha}{2-\alpha}}} H_{31}^{12}\left[\frac{2}{p^{\frac{1}{2-\alpha}}} \left| \begin{matrix} \left(\frac{1+\alpha}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2-\alpha}\right), \left(\frac{\alpha}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{2-\alpha}\right) \end{matrix} \right.\right] \\ &= \frac{\sin\left(\frac{\pi}{2-\alpha}\right)}{2\sqrt{\pi}} H_{31}^{12}\left[\frac{2}{p^{\frac{1}{2-\alpha}}} \left| \begin{matrix} \left(1, \frac{1}{2}\right), \left(\frac{1-\alpha}{2-\alpha}, \frac{1}{2-\alpha}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \\ \left(\frac{1-\alpha}{2-\alpha}, \frac{1}{2-\alpha}\right) \end{matrix} \right.\right]. \end{aligned} \tag{C.5}$$

In the important particular case $\alpha = 0$,

$$\lim_{\alpha=0} P = \frac{1}{2\sqrt{\pi}} H_{31}^{12}\left[\frac{2}{\sqrt{p}}\left|\begin{matrix}\left(1, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\\ \left(\frac{1}{2}, \frac{1}{2}\right)\end{matrix}\right.\right]$$
$$= \frac{1}{2\sqrt{\pi}} H_{20}^{02}\left[\frac{2}{\sqrt{p}}\left|\begin{matrix}\left(1, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\\ \overline{\phantom{(1, \frac{1}{2})}}\end{matrix}\right.\right]$$
$$= \exp(-\sqrt{p}). \tag{C.6}$$

Appendix D. Search reliability in the $p \gg 1$ limit

In order to find the expansion for P in the limit of large values of p we start from equation (11), where $s = 0$. The upper integral can be expressed as

$$\int_0^\infty \frac{\cos k\mathrm{d}k}{pk^\alpha + k^2} = \mathrm{Re}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}k}\mathrm{d}k}{pk^\alpha + k^2} = \mathrm{Re}\, I, \tag{D.1}$$

where

$$I = \chi^{-1}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\chi\kappa}\mathrm{d}\kappa}{\kappa^\alpha + \kappa^2} = \chi^{-1}\int_0^\infty \mathrm{d}\kappa \frac{1}{1 + \kappa^{2-\alpha}}\frac{\mathrm{e}^{\mathrm{i}\chi\kappa}}{\kappa^\alpha}, \tag{D.2}$$

such that

$$\frac{1}{\chi} = \int_0^\infty \mathrm{e}^{\mathrm{i}\chi\kappa}\kappa^{-\alpha}\phi(\kappa)\mathrm{d}\kappa, \tag{D.3}$$

with $\phi(\kappa) = \frac{1}{1 + \kappa^{2-\alpha}}$ and $\chi = p^{\frac{1}{2-\alpha}}$. Since χ is large, $\mathrm{e}^{\mathrm{i}\chi k}$ is a highly oscillating function. In addition, we have an integrated divergence of the integrand at zero. Due to these two reasons the main contribution to the integral will be given by the contribution around $\kappa \approx 0$. Therefore (and using the expression 2.3.3.1 from [66])

$$I = \frac{1}{\chi}\int_0^\infty \mathrm{e}^{\mathrm{i}\chi\kappa}\kappa^{-\alpha}\phi(\kappa)\mathrm{d}\kappa \approx \chi^{-1}\int_0^\infty \mathrm{e}^{\mathrm{i}\chi\kappa}\kappa^{-\alpha}\phi(0)\mathrm{d}\kappa$$
$$= \chi^{-(2-\alpha)}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\xi}}{\xi^\alpha}\mathrm{d}\xi$$
$$= \chi^{-(2-\alpha)}\Gamma(1-\alpha)(-\mathrm{i})^{\alpha-1} = -\frac{\Gamma(1-\alpha)}{p}\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(\alpha-1)}. \tag{D.4}$$

And thus

$$\int_0^\infty \frac{\cos k\mathrm{d}k}{pk^\alpha + k^2} = \frac{1}{p}\Gamma(1-\alpha)\sin\left(\frac{\pi\alpha}{2}\right). \tag{D.5}$$

The integral in the denominator of equation (11) can be computed analytically for any p and the result is given by equation (C.2). Hence one gets equation (19).

Appendix E. The case $\alpha = 1$

For $\alpha = 1$,

$$\wp_{\mathrm{fa}}(s) = \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{\mathrm{i}kx_0}}{s + K_\alpha\ |k| + K_{\mathrm{B}} k^2}}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}k \frac{1}{s + K_\alpha\ |k| + K_{\mathrm{B}} k^2}}. \tag{E.1}$$

Obviously the integrals can be simplified due to the symmetry in k. Then

$$\wp_{\mathrm{fa}}(s) = \frac{\displaystyle\int_{0}^{\infty} \mathrm{d}k \frac{\cos(kx_0)}{s + K_\alpha\ |k| + K_{\mathrm{B}} k^2}}{\displaystyle\int_{0}^{\infty} \mathrm{d}k \frac{1}{s + K_\alpha\ |k| + K_{\mathrm{B}} k^2}} = \frac{I_2}{I_1}, \tag{E.2}$$

where I_1 and I_2 can be computed or taken from Prudnikov [66]

$$I_1 = \frac{1}{K_{\mathrm{B}}(k_2 - k_1)} \ln \left| \frac{k_2}{k_1} \right|, \tag{E.3}$$

$$\begin{aligned} I_2 = \frac{1}{K_{\mathrm{B}}(k_2 - k_1)} & [\mathrm{ci}(x_0 k_2)\cos(x_0 k_2) - \mathrm{ci}(x_0 k_1)\cos(x_0 k_1) \\ & + \mathrm{si}(x_0 k_2)\sin(x_0 k_2) - \mathrm{si}(x_0 k_1)\sin(x_0 k_1)], \end{aligned} \tag{E.4}$$

where $k_{1,2} = \frac{K_1 \mp \sqrt{K_1^2 - 4K_{\mathrm{B}} s}}{2K_{\mathrm{B}}}$, $\mathrm{ci}(z)$ and $\mathrm{si}(z)$ are integral cosine and sine respectively, defined via $\mathrm{si}(z) = -\int_z^\infty \frac{\sin y}{y} \mathrm{d}y$ and $\mathrm{ci}(z) = -\int_z^\infty \frac{\cos y}{y} \mathrm{d}y$. Thus, for $\alpha = 1$, $v = 0$ and

$$\begin{aligned} \wp_{\mathrm{fa}}(s) &= \frac{1}{\ln \left| \frac{k_2}{k_1} \right|} [\mathrm{ci}(x_0 k_2)\cos(x_0 k_2) - \mathrm{ci}(x_0 k_1)\cos(x_0 k_1) \\ &\quad + \mathrm{Si}(x_0 k_2)\sin(x_0 k_2) - \mathrm{Si}(x_0 k_1)\sin(x_0 k_1) \\ &\quad + \pi\cos\left(\frac{k_1 + k_2}{2}\right)\sin\left(\frac{k_2 - k_1}{2}\right)\bigg] \\ &= \frac{1}{\ln \left| \frac{k_2}{k_1} \right|} [\mathrm{ci}(x_0 k_2)\cos(x_0 k_2) - \mathrm{ci}(x_0 k_1)\cos(x_0 k_1) \\ &\quad + \mathrm{si}(x_0 k_2)\sin(x_0 k_2) - \mathrm{si}(x_0 k_1)\sin(x_0 k_1)]. \end{aligned} \tag{E.5}$$

In the first expression an alternative way to represent the results is used (which corresponds to Mathematica) with

$$\mathrm{Si}(z) = \int_0^z \frac{\sin y}{y} \mathrm{d}y = \frac{\pi}{2} + \mathrm{si}(z). \tag{E.6}$$

Appendix F. Asymptotical behaviour of the search efficiency for $p \to \infty$

In the limit $p \to \infty$ the search efficiency can be expressed through the Brownian efficiency $\mathcal{E}_{\mathrm{B}}$ (i.e. for $p = 0$) in the form

$$\mathcal{E}(p) = \frac{\mathcal{E}_\mathrm{B}}{2}\int_0^\infty \wp_\mathrm{fa}(p,\, s)\mathrm{d}(st_\mathrm{B}), \tag{F.1}$$

where $\wp_\mathrm{fa}(p,\, s)$ is determined by equation (11).

F.1. $\alpha > 1$

For $\alpha > 1$ LFs have a finite search reliability. In the limit $p \to \infty$ the LFs dominate the search process, and we thus necessarily recover expression (B.4).

F.2. $\alpha < 1$

In the case $\alpha < 1$ convergence at infinity is due to the term k^2 and we cannot neglect it so easily as we did for $\alpha > 1$. Let us change the variables in (F.1) as $st_\mathrm{B} = p^\nu u$, $k = p^\mu \kappa$, where ν and μ will be specified below. Then from (F.1) we get

$$2\frac{\mathcal{E}(p)}{\mathcal{E}_\mathrm{B}} = p^\nu \int_0^\infty \mathrm{d}u \frac{\displaystyle\int_0^\infty \frac{\cos(p^\mu\kappa)}{p^\nu u + p^{1+\alpha\mu}\kappa^\alpha + p^{2\mu}\kappa^2}p^\mu \mathrm{d}\kappa}{\displaystyle\int_0^\infty \frac{1}{p^\nu u + p^{1+\alpha\mu}\kappa^\alpha + p^{2\mu}\kappa^2}p^\mu \mathrm{d}\kappa}. \tag{F.2}$$

We choose ν and μ such that

$$\nu = 1 + \alpha\mu = 2\mu, \tag{F.3}$$

i.e.

$$\mu = \frac{1}{2-\alpha}, \qquad \nu = \frac{2}{2-\alpha}. \tag{F.4}$$

Then equation (F.2) takes the form

$$2\frac{\mathcal{E}(p)}{\mathcal{E}_\mathrm{B}} = p^\nu \int_0^\infty \mathrm{d}u \frac{\displaystyle\int_0^\infty \frac{\cos(p^\mu\kappa)}{u + \kappa^\alpha + \kappa^2}\mathrm{d}\kappa}{\displaystyle\int_0^\infty \frac{1}{u + \kappa^\alpha + \kappa^2}\mathrm{d}\kappa}. \tag{F.5}$$

The integral in the denominator converges at all positive u, does not depend on p and has an upper bound at $u = 0$,

$$\begin{aligned} f(u) &= \int_0^\infty \frac{1}{u + \kappa^\alpha + \kappa^2}\mathrm{d}\kappa \leqslant f(0) \\ &= \int_0^\infty \frac{1}{\kappa^\alpha + \kappa^2}\mathrm{d}\kappa. \end{aligned} \tag{F.6}$$

As for the integral in the numerator, since $p \gg 1$ the main contribution comes from small κ. We thus neglect κ^2 in comparison with κ^α and use the approach from appendix B. Hence, for the efficiency we get

$$\begin{aligned} 2\frac{\mathcal{E}(p)}{\mathcal{E}_\mathrm{B}} &\approx p^\nu \int_0^\infty \frac{\mathrm{d}u}{f(u)} \int_0^\infty \frac{\cos(p^\mu\kappa)}{u + \kappa^\alpha}\mathrm{d}\kappa \\ &\sim p^\nu \int_0^\infty \frac{\mathrm{d}u}{f(u)} \int_0^\infty \mathrm{d}\tau \mathrm{e}^{-u\tau}\frac{1}{\tau^{1/\alpha}} l_\alpha\left(\frac{p^\mu}{\tau^{1/\alpha}}\right), \end{aligned} \tag{F.7}$$

where

$$y = p^{\mu}/\tau^{1/\alpha}, \qquad \tau = p^{\mu\alpha}/y^{\alpha}, \qquad \mathrm{d}\tau = -\frac{1}{\alpha}\frac{p^{\mu\alpha}}{y^{\alpha+1}}\mathrm{d}y.$$

Thus

$$\begin{aligned}
2\frac{\mathcal{E}(p)}{\mathcal{E}_{\mathrm{B}}} &\sim p^{\nu-\mu+\mu\alpha}\int_0^\infty \frac{\mathrm{d}u}{f(u)}\int_0^\infty \mathrm{d}y \exp\left(-\frac{up^{\mu\alpha}}{y^{\mu\alpha}}\right)y^{-\alpha}l_\alpha(y)\\
&= p^{\nu-\mu+\mu\alpha}\int_0^\infty \mathrm{d}y l_\alpha(y)y^{-\alpha}\int_0^\infty \frac{\mathrm{d}u}{f(u)}\exp\left(-\frac{up^{\mu\alpha}}{y^{\mu\alpha}}\right)\\
&= p^{\nu-\mu}\int_0^\infty \mathrm{d}y l_\alpha(y)\int_0^\infty \frac{\mathrm{e}^{-t}\mathrm{d}t}{f\left(\dfrac{y^{\alpha}t}{p^{\mu\alpha}}\right)}\\
&\sim p^{\nu-\mu} \sim p^{1/(2-\alpha)}, \qquad @p\to\infty,
\end{aligned} \tag{F.8}$$

due to relation (F.6), i.e. for $\alpha < 1$

$$\mathcal{E}(p) \sim p^{\frac{1}{2-\alpha}}. \tag{F.9}$$

F.3. $\alpha = 1$

The case $\alpha = 1$ requires a special treatment. The efficiency in this case is

$$\frac{2\mathcal{E}(p)}{\mathcal{E}_{\mathrm{B}}} = \int_0^\infty \mathrm{d}(st_{\mathrm{B}})\frac{\displaystyle\int_0^\infty \frac{\cos k\mathrm{d}k}{st_{\mathrm{B}}+pk+k^2}}{\displaystyle\int_0^\infty \frac{\mathrm{d}k}{st_{\mathrm{B}}+pk+k^2}}. \tag{F.10}$$

Making the same change of variables as in appendix F.2, we get (cf (F.5))

$$\frac{2\mathcal{E}(p)}{\mathcal{E}_{\mathrm{B}}} = p^2\int_0^\infty \mathrm{d}u\frac{\displaystyle\int_0^\infty \frac{\cos(p\kappa)\mathrm{d}\kappa}{u+\kappa+\kappa^2}}{\displaystyle\int_0^\infty \frac{\mathrm{d}\kappa}{u+\kappa+\kappa^2}}. \tag{F.11}$$

To proceed we start with the evaluation of the integral in the denominator

$$\begin{aligned}
f(u) &= \int_0^\infty \frac{1}{u+\kappa+\kappa^2}\mathrm{d}\kappa = \int_0^\infty \frac{1}{\left(\kappa+\frac{1}{2}\right)^2+u-\frac{1}{4}}\mathrm{d}\kappa\\
&= \begin{cases}
f_1(u) = \frac{1}{2\sqrt{1/4-u}}\ln\left|\frac{1/2+\sqrt{1/4-u}}{1/2-\sqrt{1/4-u}}\right|, & u<1/4,\\
f_2(u) = \frac{1}{\sqrt{u-1/4}}\left(\pi/2-\arctan\frac{1}{2\sqrt{u-1/4}}\right), & u>1/4,\\
2, & u=1/4,
\end{cases}
\end{aligned} \tag{F.12}$$

For the integral in the numerator similar to the case $\alpha < 1$ (appendix F.2) the main contribution comes from small κ values due to $p \gg 1$. Hence we can neglect κ^2 in comparison with κ. Thus

$$f(u) = \int_0^\infty \frac{\cos(p\kappa)}{u + \kappa + \kappa^2} \mathrm{d}\kappa \underset{p \gg 1}{\simeq} \int_0^\infty \frac{\cos(p\kappa)}{u + \kappa} \mathrm{d}\kappa = g(pu), \tag{F.13}$$

where g(z) can be expressed through sine and cosine integrals Si(z) and Ci(z) (for the definitions see appendix E):

$$g(z) = -\mathrm{Ci}(z)\cos(z) - (\mathrm{Si}(z) - \pi/2)\sin(z). \tag{F.14}$$

equation (F.11) can be rewritten as

$$\frac{2\mathcal{E}(p)}{\mathcal{E}_\mathrm{B}} = p^2 \int_0^{1/4} \frac{\mathrm{d}u}{f_1(u)} g(pu) + p^2 \int_{1/4}^\infty \frac{\mathrm{d}u}{f_2(u)} g(pu). \tag{F.15}$$

For the second term in the latter expression one can use an asymptotic of $g(z) \sim 1/z^2$ for $pu \gg 1$ since $p \gg 1$ (see equation 5.2.35 in [67]). This implies that the contribution from the second term does not grow with increasing p at large p values.

The first term can be rewritten as

$$p^2 \int_0^{1/4} \frac{\mathrm{d}u}{f_1(u)} g(pu) = p \int_0^{p/4} \frac{\mathrm{d}y}{f_1(y/p)} g(y). \tag{F.16}$$

The upper bound of this term is given by

$$p \int_0^{p/4} \frac{\mathrm{d}y}{f_1(y/p)} g(y) < \frac{p}{f_1(1/4)} \int_0^\infty \mathrm{d}y g(y), \tag{F.17}$$

as $g(y)$ is integrable on $[0, \infty)$ and we can replace the upper limit $p/4$ of the integral with ∞ at $p \gg 1$. Thus, the first term in equation (F.15) does not grow faster than p. To get a lower bound for the growth limit of large p we use the first mean value theorem [68] and a small argument asymptotic $f_1(u) \sim -\ln u$, yielding

$$p \int_0^{p/4} \frac{\mathrm{d}y}{f_1(y/p)} g(y) = \frac{p}{f_1(y^*/p)} \int_0^{p/4} \mathrm{d}y g(y) \sim \frac{p}{-\ln(y^*/p)} \int_0^\infty \mathrm{d}y g(y), \tag{F.18}$$

where $0 < y^* < p/4$. Hence

$$\frac{\mathcal{E}(p)}{\mathcal{E}_\mathrm{B}} \sim \frac{p}{\ln p}, \tag{F.19}$$

which is confirmed by numerical simulations.

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