

Lévy strategies in intermittent search processes are advantageous

Michael A. Lomholt^{*†}, Tal Koren[‡], Ralf Metzler^{†§}, Joseph Klafter[‡]

^{*}MEMPHYS Center for Biomembrane Physics, Department of Physics and Chemistry, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark; [†]Physics Department, Technical University of Munich, James Franck Strasse, D-85747 Garching, Germany; and [‡]School of Chemistry, Tel Aviv University, 69978 Tel Aviv, Israel

Communicated by Joshua Jortner, Tel Aviv University, Tel Aviv, Israel, and approved April 6, 2008 (received for review January 8, 2008)

Intermittent search processes switch between local Brownian search events and ballistic relocation phases. We demonstrate analytically and numerically in one dimension that when relocation times are Lévy distributed, resulting in a Lévy walk dynamics, the search process significantly outperforms the previously investigated case of exponentially distributed relocation times: The resulting Lévy walks reduce oversampling and thus further optimize the intermittent search strategy in the critical situation of rare targets. We also show that a searching agent that uses the Lévy strategy is much less sensitive to the target density, which would require considerably less adaptation by the searcher.

random processes | optimization | Lévy walk | movement ecology

Random search processes occur in many areas, from chemical reactions of diffusing reactants (1) to the foraging behavior of bacteria and animals (2, 3). Of general importance is the search efficiency. Brownian search in one and two dimensions involves frequent returns to an area, leading to oversampling. Higher efficiency, can be achieved, for instance, by facilitated diffusion in gene regulation (4) or by controlled motion in foraging (2, 3). From theoretical and data analysis Lévy strategies, in which the searching agent performs excursions whose length is drawn from distributions with a heavy tail

$$\lambda(x) \simeq |x|^{-1-\alpha}, \quad [1]$$

for $0 < \alpha < 2$ were shown to be advantageous (5–16); occasional long excursions assist in exploring previously unvisited areas and significantly reduce oversampling.

As an alternative to Lévy search, intermittent strategies have been introduced to improve the efficiency of diffusive search (17–21). Intermittent search requires that the searcher occasionally shifts focus from the search and concentrates on fast relocation. The relocation phase implies that the searcher is wasting time in the short run because the target cannot be spotted during it. However, the overall search efficiency is improved by introducing the searcher to previously unexplored areas (17–21).

In refs. 18 and 20 relocation events were assumed to occur in a random direction for exponentially distributed time spans, giving rise to a Markovian process. We show here analytically and numerically in one dimension that this is only a partial solution to oversampling, as eventually the central limit theorem (CLT) reduces the process to a Brownian random walk with jumps on the scale of $v\tau_2$, where τ_2 is the typical time spent in a relocation event. In practice, revisits can be reduced by adjusting the average time spent in search and relocation phases to the density of targets. Lévy strategies, on the other hand, fundamentally circumvent the CLT, and we here demonstrate a twofold advantage of them over the exponential distribution: Lévy walk intermittent processes find the target faster than exponential strategies in the critical case of rare targets, and their performance is much less dependent on adapting to the target density.

Intermittent Search with Lévy Relocations

Generalizing the model from ref. 20, we consider two phases: In the search phase the searcher scans for the target by diffusive motion with diffusivity D . With probability per time τ_1^{-1} the searcher switches to the relocation phase, during which it moves ballistically with velocity v in a random direction (20). The relocation time is drawn from the waiting time distribution $\psi(t)$, which will be considered to be exponential or Lévy stable. The purpose of relocations is to move as quickly as possible away from the area that has just been searched, and thus the searcher is not scanning for the target in this phase. To compare with previous results we take a closed cell approach: the search is performed on an interval of length L with periodic boundary conditions, corresponding to regularly spaced targets with density $1/L$. The model can be formulated as an equation for the probability density $P(x, t)$ for the position x of the searcher in the search phase:

$$\frac{\partial P}{\partial t} = \frac{1}{\tau_1} \int_{-L/2}^{L/2} dx' \int_0^\infty dt' W(x-x', t-t') P(x', t') - \frac{1}{\tau_1} P(x, t) + D \frac{\partial^2 P}{\partial x^2} - p_{\text{fa}}(t) \delta(x). \quad [2]$$

The role of the last term on the right-hand side is to remove the particle when it arrives at the target placed at $x = 0$. The density $p_{\text{fa}}(t)$ thus represents the first arrival time at the target, which is determined implicitly by the absorbing boundary condition $P(x = 0, t) = 0$. The term proportional to the diffusivity D describes the local Brownian motion in the search phase. The term $-\tau_1^{-1} P(x, t)$ removes the searcher from location x with rate τ_1^{-1} . The searcher is then relocated according to the integral expression in which the kernel $W(x, t)$ is the joint probability density of making a relocation of length x during a time t . It is defined by

$$W(x, t) = \frac{\psi(t)}{2} \sum_{n=-\infty}^{\infty} \delta(|x + nL| - vt). \quad [3a]$$

Here the δ -coupling enforces that the distance traveled in time t is vt , and the sum over n renders $W(x, t)$ L -periodic in x . $\psi(t)$ is related to the spatial distribution of the relocations $\lambda(x)$ by

$$\psi(t) = 2v\lambda(vt). \quad [3b]$$

The jump length distribution $\lambda(x)$ is assumed to be symmetric around $x = 0$ (no orientational memory).

The search efficiency is quantified by the mean search time

$$\langle t \rangle = \int_0^\infty dt t p_{\text{fa}}(t). \quad [4]$$

Author contributions: M.A.L., T.K., R.M., and J.K. designed research, performed research, analyzed data, and wrote the paper.

The authors declare no conflict of interest.

[§]To whom correspondence should be addressed. E-mail: metzler@ph.tum.de

© 2008 by The National Academy of Sciences of the USA

To obtain $\langle t \rangle$ we Fourier expand

$$P(n, t) = \int_{-L/2}^{L/2} dx e^{ik_n x} P(x, t), \quad [5]$$

where n is an integer with corresponding wavenumber $k_n = 2\pi n/L$, and Laplace transform, where

$$P(n, u) = \int_0^\infty dt e^{-ut} P(n, t). \quad [6]$$

We find

$$uP(n, u) - \delta_{n,0} = \frac{1}{\tau_1} W(n, u)P(n, u) - \frac{1}{\tau_1} P(n, u) - Dk_n^2 P(n, u) - p_{fa}(u). \quad [7]$$

The initial distribution is uniform, $P(x, t = 0) = 1/L$, because the searcher initially has no information on the target position. Isolating $P(n, u)$, summing over n (note that $\sum_n P(n, u) = P(x = 0, u) = 0$), we find for $p_{fa}(u)$,

$$p_{fa}(u) = \left\{ \sum_{n=-\infty}^{\infty} \frac{u + [1 - \psi(u)]/\tau_1}{u + Dk_n^2 + [1 - W(n, u)]/\tau_1} \right\}^{-1}. \quad [8]$$

In Laplace space the mean search time $\langle t \rangle$ yields from expansion of p_{fa} at small u because $p_{fa}(u) \sim 1 - \langle t \rangle u + \dots$. From the average time τ_2 spent in a single relocation event ($\psi(u) \sim 1 - \tau_2 u + \dots$), one obtains

$$\langle t \rangle = \sum_{n=1}^{\infty} \frac{2(\tau_1 + \tau_2)}{D\tau_1 k_n^2 + 1 - \lambda(k_n)}. \quad [9a]$$

Here,

$$\lambda(k_n) = W(n, u = 0) = \int_{-\infty}^{\infty} dx e^{ik_n x} \lambda(x) \quad [9b]$$

is the Fourier transform of the relocation length distribution $\lambda(k) = \int_{-\infty}^{\infty} dx e^{ikx} \lambda(x)$, at the discrete wavenumbers $k_n = 2\pi n/L$. We now use Eq. 9a to determine the search efficiency of (i) Lévy and (ii) exponentially distributed relocations:

(i) For Lévy distributed relocations we use the symmetric Lévy stable law with characteristic function (22)

$$\lambda(k) = \exp\{-\sigma^\alpha |k|^\alpha\}, \quad \sigma = \frac{\pi v \tau_2}{[2\Gamma(1 - 1/\alpha)]}. \quad [10]$$

From this closed expression the asymptotic form 1 follows. The index α is restricted to $1 < \alpha < 2$ so that the mean relocation time τ_2 is finite. Fig. 1 depicts trajectories for cases of exponential and Lévy relocations, distinguishing the Lévy case with its occasional long relocations.

We introduce three approximations valid for large L :

(a) Assume that $v\tau_2 \gg \sqrt{D\tau_1}$, i.e., that the mean relocation distance is much longer than the average distance scanned in a typical search phase. We will see that this is self-consistent with the obtained optimal values of τ_1 and τ_2 that have the same L -scaling for large L . This assumption means that $D\tau_1 k_n^2$ and $\lambda(k_n)$ are to a good approximation nonzero at different n , and we expand

$$\frac{1}{D\tau_1 k_n^2 + 1 - \lambda(k_n)} \sim \frac{1}{D\tau_1 k_n^2 + 1} + \frac{1}{1 - \lambda(k_n)} - 1. \quad [11]$$

(b) Assuming that the search range $\sqrt{D\tau_1}$ is much smaller than L , we replace the sum over the first term on the right-hand side of Eq. 11 by an integral, yielding

$$\sum_{n=1}^{\infty} \frac{1}{D\tau_1 k_n^2 + 1} \sim \int_0^\infty \frac{1}{D\tau_1 k_n^2 + 1} dn = \frac{L}{4\sqrt{D\tau_1}}. \quad [12]$$

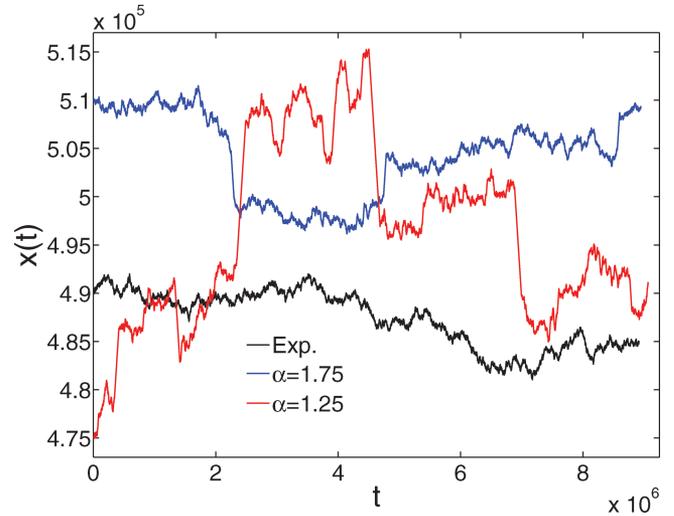


Fig. 1. x - t diagram with exponential and Lévy relocations, with $\tau_1 = 37$, $\tau_2 = 200$, $D = 1$, $v = 0.1$, and $L = \infty$.

(c) Because $\lambda(k_n) \sim 1 - \sigma^\alpha |k_n|^\alpha$ at small values of k_n ($k_n \rightarrow 0$ at $n = 1$ in the limit of large L) we approximate the last two terms of Eq. 11. Namely, the contribution from the singularity at small n dominates the sum,

$$\sum_{n=1}^{\infty} \left[\frac{1}{1 - \lambda(k_n)} - 1 \right] \sim \left(\frac{L}{2\pi\sigma} \right)^\alpha \zeta(\alpha). \quad [13]$$

Here $\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}$ is the Riemann ζ function.

Collecting a to c , Eq. 9a is approximated by

$$\langle t \rangle \sim 2(\tau_1 + \tau_2) \left[\frac{L}{4\sqrt{D\tau_1}} + \left(\frac{L}{2\pi\sigma} \right)^\alpha \zeta(\alpha) \right]. \quad [14]$$

For honest comparison between Lévy and exponential strategies, we determine the respective optimal τ_1 and τ_2 . Solving $\partial\langle t \rangle/\partial\tau_1 = 0$ and $\partial\langle t \rangle/\partial\tau_2 = 0$ simultaneously, we obtain from Eq. 14 that at large L

$$\tau_1 \sim (b/a^\alpha)^{1/(\alpha-1/2)}, \quad \tau_2 \sim (b/\sqrt{a})^{1/(\alpha-1/2)}, \quad [15]$$

where (using $\Omega \equiv \sqrt{1 + 4(\alpha - 1)}$)

$$a = (1 + \Omega)/(2[\alpha - 1]), \quad [16a]$$

$$b = 2\sqrt{D}[2\alpha + \Omega - 3]\zeta(\alpha)L^{\alpha-1} \left[\frac{\Gamma(1 - \alpha^{-1})}{\pi^2 v} \right]^\alpha \quad [16b]$$

such that the optimal τ_i scale with L as $L^{(\alpha-1)/(\alpha-1/2)}$. According to Eq. 14, $\langle t \rangle$ will then scale as $L^{(3\alpha-2)/(2\alpha-1)}$, implying that for large L the more efficient search will occur for α close to 1. However, the prefactor to the L -scaling diverges as $\alpha \rightarrow 1$, so the optimal choice of α will be somewhat larger than 1 for any finite L , as demonstrated in Fig. 2. The inset of Fig. 2 shows the validity of the approximate $\langle t \rangle$ for optimal τ_i .

(ii) For exponentially distributed relocation with

$$\psi(t) = \tau_2^{-1} e^{-t/\tau_2}, \quad [17]$$

approximations a to c also apply, with $\sigma = v\tau_2$. The corresponding results for $\langle t \rangle$ and optimal τ_i obtain by replacing $\Gamma(1 - 1/\alpha)$ by $\pi/2$ and taking $\alpha = 2$:

$$\langle t \rangle \sim \frac{\tau_1 + \tau_2}{12} \left[\frac{6L}{\sqrt{D\tau_1}} + \left(\frac{L}{v\tau_2} \right)^2 \right], \quad [18]$$

$$\tau_1 \sim (D/[18v^4])^{1/3} L^{2/3} / 2, \quad \tau_2 \sim 2\tau_1. \quad [19]$$

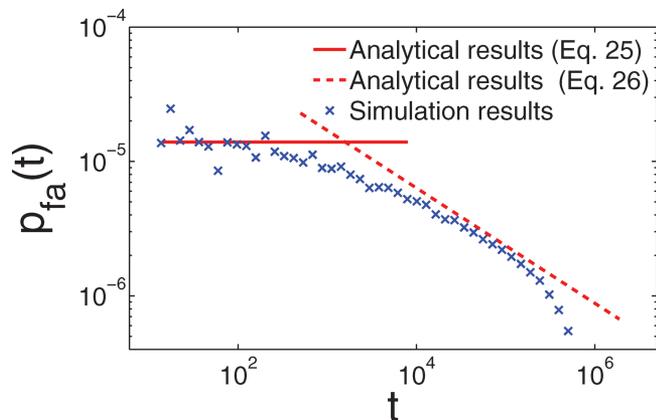


Fig. 4. First arrival density versus time. The crosses are simulation data, and the straight lines are the intermediate regimes of Eq. 25 and Eq. 26. Parameters are $\tau_1 = 35$, $\tau_2 = 50$, $L = 10^4$, $\alpha = 1.75$, $\nu = 1$, and $D = 1$.

as a result. This causes the crossover to the power-law behavior 26. Fig. 4 shows the turnover from plateau to inverse power-law of the first arrival. At even longer times, finite size effects cause a turnover to an exponential decay.

From Eq. 26 the advantage of having α close to unity at large L becomes evident: the presence of rare but long relocation events reduces the risk of rescanning already visited areas, which will be important for large L . However, the downside to choosing an α -value too close to 1 is that an increased amount of very long relocations implies an increased amount of very short ones too, because the average distance is fixed by $\nu\tau_2$ (24). This means that the crossover to the less favorable situation described by Eq. 26 happens earlier, so that larger α becomes more efficient for shorter search times relevant at smaller L .

Intermittent strategies are beneficial when purely diffusive search would slow down over time due to the increasing returns to previously scanned areas (oversampling). Choosing an exponential strategy for relocations, however, only partially solves this problem: At times $t \gg \tau_2$, the CLT governs, leading to oversampling on a typical scale $\nu\tau_2$. Conversely, Lévy-intermittent strategies are not bound to the CLT, rendering them a more amenable solution to reduce oversampling and therefore advantageous in the search for rare targets. Although less pronounced, the problem of oversampling still occurs in two-dimensional search studied in ref. 19. Lévy strategies are expected to improve the search efficiency in this case, as well; however, as to what extent remains to be established quantitatively.

On the basis of our results we advocate that intermittent strategies should not be thought of as alternatives to Lévy strategies. In contrast, the synergistic combination of intermittent search and Lévy relocation strategies turns out to be beneficial. Moreover, a given Lévy walk intermittent search strategy (with fixed τ_i) is almost optimal over a wide range of sparse target densities, which

might be a strategic advantage for creatures that have limited abilities to adjust their search parameters.

We note, however, that the small scaling exponent of $\langle t \rangle$ with L for the Lévy strategy is not a result of the Lévy part of the strategy alone. To explain what we mean by this we will define the pure Lévy strategy as a strategy where the searcher only quickly tests his immediate neighborhood for the target at the end of each relocation. Thus we assume that τ_1 has a small finite value (alternatively the target could have a small finite size and $\tau_1 = 0$) and only consider optimization of the strategy with respect to τ_2 . Doing this, we find from our analytic asymptotic result that the optimal τ_2 scales with L as $L^{1-1/\alpha}$ and that this results in a scaling $\langle t \rangle \simeq L^{2-1/\alpha}$, a scaling that increases faster with L for any $\alpha > 1$ compared with the result where τ_1 is also optimized. And it is only an improvement over the optimized exponential strategy when $\alpha < 3/2$. Without any optimization the Lévy strategy would result in $\langle t \rangle \simeq L^\alpha$, a scaling that is still better than for the optimized exponential strategy when $\alpha < 4/3$.

A remark on the recent discussion about the empirical observation of Lévy distributions of relocation lengths in animal foraging is in order. Thus, while the original publications provided evidence of long-tailed relocation lengths in accordance with theoretical considerations (6–8), a reanalysis of the data reveals that the original data contained few extreme events for the flight times, after removal of which the data no longer unequivocally allow an interpretation as Lévy pattern (23). In that paper also a few other previous claims of Lévy foraging patterns were invalidated. This has caused some uncertainty about the general relevance of Lévy search patterns in animal foraging (24). Among the recent criticisms of ref. 23 we refer to the consideration of finite size effects of real trajectories in ref. 25 that were shown to reestablish the validity of a Lévy-based search mechanism for the albatross flight. It is our belief that Lévy search models show a distinct advantage over strategies governed by the central limit theorem. However, it will require considerably larger data sets to be able to tell for sure whether typically animals use a specific search strategy. The value of this and similar theoretical studies is to provide a framework for the analysis of data that are being collected now or in the future. The robustness of the search efficiency of Lévy strategies to changing target densities, as demonstrated here, appears to be a key concept in the discussion of search mechanisms, and potentially an important evolutionary advantage.

Our analysis relies on the assumption that each relocation is pointed toward a random direction. This will be a good model for “nonintelligent” search, similar to bacterial movement in the absence of chemical or temperature gradients, during which tumbling motion changes with directed motion (2). Intelligent creatures will improve the target search by partial or complete memory, avoiding previously visited locations. It will be interesting to study in more detail models with search memory.

ACKNOWLEDGMENTS. Part of this research was funded by the Natural Sciences and Engineering Research Counsel of Canada and the Canada Research Chairs programme.

- von Smoluchowski M (1916) Drei Vorträge über Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen (Three presentations on diffusion, Brownian motion, and coagulation of colloidal particles). *Phys Z* 17:557–585.
- Berg HC (1993) *Random Walks in Biology* (Princeton Univ Press, Princeton).
- Bell WJ (1991) *Searching Behaviour* (Chapman & Hall, London).
- von Hippel PH, Berg OG (1989) Facilitated target location in biological systems. *J Biol Chem* 264:675–678.
- Shlesinger MF, Klafter J (1986) Lévy walks versus Lévy flights. *On Growth and Form*, eds Stanley HE, Ostrowsky N (Nijhoff, Amsterdam), pp 279–283.
- Viswanathan GM, et al. (1996) Lévy flight search patterns of wandering albatrosses. *Nature* 381:413–415.
- Viswanathan GM, et al. (1999) Optimizing the success of random searches. *Nature* 401:911–914.
- Faustino CL, Silva LR, Luz MG, Raposo EP, Viswanathan GM (2007) Search dynamics at the edge of extinction: Anomalous diffusion as a critical survival state. *Europhys Lett* 77:30002.
- Reynolds AM, Frye MA (2007) Odor tracking in *Drosophila* is consistent with an optimal intermittent scale-free search. *PLoS ONE* 2(4):e354.
- Brown CT, Liebovitch LS, Glendon R (2007) Lévy flights in Dobe Ju/hoanso foraging patterns. *Hum Ecol* 35:129–138.
- Atkinson RPD, Rhodes CJ, Macdonald DW, Anderson RM (2002) Scale-free dynamics in the movement patterns of jackals. *OIKOS* 98:134–140.
- Ramos-Fernandez G, et al. (2003) Lévy walk patterns in the foraging movements of spider monkeys (*Ateles geoffroyi*). *Behav Ecol Sociobiol* 55:223–230.
- Bartumeus F (2007) Lévy processes in animal movement: An evolutionary hypothesis. *Fractals* 15:151–162.
- Lomholt MA, Ambjörnsson T, Metzler R (2005) Optimal target search on a fast-folding polymer chain with volume exchange. *Phys Rev Lett* 95:260603.
- Bartumeus F, Catalan J, Fulco UL, Lyrá ML, Viswanathan GM (2002) Optimizing the encounter rate in biological interactions: Lévy versus Brownian strategies. *Phys Rev Lett* 88:097901.
- Sims DW, et al. (2008) Scaling laws of marine predator search behaviour. *Nature* 451:1098–1102.

17. Bénichou O, Coppey M, Moreau M, Suet P-H, Voituriez R (2005) Optimal search strategies for hidden targets. *Phys Rev Lett* 94:198101.
18. Bénichou O, Loverdo C, Moreau M, Voituriez R (2006) Two-dimensional intermittent search processes: An alternative to Lévy flight strategies. *Phys Rev E* 74:020102R.
19. Shlesinger MF (2006) Mathematical physics: Search research. *Nature* 443:281–282.
20. Bénichou O, Coppey M, Moreau M, Voituriez R (2006) When losing time becomes efficient. *Europhys Lett* 75:349–354.
21. Oshanin G, Wio HS, Lindenberg K, Burlatsky SF (2007) Intermittent random walks for an optimal search strategy: One-dimensional case. *J Phys Cond Mat* 19:065142.
22. Metzler R, Klafter J (2000) The random walk's guide to anomalous diffusion: A fractional dynamic approach. *Phys Rep* 339:1–77.
23. Edwards AM, et al. (2007) Revisiting Lévy flight search patterns of wandering albatrosses, bumblebees and deer. *Nature* 449:1044–1048.
24. Travis J (2007) Do wandering albatrosses care about math? *Science* 318: 742–743.
25. Boyer D, Miramontes O, Ramos-Fernández G (2008) Evidence for biological Lévy flights stands. E-print arXiv:0802.1762.