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# The generalized Cattaneo equation for the description of anomalous transport processes

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**Abstract.** The Cattaneo equation, which describes a diffusion process with a finite velocity of propagation, is generalized to describe anomalous transport. Three possible generalizations are proposed, each one supported by a different scheme: continuous time random walks, non-local transport theory, and delayed flux-force relation. The properties of these generalizations are studied in both the long-time and the short-time regimes. In the long-time limit, we recover the mean-square displacement which is characteristic for these anomalous processes. As expected, the short-time behaviour is modified in comparison to generalized diffusion equations.

## 1. Introduction

Normally, Fick's second law is used to describe standard diffusive processes. This law can be derived by the combination of a continuity equation:

$$\partial_t \varrho(x,t) = -\jmath_x(x,t) \tag{1}$$

and a constitutive equation (Fick's first law):

$$\jmath(x,t) = -D\varrho_x(x,t). \tag{2}$$

Here, $\jmath(x,t)$ denotes the flux, $\varrho(x,t)$ the distribution function of the diffusing quantity, and $D$ the diffusion constant. Henceforth, $\partial_t$ is the time derivative operator and the subscript $x$ stands for $\partial/\partial x$. Taking the gradient of equation (2) and inserting the result in equation (1), one arrives at the (phenomenological) diffusion equation, or Fick's second law [1]:

$$\partial_t \varrho(x,t) = D\varrho_{xx}(x,t) \tag{3}$$

assuming $D$ constant. For an initial delta distribution $\varrho(x,0) = \delta(x)$, one finds a typical Gaussian solution of equation (3), namely

$$\varrho(x,t) = \frac{1}{(4\pi Dt)^{1/2}} \exp\left(-\frac{x^2}{4Dt}\right). \tag{4}$$

Thus, even for very small times, there exists a finite amount of the diffusing substance at large distances from the origin. It is, therefore, an intrinsic property of equation (3) that it issues an infinite velocity of propagation. This is, mathematically speaking, due to the fact that equation (3) is a parabolic partial differential equation. From a physical point of view, this property is unphysical.

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To overcome the infinitely fast propagation, Cattaneo in 1948 proposed a modified approach [2, 3]. He replaced the constitutive equation (2) by

$$\jmath(x,t) + \tau\partial_t \jmath(x,t) = -D\varrho_x(x,t) \tag{5}$$

where now the flux relaxes, with some given characteristic time constant $\tau$. Combining (5) with the equation of continuity (1), one is led to the so-called Cattaneo equation (or modified diffusion equation)

$$\partial_t\varrho(x,t) + \tau\partial_t^2\varrho(x,t) = D\varrho_{xx}(x,t) \tag{6}$$

for constant $D$ and $\tau$. This extension to the diffusion equation (3) turns the parabolic into a hyperbolic equation. Consequently, the propagation velocity is finite, namely $v = (D/\tau)^{1/2}$ [4]. Note that in the diffusion limit, $\tau \to 0$, one recovers Fick's second law with an infinite $v$. Equation (6) is of a damped wave or telegrapher's equation type.

Applications of the Cattaneo equation in the physical sciences are, due to its hyperbolic character, widely spread. They comprise both instances of heat and of particle transport, since Cattaneo's equation is a generalization both for a heat diffusion equation (Fourier's law) and for a particle diffusion equation (Fick's law). The Cattaneo equation finds applications in extended irreversible thermodynamics [3], in heat transfer in Bénard convection [5, 6], in (inflationary) cosmological models [7], in shock waves in rigid heat conductors [8], or in the theory of diffusion in crystalline solids [9]. On the other hand, it can be explicitly derived from the Boltzmann equation [10] and is thus applied to generalize hydrodynamic equations [11, 12]. The characteristic time constant $\tau$ of the Cattaneo equation is discussed in [13].

However, from many experiments and theoretical considerations we know that, in many situations of physical interest, diffusive transport processes are of anomalous nature [14, 15]. First above all, this anomaly is manifested in a mean-square displacement (MSD) of the form

$$\langle x^2 \rangle = Kt^\gamma \tag{7}$$

which deviates from the standard linear behaviour. The exponent $\gamma$, often written as $2/d_w$, is called the anomalous diffusion exponent. In this paper we investigate possible generalizations of the Cattaneo equation (6) which describe this required anomalous property in a consistent way. A first version of a fractional Cattaneo-type constitutive equation was formulated in [16]. We show here that some fractional Cattaneo equations reproduce the features that can be derived from other extended schemes like the continuous time random walk (CTRW) picture or a non-local flux concept.

This connection with other schemes is especially interesting because it enables the identification of concrete examples where a generalized Cattaneo equation might be of interest. The CTRW scheme, for instance, is a stochastic model to describe diffusive phenomena, where the random walkers perform jumps of length $l$ between waiting periods of duration $\tau$, $l$ and $\tau$ being newly drawn at each step from a probability distribution $\psi(l,\tau)$. This model has been successfully used to account for anomalous dispersion in amorphous materials [17] or for turbulent diffusion in plasmas [18], among many other applications [19]. On the other hand, a non-local theory of transport proves necessary in materials with microscale inhomogeneities whenever the mean residence time of a tracer in the material is comparable to the correlation time (i.e. to the time required for the tracer to sample the microstructures) [20]. Examples of such systems can be found, for instance, in environmental situations like the initial dispersion of a pollutant in the atmosphere, in rivers or in ground-water flows [21].

In section 2 we present a purely phenomenological approach to an anomalous theory. In section 3, we take a stochastic approach from CTRW theory. Section 4 presents ideas dealing with non-local transport theory, i.e. memory effects. Section 5 discusses the properties of the generalized Cattaneo equation(s). Finally, we draw our conclusions in section 6.

We do not hesitate to remark here that we discuss domains of dispersive transport or subdiffusion characterized by $0 < \gamma < 1$, which means transport slower than normal diffusion, as well as super- or enhanced diffusion with $1 < \gamma < 2$. Other cases, in which superdiffusion occurs, are discussed in connection with an extended CTRW scheme in [22, 23], and also in [24, 25].

## 2. Phenomenological theory

Recently, some papers discussed the generalization of the diffusion equation by the introduction of fractional derivatives, i.e. convolutions of the distribution function with a power-law memory kernel [26–29], see also section 4. From this theory we know that the order of the introduced fractional temporal derivative equals the anomalous diffusion exponent in the MSD. Also in fractional theory, we encounter an infinite speed of propagation. (In this paper, we do not discuss fractality in the underlying geometry, however, as we want to introduce the CTRW picture later.) The connection of fractional diffusion equations to CTRW theory was investigated in [25, 30].

By a fractional derivative or integral we mean, respectively,

$$\partial_t^\alpha \varrho(x,t) \equiv {}_0D_t^\alpha \varrho(x,t) \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\partial}{\partial t} \displaystyle\int_0^t \mathrm{d}t' \, \dfrac{\varrho(x,t')}{(t-t')^\alpha} & \text{for } 0 < \alpha < 1 \\ \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^t \mathrm{d}t' \, \dfrac{\varrho(x,t')}{(t-t')^{1+\alpha}} & \text{for } \alpha < 0 \end{cases} \tag{8}$$

which is referred to as the Riemann–Liouville fractional derivative [31]. In (8) we also introduced the short-hand notation $\partial_t^\alpha$, which we will use in the further procedures.

In this section, we want to show how the phenomenological introduction of fractional derivatives changes the original Cattaneo equation (6). This section being of a somewhat speculative character, we will not consider complying with the subtleties of fractional calculus for the sake of readability, and we will equate $\partial_t^\alpha \partial_t^\beta$ to $\partial_t^{\alpha+\beta}$ even though, generally, these two operators differ by initial value terms [31]. In the following sections we will investigate whether this *ad hoc* generalization can be interpreted by means of other approaches and there we shall be more accurate with respect to fractional differintegrals. The properties of the derived modified equations will be discussed in section 5.

In contrast to the diffusion equation, we encounter two different derivatives in time for the case of the Cattaneo equation in the basic equations (1) and (5). Thus we find, in principle, a large variety of possible generalizations. However, we concentrate on three different equations which will be corroborated in the following. First, we choose to introduce a fractional derivative (memory) in the constitutive equation (5) and obtain

$$\jmath(x,t) + \tau^\gamma \partial_t^\gamma \jmath(x,t) = -D\varrho_x(x,t) \tag{9}$$

which, combined with the unchanged equation of continuity (1), results in the generalized Cattaneo equation (GCE):

$$\partial_t \varrho(x,t) + \tau^\gamma \partial_t^{1+\gamma} \varrho(x,t) = D\varrho_{xx}(x,t). \tag{10}$$

Here, $\tau^\gamma$ was introduced to keep the dimensions in order. Equation (10) reduces to the normal Cattaneo equation (6) for $\gamma \to 1$, as it should be. However, we will discard equation (10) in our further discussion, as none of the other approaches leads to this generalization.

*Ad hoc*, we could also imagine the generalization of the continuity equation to obtain

$$\partial_t^\gamma \varrho(x,t) = -\jmath_x(x,t). \tag{11}$$

This is not just a formal trick altering the intuitive meaning of the continuity equation, but stands in direct connection to Hilfer's discussion of 'fractional stationarity' and a fractional Liouville equation causing a decreasing phase space in statistical systems [33]. Concerning diffusion theory, this would imply a non-conservation of heat or particle number, i.e. the transfer of heat or particles to a corresponding reservoir.

Thus, introducing the modified continuity equation (11) into the generalized constitutive equation (9), we arrive at

$$\partial_t^\gamma \varrho(x,t) + \tau^\gamma \partial_t^{2\gamma} \varrho(x,t) = D\varrho_{xx}(x,t) \qquad \text{(GCE I)}. \tag{12}$$

At this point it is important to note that this same derivation of the generalization GCE I was obtained from a linearized fractional two-velocity Boltzmann equation [16]. From this formalism, Nonnenmacher [10] obtained both a generalized constitutive equation of the form (9) and a corresponding fractional equation of continuity, which matches equation (11). This corroborates our *ad hoc* generalization GCE I, especially as equation (12) also results from our stochastic flux scheme in the next section.

Note that equation (12) can also be obtained through the combination of the standard continuity equation (1) and the modified constitutive equation

$$\jmath(x,t) + \tau^\gamma \partial_t^\gamma \jmath(x,t) = -D\partial_t^{1-\gamma}\varrho_x(x,t). \tag{13}$$

We will find this last extension again in section 3 where we give a definition of the flux in terms of CTRW theory. Equation (12) can also be obtained by replacing the time derivatives of order $1 \to \gamma$ and $2 \to 2\gamma$ in the original Cattaneo equation (6), which would be the direct phenomenological generalization if one starts off from equation (6) instead of equations (1) and (5).

In (13) it is clear that the constant $D$ now no longer has the usual dimensions of a standard diffusivity. Allowing then other possible generalizations of $D$ into our scheme, we could write

$$\jmath(x,t) + \tau^\gamma \partial_t^\gamma \jmath(x,t) = -D\partial_t^{\gamma-1}\varrho_x(x,t) \tag{14}$$

where we now take a fractional integral on the right-hand side rather than a fractional derivative as in (13). The constitutive equation (14) combined with the continuity equation (1) yields a new generalization of Cattaneo's equation

$$\partial_t^{2-\gamma} \varrho(x,t) + \tau^\gamma \partial_t^2 \varrho(x,t) = D\varrho_{xx}(x,t) \qquad \text{(GCE II)} \tag{15}$$

which has the virtue of being reversible for short times, similarly to the standard Cattaneo equation.

On the other hand, combining equation (11) with the standard constitutive equation (5), results in the GCE:

$$\partial_t \varrho(x,t) + \tau \partial_t^2 \varrho(x,t) = D\partial_t^{1-\gamma}\varrho_{xx}(x,t) \qquad \text{(GCE III)} \tag{16}$$

which is equivalent to the equation

$$\partial_t^\gamma \varrho(x,t) + \tau \partial_t^{1+\gamma} \varrho(x,t) = D\varrho_{xx}(x,t). \tag{17}$$

This same generalization can also be obtained from the continuity equation (1) by using the constitutive equation

$$\jmath + \tau\partial_t \jmath = -D\partial_t^{1-\gamma}\varrho_x(x,t). \tag{18}$$

Interesting from this last generalization is the fact that it can be interpreted as a delayed equation relating the flux to a generalized force (as we shall see, the same force that appears in the CTRW scheme) as

$$\jmath(x, t+\tau) = -D\partial_t^{1-\gamma}\varrho_x(x,t).$$

In section 3 we now give a stochastic foundation for the generalized equations GCE I and GCE III and in section 4 we interpret equation GCE II within a non-local transport theory.

## 3. Stochastic approach: Lévy asymptotics in the waiting time distribution

In this section we will derive constitutive equations out of the CTRW scheme by defining the flux in terms of the probability density $\varrho(x,t)$ of being at point $x$ at time $t$, and the distribution of step lengths and waiting times between steps $\psi(x,t)$. As in the previous sections we will restrict our approach for simplicity to the unidimensional case.

We propose here two different situations. One possibility is to define the flux as the balance between random walkers *landing* on $x$ at time $t$ coming from the left and random walkers *landing* on $x$ at time $t$ coming from the right. We then have

$$\begin{aligned} \jmath(x,t) = \ell \int_0^t \mathrm{d}t' \int_{-\infty}^{x} \mathrm{d}x'\, P(x', t-t')\psi(x-x', t') \\ -\ell \int_0^t \mathrm{d}t' \int_x^{\infty} \mathrm{d}x'\, P(x', t-t')\psi(x-x', t') \end{aligned} \tag{19}$$

$$\varrho(x,t) = \int_0^t \mathrm{d}t'\, P(x, t-t')\Psi(t') \tag{20}$$

where $P(x,t)$ is the probability density of arriving at $x$ exactly at time $t$, $\Psi(t)$ is the probability of waiting at least a time $t$ at a site, and $\ell$ is a microscopic length scale necessary to obtain the correct dimensions for the flux. Equations (19) and (20) can now be Fourier–Laplace transformed and the following relationship between $\jmath(k,u)$ and $\varrho(k,u)$ is obtained (in the following we will indicate that an expression has been Fourier—or Laplace—transformed only by explicitly showing its dependence on $k$—or on $u$)

$$\jmath(k,u) = -2\mathrm{i}\ell \frac{u}{1-\varphi(u)}\varrho(k,u)\int_0^{\infty} \mathrm{d}x\, \psi(x,u)\sin(kx) \tag{21}$$

$\varphi(u)$ being the Laplace transform of the distribution of waiting times $\varphi(t) = \int_{-\infty}^{\infty}\psi(x,t)\,\mathrm{d}x = \psi(k=0,t)$. To proceed we now need to introduce a particular form for our $\psi(x,t)$ and thus obtain the constitutive equation relating $\jmath(k,u)$ and $\varrho(k,u)$ by inverting the Fourier–Laplace transform in (21).

If we choose the distribution of step lengths to be a Gaussian

$$\psi(x,t) = \frac{1}{\sqrt{4\sigma^2\pi}}\exp\left(-\frac{x^2}{4\sigma^2}\right)\varphi(t) \tag{22}$$

we then have from (21)

$$\jmath(k,u) = -\mathrm{i}\frac{2\ell}{\sqrt{\pi}}k\sigma\frac{u\varphi(u)}{1-\varphi(u)}\,{}_1F_1(1; \tfrac{3}{2}; -k^2\sigma^2)\varrho(k,u) \tag{23}$$

${}_1F_1(x;\, y;\, z)$ being a confluent hypergeometric function [32]. For distances much larger than the mean-square step length $2\sigma^2$ ($k\sigma \ll 1$) and setting $\ell = \sigma\sqrt{\pi}/2$ to avoid unnecessary constants, we can approximate (23) as

$$\jmath(k,u) = -\mathrm{i}\sigma^2 \frac{u\varphi(u)}{1-\varphi(u)} k \left(1 - \frac{2}{3}k^2\sigma^2 + \cdots\right) \varrho(k,u). \tag{24}$$

We now consider two different waiting time distributions $\varphi(t)$: one corresponding to a Brownian random walker and the other defining a fractal time random walker.

*Brownian walker*. For this case we choose a delta distribution of waiting times in the form $\varphi(t) = \delta(t-\tau)$, where $\tau$ stands for the microscopic but finite waiting time between successive steps. We, therefore, have $\varphi(u) = \exp(-u\tau)$ in the Laplace domain and we insert this into (24) to get

$$\frac{\mathrm{e}^{u\tau} - 1}{u\tau} \jmath(k,u) = D\left(-\mathrm{i}k + \frac{2}{3}\mathrm{i}k^3\sigma^2 + \cdots\right)\varrho(k,u) \tag{25}$$

where we define the diffusion constant as is customary $D = \sigma^2/\tau$. Reverting now to the $x$–$t$ picture, (25) yields the usual constitutive relation for Brownian diffusion (Cattaneo's constitutive equation) for distances much larger than $\sigma$ (limit $k\sigma \to 0$) and to first order in the microscopic time $\tau$:

$$\jmath(x,t) + \frac{\tau}{2}\partial_t \jmath(x,t) = -D\varrho_x(x,t). \tag{26}$$

*Fractal time random walker*. We now try a distribution of waiting times with a divergent first moment, thereby describing a fractal time random walk. Our choice might, for instance, be $\varphi(u) = \exp[-(u\tau)^\gamma]$, with $0 < \gamma < 1$. In this case we get from (24)

$$\frac{\mathrm{e}^{u^\gamma\tau^\gamma} - 1}{u^\gamma\tau^\gamma} \jmath(k,u) = Du^{1-\gamma}\left(-\mathrm{i}k + \frac{2}{3}\mathrm{i}k^3\sigma^2 + \cdots\right)\varrho(k,u) \tag{27}$$

defining now the diffusion constant as $D = \sigma^2/\tau^\gamma$ as seems to be natural for these random walks [22, 23]. Neglecting the terms in $k\sigma$, taking just the lower order in $\tau$ and inverting the Fourier and Laplace transforms in (27), we now obtain the constitutive relation for fractal time random walks with our definition for the flux,

$$\jmath(x,t) + \frac{\tau^\gamma}{2}\partial_t^\gamma \jmath(x,t) = -D\partial_t^{1-\gamma}\varrho_x(x,t) \tag{28}$$

where the operator $\partial_t^\alpha$ stands for the Riemann–Liouville differintegration operator as defined in (8). Equation (28) is to be compared with the phenomenological generalization (13).

It is also to be noted here that, in the limit $\tau \to 0$, equation (28) reduces to a linear relation between the flux $\jmath$ and a generalized thermodynamical force $\partial_t^{1-\gamma}\varrho_x$. As we observed in section 2, a delayed equation relating linearly the flux and this same generalized thermodynamical force results in the proposed generalization GCE III.

Combining equation (28) with the continuity equation (1), one is led to the following generalization of the Cattaneo equation

$$\partial_t\varrho(x,t) + \frac{\tau^\gamma}{2}\partial_t^{1+\gamma}\varrho(x,t) = D\partial_t^{1-\gamma}\varrho_{xx}(x,t)$$

which coincides with the generalization GCE I except for a rescaling of the time $\tau$ and some initial value terms. As we observed in section 2, the generalization GCE I can also be obtained from a fractional continuity equation (11) and a different fractional constitutive equation (9). As should be expected, the path connecting a stochastic model such as the CTRW scheme to generalization GCE I involves the standard continuity equation (1), since such a microscopic picture assumes conservation of the particles. However, in a phenomenologically generalized model such as the fractional Boltzmann equation in [16], this restriction is no longer applicable and a counter-intuitive (though possibly well founded, as argued in [33]) generalization of the continuity equation might ensue, as it indeed does in the form (11).

We now propose an alternative second definition for the flux: let it be equal to the difference of the particles *going through* point $x$ at time $t$ coming from the left and those *going through* point $x$ at time $t$ coming from the right:

$$\jmath(x,t) = \int_0^t \mathrm{d}t' \int_x^\infty \mathrm{d}x' \int_{-\infty}^x \mathrm{d}x'' \, P(x'', t-t')\psi(x'-x'', t') - \int_0^t \mathrm{d}t' \int_{-\infty}^x \mathrm{d}x' \times \int_x^\infty \mathrm{d}x'' \, P(x'', t-t')\psi(x'-x'', t'). \tag{29}$$

We again combine this definition with (20) and transform to the Fourier–Laplace domain to obtain

$$\jmath(k,u) = \mathrm{i}\frac{u}{k}\frac{\psi(k,u) - \varphi(u)}{1-\varphi(u)}\varrho(k,u). \tag{30}$$

Proceeding now similarly to the previous case, we introduce a Gaussian step-length distribution as (22) and then approximate for very long distances ($k\sigma \ll 1$) to obtain

$$\jmath(k,u) = -\mathrm{i}\sigma^2 \frac{u\varphi(u)}{1-\varphi(u)} k \left(1 - \frac{1}{2}k^2\sigma^2 + \cdots\right)\varrho(k,u). \tag{31}$$

We observe that (31) is exactly the same as (24) to lowest order in $k\sigma$, the limit we are interested in, whence we conclude that the constitutive equations are also the same as in the previous case, namely equation (26) for standard Brownian diffusion and equation (28) for fractal time random walks. Our definitions of the flux thus lead us to GCE I for a Lévy model, whereas the usual Cattaneo equation results for the Brownian case.

## 4. Non-local transport theory: memory effects

We see here that the generalization GCE II that we proposed phenomenologically in section 2 can be obtained and interpreted within a non-local transport theory with memory effects [20, 21, 34]. Following this theory, in media with memory the flux $\jmath$ is related to the previous history of the density $\varrho$ through a relaxation function $K(t)$ as

$$\jmath(x,t) = -\int_0^t K(t-t')\varrho_x(x,t')\,\mathrm{d}t'. \tag{32}$$

We will first see that, with a suitable choice for $K(t)$, the standard Cattaneo equation is obtained. Indeed, let us compute the left-hand side of the Cattaneo equation (5) for our generalization (32):

$$\jmath(x,t) + \tau\partial_t \jmath(x,t) = -(\tau\partial_t + 1)\int_0^t K(t-t')\varrho_x(x,t')\,\mathrm{d}t'.$$

By using the Leibniz's formula for the differentiation of an integral we obtain

$$\jmath(x,t) + \tau\partial_t \jmath(x,t) = -\tau K(0)\varrho(x,t) - \int_0^t [\tau\partial_t K(t-t') + K(t-t')]\varrho_x(x,t')\,\mathrm{d}t'.$$

Hence by comparing with the Cattaneo equation (5) it appears clear that we must have $\tau K(0) = D$ and $\tau\partial_t K(t) + K(t) = 0$. Solving this differential equation, we obtain the relaxation function that makes the non-local theory of transport compatible with the Cattaneo equation

$$K(t) = \frac{D}{\tau} \exp\left(-\frac{t}{\tau}\right).$$

Our objective now is to embed by a similar computation a generalized Cattaneo equation within a theory of transport with memory kernels. To this aim we will take the left-hand side of the generalized constitutive equations we proposed in section 2, equations (9) and (13), and introduce our flux (32). Thus, we first consider the following fractional derivative

$$\tau^\gamma \partial_t^\gamma \jmath(x,t) = -\frac{\tau^\gamma}{\Gamma(1-\gamma)} \partial_t \int_0^t \mathrm{d}t'\,(t-t')^{-\gamma} \int_0^{t'} \mathrm{d}t''\, K(t'-t'')\varrho_x(x,t'')$$

where we take as before $0 < \gamma < 1$. The right-hand side may now be manipulated by inverting the order of the integrals and changing variables with $t' = t'' + z$ to obtain

$$\tau^\gamma \partial_t^\gamma \jmath(x,t) = -\frac{\tau^\gamma}{\Gamma(1-\gamma)} \partial_t \int_0^t \mathrm{d}t''\,\varrho_x(x,t'') \int_0^{t-t''} \mathrm{d}z\,(t-t''-z)^{-\gamma} K(z).$$

We integrate the innermost integral once by parts

$$\tau^\gamma \partial_t^\gamma \jmath(x,t) = -\frac{\tau^\gamma K(0)}{\Gamma(1-\gamma)} \int_0^t \mathrm{d}t'\,(t-t')^{-\gamma} \varrho_x(x,t') - \frac{\tau^\gamma}{\Gamma(2-\gamma)} \partial_t \int_0^t \mathrm{d}y\,\varrho_x(x,t-y) \times \int_0^y \mathrm{d}z\,(y-z)^{1-\gamma} \partial_z K(z) \tag{33}$$

where, on the right-hand side, we have already taken the temporal derivative on the integral of the first summand (Leibniz's formula) and we have changed the variable $t''$ to $t-y$ in the integrals of the second summand. We can now recognize in the first summand of the right-hand side of equation (33) a fractional integral of order $1-\gamma$ (8), symbolized by $\partial_t^{\gamma-1}$. To proceed, it is now convenient to apply Leibniz's formula to differentiate the integral in the second summand. Then we integrate the resulting expression once by parts, in order to obtain the following generalized constitutive Cattaneo equation, where we use the definition of fractional differintegrals where applicable:

$$\jmath(x,t) + \tau^\gamma \partial_t^\gamma \jmath(x,t) = -\tau^\gamma K(0) \partial_t^{\gamma-1} \varrho_x(x,t) - \int_0^t [\tau^\gamma \partial_y^{\gamma-1} \partial_y K(y) + K(y)]_{y=t-t'} \times \varrho_x(x,t')\,\mathrm{d}t'.$$

We now choose the relaxation function $K(t)$ to fulfil $\tau^\gamma K(0) = D$ and

$$\tau^\gamma \partial_t^{\gamma-1} \partial_t K(t) + K(t) = 0. \tag{34}$$

We therefore end up with the following generalized constitutive Cattaneo equation:

$$\jmath(x,t) + \tau^\gamma \partial_t^\gamma \jmath(x,t) = -D \partial_t^{\gamma-1} \varrho_x(x,t). \tag{35}$$

We can now compute the relaxation function $K(t)$ by solving equation (34) in the Laplace domain:

$$K(u) = \frac{D}{\tau^\gamma} \frac{u^{-1}}{1 + \tau^{-\gamma} u^{-\gamma}}. \tag{36}$$

The expression in (36) can be inverted in terms of a generalized Mittag–Leffler function (see [23]) to yield

$$K(t) = \frac{D}{\tau^\gamma} E_{\gamma,1}\left[-\left(\frac{t}{\tau}\right)^\gamma\right]. \tag{37}$$

We have, therefore, seen that generalization (35) is associated with a flux relaxation function of the form (37), which for long times has a power-law behaviour $K(t) \sim t^{-\gamma}$ with $0 < \gamma < 1$. In media where relaxation follows such a power law for long times, diffusion can be quite accurately described by a generalized Cattaneo equation of the form

$$\partial_t \varrho(x,t) + \tau^\gamma \partial_t^{\gamma-1} \partial_t^2 \varrho(x,t) = D \partial_t^{\gamma-1} \varrho_{xx}(x,t)$$

which is equivalent to generalization GCE II, except possibly for some initial value terms.

## 5. Properties of the GCEs

In the preceding sections, we introduced different generalized Cattaneo equations and argued that there are three different, physically reasonable possibilities, i.e. GCE I, GCE II and GCE III. Here we want to discuss their properties by means of the MSD and the velocity of propagation. We could not find exact solutions of these equations, however. This is no surprise, as the solution of even the standard Cattaneo equation becomes quite intricate [35]. Nevertheless, in this section we will extract as much information as possible.

Using the standard Fourier–Laplace transform technique, we can recover GCE I from equation (12),

$$\varrho_{\mathrm{I}}(k,u) = \frac{\tau^\gamma u^{-1} + u^{-(\gamma+1)}}{\tau^\gamma + u^{-\gamma} + D u^{-2\gamma} k^2} \tag{38}$$

where we have used the subscript I to denote the solution of GCE I and we take an initial delta distribution at the origin. The same can be done for our second example (15), GCE II, to obtain

$$\varrho_{\mathrm{II}}(k,u) = \frac{\tau^\gamma u^{-1} + u^{-(\gamma+1)}}{\tau^\gamma + u^{-\gamma} + D u^{-2} k^2}. \tag{39}$$

Analogously, for GCE III, equation (16), one finds

$$\varrho_{\mathrm{III}}(k,u) = \frac{\tau u^{-1} + u^{-2}}{\tau + u^{-1} + D u^{-\gamma-1} k^2}. \tag{40}$$

In $(x,t)$-space, equations (38)–(40) lead to a modified Gaussian behaviour (stretched Gaussian), a situation which is well known from relaxation theory under the keyword Kohlrausch–Williams–Watts relaxation, see, for example, [36] where the occurrence of compressed exponentials is also discussed.

Due to the occurrence of $k^2$ in the denominator of (38)–(40), all the even spatial moments of $\varrho_{\mathrm{I/II/III}}(x,t)$ exist, whereas all uneven moments vanish. We recover for the MSD $\langle x^2(t)\rangle = -\varrho_{kk}(k,t)|_{k=0}$, and hence from (38)

$$\langle x^2 \rangle_{\mathrm{I/III}} \sim \frac{2Dt^\gamma}{\Gamma(1+\gamma)} \qquad t \gg \tau \tag{41}$$

which is exactly equivalent to the long-time behaviour for $\langle x^2\rangle_{\mathrm{III}}$. Thus, both definitions for $\varrho_{\mathrm{I}}$ and $\varrho_{\mathrm{III}}$ are asymptotically equivalent for $t \gg \tau$ and correspond to anomalous diffusion (subdiffusion), as one would expect. For GCE II, however, we obtain a different result, namely

$$\langle x^2 \rangle_{\mathrm{II}} \sim \frac{2Dt^{2-\gamma}}{\Gamma(3-\gamma)} \qquad t \gg \tau \tag{42}$$

which corresponds to superdiffusion since $0 < \gamma < 1$. In the short-time limit, we find the most diverse behaviour:

$$\langle x^2\rangle_{\mathrm{I}} \sim \frac{2Dt^{2\gamma}}{\tau\Gamma(1+2\gamma)} \qquad t \ll \tau \tag{43}$$

$$\langle x^2\rangle_{\mathrm{II}} \sim \frac{D}{\tau^{\gamma}} t^2 \qquad t \ll \tau \tag{44}$$

and

$$\langle x^2\rangle_{\mathrm{III}} \sim \frac{2Dt^{1+\gamma}}{\tau\Gamma(2+\gamma)} \qquad t \ll \tau \tag{45}$$

respectively. Note that only in the short-time limit does $\tau$ occur. Thus, as in standard theory, only the short-time behaviour is affected by Cattaneo's modified theory, whereas the long-time behaviour is purely diffusive in nature. Indeed, for $\tau \to 0$, we find from equations (38)–(40) that we recover the diffusion-dominated regime (41) and (42), respectively, for all $t$. It is worth noting here that the short-time behaviour for GCE II (44) has the intuitive property of being ballistic in nature, just as for the standard Cattaneo equation. This can be interpreted as the most sensible behaviour for times much shorter than the mean collision time, since we then have a cloud of particles advancing ballistically independently of each other before the diffusion mechanism sets in. In contrast, this interpretation is inapplicable to results (43) and (45), which only recover this behaviour for $\gamma \to 1$.

We see that the fractional Cattaneo equations (12) and (16) both lead to slower transport, whereas for GCE II we obtain superdiffusion. To gain further insight into the meaning of this observation, we calculate the propagation velocity. As defined in [3], we find, after assuming a plane-wave solution and some tedious calculations, the phase velocity $v_{\mathrm{Ph}} \stackrel{\mathrm{def}}{=} \omega/\mathrm{Re}k$:

$$v_{\mathrm{Ph,I}} = \frac{(2D)^{1/2}\omega^{1-\gamma/2}}{((1+2\tau^{\gamma}\omega^{\gamma}\cos(\pi\gamma/2)+\tau^{2\gamma}\omega^{2\gamma})^{1/2} - \tau^{\gamma}\omega^{\gamma}\cos\pi\gamma - \cos(\pi\gamma/2))^{1/2}} \tag{46}$$

$$v_{\mathrm{Ph,II}} = \frac{(2D)^{1/2}\omega^{\gamma/2}}{((1-2\tau^{\gamma}\omega^{\gamma}\cos(\pi\gamma/2)+\tau^{2\gamma}\omega^{2\gamma})^{1/2} + \tau^{\gamma}\omega^{\gamma} - \cos(\pi\gamma/2))^{1/2}} \tag{47}$$

and

$$v_{\mathrm{Ph,III}} = \frac{(2D)^{1/2}\omega^{1-\gamma/2}}{\sin(\pi(\gamma+1)/4)((1+\tau^2\omega^2)^{1/2}+\tau\omega)^{1/2} - \cos(\pi(\gamma+1)/4)((1+\tau^2\omega^2)^{1/2}-\tau\omega)^{1/2}} \tag{48}$$

for each case. Let us consider the limiting cases. In the small-$\omega$ limit corresponding to the long-time limit, we find, again, that versions (46) and (48) lead to the same result,

$$v_{\mathrm{Ph,I/III}} \sim \frac{(2D)^{1/2}\omega^{1-\gamma/2}}{(1-\cos(\pi\gamma/2))^{1/2}} \qquad \tau\omega \ll 1 \tag{49}$$

whereas GCE II, (47), differs

$$v_{\mathrm{Ph,II}} \sim \frac{(2D)^{1/2}\omega^{\gamma/2}}{(1-\cos(\pi\gamma/2))^{1/2}} \qquad \tau\omega \ll 1. \tag{50}$$

All of them, however, reduce for $\gamma \to 1$ to the standard result $v_{\mathrm{Ph}} \sim (2D\omega)^{1/2}$. Also, we find that this result is not dependent on $\tau$, as one would expect. The expressions (49) and (50) vanish in the limit $\omega \to 0$, which indicates that the velocity of dispersion asymptotically vanishes, as one would expect for long-time sub-ballistic behaviour as in (41) and (42). The trace of subdiffusion and superdiffusion can also be seen here, since the phase velocity vanishes more rapidly in (49) than for standard diffusion $v_{\mathrm{Ph}} \sim \omega^{1/2}$, whence subdiffusion, and conversely in the case GCE II (50), whence superdiffusion.

It is, like before, in the large-$\omega$ limit that we encounter the most diverse behaviour, namely

$$v_{\mathrm{Ph,I}} \sim \frac{(2D)^{1/2}\omega^{1-\gamma}}{(\tau^{\gamma}(1-\cos\pi\gamma))^{1/2}} \qquad \tau\omega \gg 1 \tag{51}$$

$$v_{\mathrm{Ph,II}} \sim \sqrt{\frac{D}{\tau^{\gamma}}} \qquad \tau\omega \gg 1 \tag{52}$$

and

$$v_{\mathrm{Ph,III}} \sim \frac{(D\omega^{1-\gamma})^{1/2}}{\tau^{1/2}\sin((\pi/4)(\gamma+1))} \qquad \tau\omega \gg 1. \tag{53}$$

They all reduce to the same $v_{\mathrm{Ph}} \sim (D/\tau)^{1/2}$ for $\gamma \to 1$, i.e. to a finite and constant velocity. However, in (51) and (53) the velocity increases unboundedly with increasing $\omega$ when $0 < \gamma < 1$. For $\omega \to \infty$, both results lead to an infinite propagation velocity, as also becomes clear from (43) and (45) by differentiating $(\langle x^2\rangle)^{1/2}$. For GCE II, however, the situation is completely different since (52) is a finite constant, which is interpreted as the finite initial velocity of propagation of the perturbation.

Similar expressions, as cumbersome as (46)–(48), can be calculated for the attenuation coefficient $\alpha = -1/\mathrm{Im}k$. The most interesting result is that for short times ($\tau\omega \gg 1$) the attenuation coefficient associated with GCE II, $\alpha_{\mathrm{II}}(\omega)$, is not a constant but vanishes as $\omega^{-(1-\gamma)}$ when $\omega \to \infty$. This is in contrast to the short-time behaviour of $\alpha$ for the standard Cattaneo equation, where it goes asymptotically to a constant $2(D\tau)^{1/2}$. This is the only point where the generalized Cattaneo equation GCE II differs qualitatively from the standard Cattaneo equation at short times and, interestingly enough, it mimics the behaviour of Fick's equation $\alpha = (2D/\omega)^{1/2}$.

## 6. Conclusions

Starting off from the interest in extending the Cattaneo equation to anomalous transport processes, we proposed three different, possible and physically meaningful generalized Cattaneo equations (GCEs) and deduced them from well established alternative schemes, all of which are related to anomalous properties in complex materials. Anomalous diffusion being a very widely open concept with many different kinds of implementations, it must not come as a surprise that so many different generalizations of one single equation appear. The same phenomenon occurred for the generalizations of Fick's equation [26–30].

The proposed equations describe an anomalous MSD, both in the long-time and in the short-time limits. In the short-time limit we encounter the peculiarity of GCE II which describes pure ballistic transport with a well defined finite velocity, whereas the other generalizations, GCE I and GCE III, show an infinite signal propagation velocity, much as in standard Fickian diffusion. In the long-time limit diffusion is anomalous, in the form derived previously for analogous generalizations of Fick's equation [26–29]. At this point it is only to be noted that generalizations GCE I and GCE III show subdiffusion, while GCE II displays a superdiffusive behaviour.

As we have seen, GCE I is obtained from the generalized flux scheme proposed in section 3 and therefore seems to be quite an accurate diffusion equation for fractal time random walks, at least to the lowest order in $\tau$. Further support for GCE I comes from a previous phenomenological generalization of the Boltzmann equation [16], which also requires the concept of 'fractional stationarity' [33] to interpret the fractional continuity equation that ensues. On the other hand, generalization GCE II has a beautiful derivation in the frame of non-local transport theory with memory effects, where we have proved that it is associated with diffusion with long-tail correlations or, more specifically, where the flux relates to the gradient of the concentration by means of a memory kernel vanishing as a power law. Furthermore, this GCE has the interesting properties of yielding a finite perturbation propagation and of being time reversible for sufficiently small times, both properties shared by the standard Cattaneo equation. As for generalization GCE III, the main support comes again from CTRW theory since this theory yields a generalization of the linear flux-force thermodynamical relation which is then combined with a delayed equation to yield GCE III. This generalization has proven quite similar to GCE I, differing mainly in the short-time regime. One further difference between these two generalizations comes from the fact that in each case a different $\tau$ appears with very different interpretations: for GCE I $\tau$ is a temporal constant associated with the waiting time distribution of fractal time random walks, whereas for GCE III $\tau$ is the delay time introduced in our constitutive equation and need not be related to the internal mechanism of the random walks.

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## References

[1] Crank J 1970 *The Mathematics of Diffusion* (Oxford: Clarendon)
[2] Cattaneo G 1948 *Atti. Sem. Mat. Fis. Univ. Modena* **3** 83
[3] Casas-Vázquez, Jou D and Lebon 1996 *Extended Irreversible Thermodynamics* (Berlin: Springer)
[4] Nonnenmacher T F 1980 *J. Non-Equilib. Thermodyn.* **5** 361
[5] Straughan B 1984 *Proc. R. Soc.* A **96** 175
[6] McTaggart C L and Lindsay K A 1985 *SIAM J. Appl. Math.* **45** 70
[7] Zakari M and Jou D 1993 *Phys. Rev.* D **48** 1597
[8] Ruggeri T, Muracchini A and Seccia L 1990 *Phys. Rev. Lett.* **64** 2640
[9] Godoy S and García-Colín L S 1996 *Phys. Rev.* E **53** 5779–85
[10] Nonnenmacher T F 1984 *J. Non-Equil. Thermod.* **9** 171
[11] Grmela M and Teichmann J 1983 *Int. J. Eng. Sci.* **21** 297
[12] Velasco R M and Garcia Colin L S 1991 *Phys. Rev.* A **44** 4961
[13] Kaminski W 1990 *Trans. ASME, J. Heat Transfer* **112** 555
[14] Havlin S and Ben-Avraham D 1987 *Adv. Phys.* **36** 695
[15] Bouchaud J-P and Georges A 1990 *Phys. Rep.* **195** 127
[16] Nonnenmacher T F and Nonnenmacher D J F 1989 *Acta Phys. Hung.* **66** 145
[17] Montroll E W and Scher H 1973 *J. Stat. Phys.* **9** 101
Scher H and Montroll E W 1973 *Phys. Rev.* B **7** 4491
[18] Zimbardo G, Veltrei P, Basile G and Principato S 1995 *Phys. Plasma* **2** 3653
Balescu R 1995 *Phys. Rev.* E **51** 4807
[19] Shlesinger M F, Zaslavsky G M and Frisch U (eds) 1995 *Lévy Flights and Related Topics in Physics* (Berlin: Springer)
[20] Koch D L and Brady J F 1987 *Chem. Eng. Sci.* **42** 1377–92

[21] Brady J F and Koch D L 1988 *Disorder and Mixing* ed E Guyon *et al* (Dordrecht: Kluwer) pp 107–22
[22] Compte A 1997 *Phys. Rev.* E **55** 6821
[23] Compte A, Metzler R and Camacho J 1997 *Phys. Rev.* E **56** 1445
[24] Klafter J, Zumofen G and Blumen A 1991 *J. Phys. A: Math. Gen.* **25** 4835
[25] West B J, Grigolini P, Metzler R and Nonnenmacher T F 1996 *Phys. Rev.* E **55** 99
[26] Schneider W and Wyss 1989 *J. Math. Phys.* **30** 134
[27] Giona M and Roman H E 1992 *Physica* **185A** 87
[28] Metzler R, Glöckle W G and Nonnenmacher T F 1994 *Physica* **211A** 13
[29] Metzler R and Nonnenmacher T F 1996 *J. Phys. A: Math. Gen.* **30** 1089
[30] Compte A 1996 *Phys. Rev.* E **53** 4191
[31] Oldham K B and Spanier J 1974 *The Fractional Calculus* (New York: Academic)
[32] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* (Orlando: Academic)
[33] Hilfer R 1995 *Fractals* **3** 549–56
[34] Sobolev S L 1991 *Sov. Phys.–Usp.* **34** 217–29
[35] De Jagher P C 1980 *Physica* **101A** 629
[36] Vlad M O, Metzler R M, Nonnenmacher T F and Mackey M C 1996 *J. Math. Phys.* **37** 2279