Alpha-Stable (α-Stable) Probability Laws and Generalized Central Limit Theorem

Classical treatise: B:V. Gnedenko and A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*.
Classical treatise: W. Feller, An An Introduction to Probability Theory and Its Applications, Volume 2 ..
Modern exposition: G. Samorodnitsky and M. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*.
Practical guide: J. Nolan, *Stable Distributions*.

• Limit Theorems: How large-scale random phenomena create strict, nonrandom regularity in their collective action.

1. Stable probability laws

1.1. Definition of stable r.v.'s.

X stable if $\exists\, b_n > 0$, a_n

$$S_n = \sum_{j=1}^{n} X_j \overset{d}{=} a_n + b_n X \qquad (1.1)$$

where X_1, X_2, … i.i.d., $X_i \overset{d}{=} X$. X strictly stable if $a_n = 0$.

Theorem (Feller). $b_n = n^{1/\alpha}$, $0 < \alpha \le 2$.

Example 1. Gaussian, zero mean.

$$p_X(x) = \frac{1}{\sqrt{4\pi\sigma}} e^{-x^2/4\sigma^2} \quad , \quad \hat{p}_X(k) = e^{-\sigma^2 k^2} \quad . \qquad (1.2)$$

Note: *Var*{*X*} = 2σ^2;. We compare characteristic functions (CF) on the left and right hand sides of Eq.(1.1) **Remainder: CF as the Fourier transform of the PDF**

l.h.s.: $\left\langle e^{ik\sum_{j=1}^{n} X_j} \right\rangle = \left\langle e^{ikX_1 + ikX_2 + \ldots} \right\rangle = \left\langle e^{ikX} \right\rangle^n = e^{-n\sigma^2 k^2}$

r.h.s.: $\left\langle e^{ik(a_n + b_n X)} \right\rangle = e^{ika_n} \hat{p}_X(b_n k) = e^{ika_n - \sigma^2 (b_n k)^2}$

$$\Rightarrow a_n = 0 \quad , \quad b_n = \sqrt{n}$$

Strictly stable.

Example 2. Gaussian, non-zero mean.

$$p_X(x) = \frac{1}{\sqrt{4\pi\sigma}} e^{-(x-a)^2/4\sigma^2} \quad , \quad \hat{p}_X(k) = e^{ika - \sigma^2 k^2} \quad . \qquad (1.3)$$

Stable.

Example 3. *Cauchy*.

$$p_X(x) = \frac{1}{\pi}\frac{\sigma}{x^2+\sigma^2} \quad \Rightarrow \quad \hat{p}(k) = e^{-\sigma|k|} \tag{1.4}$$

l.h.s. $\left\langle e^{ik\sum_{j=1}^{n} X_j} \right\rangle = e^{-n\sigma|k|}$

r.h.s. $\left\langle e^{ik(a_n+b_n X)} \right\rangle = e^{ika_n}\,\hat{p}(b_n k) = e^{ika_n-\sigma b_n|k|}$

$$\Rightarrow \quad a_n = 0 \quad , \quad b_n = n$$

Stable.

Example 4. *Laplace distribution*.

$$p_X(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}} \quad \Rightarrow \quad \hat{p}(k) = \frac{1}{1+\sigma^2 k^2} \quad . \tag{1.5}$$

l.h.s. $\left\langle e^{ik\sum_{j=1}^{n} X_j} \right\rangle = \frac{1}{\left(1+\sigma^2 k^2\right)^n}$

rhs. $\left\langle e^{ik(a_n+b_n X)} \right\rangle = e^{ika_n}\,\hat{p}(b_n k) = \frac{e^{ika_n}}{1+\sigma^2 b_n^2 k^2}$

Non-stable.

1.2. Equivalent definition of strictly stable r.v.'s

X strictly stable if $\forall\ \sigma_1,\ \sigma_2\ \exists\ \sigma,\ \sigma_1^\alpha + \sigma_2^\alpha = \sigma^\alpha$,

$$\sigma_1 X_1 + \sigma_2 X_2 \overset{d}{=} \sigma X \quad , \tag{1.6}$$

where $X_1 \overset{d}{=} X_2 \overset{d}{=} X$.

1.3. Solution for strictly stable

$$\hat{p}(\sigma_1 k)\,\hat{p}(\sigma_2 k) = \hat{p}(\sigma k) \quad . \tag{1.7}$$

Solution:

$$\hat{l}_\alpha(k) = e^{-|k|^\alpha} \quad . \tag{1.8}$$

Remark. The function e^{-k^α} is not possible, since CF must obey $\hat{p}^*(k) = \hat{p}(-k)$.

Question: Which α's are possible ?

1. Normalization of the PDF. $\hat{l}_\alpha(0)=1 \quad \Rightarrow \quad \alpha>0$

2. Suppose $\alpha > 2$. Then $\left\langle x^2 \right\rangle = \int\limits_{-\infty}^{\infty} x^2 l_\alpha(x)dx = -\left.\frac{d^2\hat{l}_\alpha(k)}{dk^2}\right|_{k=0} = 0.$

Conclusion. $0<\alpha\le 2$.

Examples. $\alpha=2, \quad \alpha=1$ (Gauss and Cauchy).

1.4. General solution of Eq.(1.1) (Lévy; Khintchine; Gnedenko, Kolmogorov):

$$p_{\alpha,\beta}(k;\mu,\sigma)=\exp\left[i\mu k-\sigma^\alpha\,|k|^\alpha\,\left(1-i\beta sign(k)\omega(|k|,\alpha)\right)\right] \quad , \tag{1.9}$$

where

$$\omega(k,\alpha)=\begin{cases}\tan(\pi\alpha/2) & if \quad \alpha\ne 1, \quad 0<\alpha<2 \\ -\left(2/\pi\right)\ln|k| & if \quad \alpha=1\end{cases} \quad .$$

4-parametric: 0 < α ≤ 2 (**Levy index**), -1 ≤ β ≤ 1 (**skewness parameter**), σ > 0 (**scale parameter**), μ real (**shift parameter**).

1.5. Properties of the PDF

1.5.1. **α and β are important ! μ and δ are less important** (γ and δ can be eliminated by proper shift and scale transformation:

$$p_{\alpha,\beta}(x;\mu,\sigma)=\frac{1}{\sigma}\,p_{\alpha,\beta}\left(\frac{x-\mu}{\sigma};0,1\right) \quad , \quad \alpha\ne 1\, or \quad \alpha=1,\beta=0 \quad . \tag{1.10}$$

Notation: $p_{\alpha,\beta}(x;0,1)\equiv p_{\alpha,\beta}(x)$

1.5.2. **Reflection property**:

$$p_{\alpha,-\beta}(x)=p_{\alpha,\beta}(-x) \quad \text{(trivially checked with Eq.(1.9))} \; . \tag{1.11}$$

1.5.3. **The sum of two stable variables with the same Levy index α**

$$\mu=\mu_1+\mu_2 \tag{1.12a}$$

$$\sigma^\alpha=\sigma_1^\alpha+\sigma_2^\alpha \tag{1.12b}$$

(generalization of the rule of summing variances)

$$\beta=\frac{\beta_1\sigma_1^\alpha+\beta_2\sigma_2^\alpha}{\sigma^\alpha} \quad , \quad -1\le\beta\le 1 \; . \tag{1.12c}$$

1.5.4. **Stable laws are continuous and have derivatives of all orders at every point** .

1.5.5. **Unimodality** (single max)

1.6. *Symmetric stable laws*: $\beta = 0$ $p_{\alpha,0}(x) = p_{\alpha,0}(-x)$

1.6.1. CF for symmetric stable law:

$$p_{\alpha,0}(k) \equiv p_\alpha(k) = \exp\left(-|k|^\alpha\right) \quad .$$

$$p_\alpha(k) \simeq 1 - |k|^\alpha + \frac{1}{2}|k|^{2\alpha} - \ldots \quad k \to 0 \quad .$$

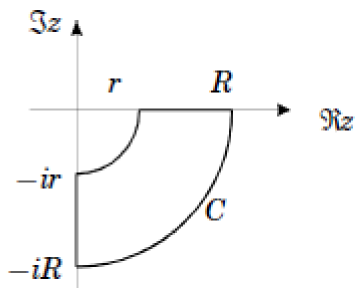
1.6.2. Asymptotics of the PDF

Due to symmetry we may consider $x > 0$, only.

$$l_\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - |k|^\alpha} dk = \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{-ikx - k^\alpha} dk \quad . \tag{1.13}$$

Let us consider the integral in the complex plane,

$$\oint_C e^{-ikz - z^\alpha} dz = 0 \quad .$$



Let $r \to 0$, $R \to \infty$. Then, according to *Jordan's lemma*, the integral along the large arc vanishes. The integral along the small arc vanishes as well. As the result, for our integral along the real axis we have

$$\int_0^\infty e^{-ikx - k^\alpha} dk = -\int_{-i\infty}^{-i0} e^{-ikx - k^\alpha} dk = \quad < ik = \kappa > \quad = -\frac{1}{i}\int_\infty^0 e^{-x\kappa - \kappa^\alpha e^{-i\pi\alpha/2}} d\kappa =$$

$$-i\int_0^\infty e^{-x\kappa - \kappa^\alpha e^{-i\pi\alpha/2}} d\kappa \quad . \tag{1.14}$$

Plug (1.14) into (1.13), change variable $x\kappa = y$, and expand exponential function in the integrand. As the result, Eq.(1.13) reads

$$l_\alpha(x) = \frac{1}{\pi x}\mathrm{Re}\left\{-i\int_0^\infty e^{-y - \frac{y^\alpha}{x^\alpha}e^{-i\pi\alpha/2}} dy\right\} = \frac{1}{\pi x}\mathrm{Re}\left\{-i\int_0^\infty e^{-y}\left[1 - \frac{y^\alpha}{x^\alpha}e^{-i\pi\alpha/2} + O\left(\frac{y^{2\alpha}}{y^{2\alpha}}\right)\right] dy\right\} =$$

Integrate term by term, the first term gives zero

$$\simeq \frac{1}{\pi x^{1+\alpha}} \mathrm{Re}\left\{ +ie^{-i\pi\alpha/2} \int_0^\infty e^{-y} y^\alpha dy \right\} \Rightarrow$$

Finally,

$$\boxed{p_\alpha(x) \approx \frac{C(\alpha)}{|x|^{1+\alpha}} \quad , \quad x \to \pm\infty} \tag{1.15}$$

$$C(\alpha) = \frac{1}{\pi} \sin\left(\frac{\pi\alpha}{2}\right) \Gamma(1+\alpha) \quad . \tag{1.16}$$

1.6.3. Asymptotics of the cumulative probability

$$L_\alpha(x) = \int_{-\infty}^{x} l_\alpha(y) dy \quad \sim \begin{cases} \dfrac{C(\alpha)}{\alpha |x|^\alpha} & , \quad x \to -\infty \\ 1 - \dfrac{C(\alpha)}{\alpha x^\alpha} & , \quad x \to \infty \end{cases} \quad . \tag{1.17}$$

1.6.4. Moments

$$\left\langle |x|^q \right\rangle = \int_{-\infty}^{\infty} |x|^q l_\alpha(x) dx = \begin{cases} C(q;\alpha) & , \quad q < \alpha \\ \infty & , \quad q \geq \alpha \end{cases} \quad . \tag{1.18}$$

Examples. In terms of elementary functions:
Example 1. Gauss

$$l_2(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4} \quad .$$

Example 2. Cauchy

$$l_1(x) = \frac{1}{\pi(1+x^2)} \quad .$$

Remark **2**. *Asymmetric densities* $0 < \beta < 1$. The decay at $x \to \infty$ is the same, but $C(\alpha) \to C(\alpha, \beta)$.

1.7. Extremal or totally skewed stable laws: $\beta = \pm 1$.

Go to Eq.(1.9), take β=1 and put (just for simplicity) $\mu = 0$.

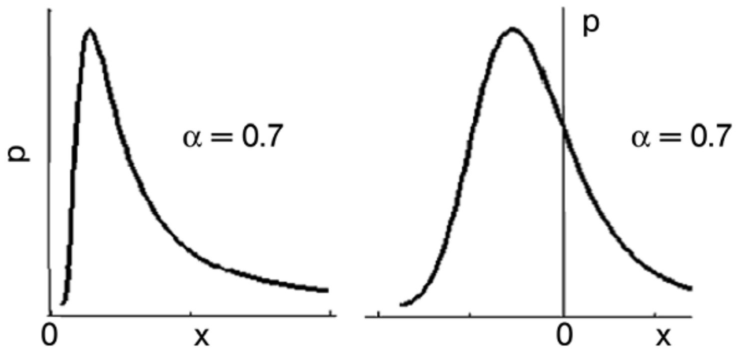
$$\hat{l}_{\alpha,1}(k) = \exp\left[-\sigma^\alpha |k|^\alpha \left(1 - i sign(k) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right] =$$

$$= \exp\left[-\frac{\sigma^{\alpha} |k|^{\alpha}}{\cos\left(\frac{\pi\alpha}{2}\right)} \left(\cos\left(\frac{\pi\alpha}{2}\right) - i sign(k) \sin\left(\frac{\pi\alpha}{2}\right)\right)\right] =$$

$$= \exp\left[-\frac{\sigma^{\alpha} |k|^{\alpha}}{\cos\left(\frac{\pi\alpha}{2}\right)} \left(\cos\left(\frac{\pi\alpha}{2} sign(k)\right) - i \sin\left(sign(k) \frac{\pi\alpha}{2}\right)\right)\right] =$$

$$= \exp\left[-\frac{\sigma^{\alpha} |k|^{\alpha}}{\cos(\pi\alpha/2)} e^{-i sign(k)\pi\alpha/2} \right] \quad . \qquad (1.19)$$

Similar to derivation in (1.13-1.15) it can be shown that $l_{\alpha,1}(x) \sim x^{-1-\alpha} \quad , \quad x \to \infty$



Sketch of one-sided (left) and two-sided (right) totally skewed (extremal) α-stable PDFs with $\beta = 1$.

In terms of elementary function: only for $\alpha = 1/2$, $\beta = 1$ (Lévy-Smirnov distribution).

$$p_{1/2,1}(x) = \begin{cases} \frac{1}{\sqrt{2\pi x^3}} e^{-1/2x} & , \quad x \ge 0 \\ 0 & , \quad x < 0 \end{cases} \quad .$$

2. Generalized Central Limit Theorem

Under what conditions on identically distributed independent variables X_1, … X_n, … can a limit relation

$$\Pr\left\{ \frac{\sum_{i=1}^{n} X_i - A_n}{B_n} < z \right\} \to L(z) \quad , \quad n \to \infty \qquad (2.1)$$

hold, where A_n and B_n are constants, and what kind of limit laws can appear ?

2.1. Reminder: Central Limit Theorem

2.1.1. A pedestrian approach to CLT. Random walk picture.

Random walk is succession of random steps.

What is the distribution of the sum of r.v.'s ?

Example 1. Gaussian. $X_1, \dots X_n$ i.i.d. Gaussian

$$p_X(x) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp\left[-\frac{x^2}{4\sigma^2}\right] \qquad \hat{p}(k) = e^{-\sigma^2 k^2} \qquad p_n(x) = ? \quad . \tag{2.2}$$

$$E[X] = 0 \quad , \quad Var[X] := E\left[\left(X - E[X]\right)^2\right] := \left\langle \left(x - \langle x \rangle\right)^2 \right\rangle = 2\sigma^2 \qquad .$$

Let us calculate the PDF $p_n(x)$ of the sum $\sum_{j=1}^{n} X_j$. We have already seen that it can be easily done by making use of the CFs:

$$\hat{p}_n(k) = \left\langle e^{ik\sum_{j=1}^{n} X_j} \right\rangle = \left\langle e^{ikX_1 + ikX_2 + \dots + ikX_n} \right\rangle = \prod_{j=1}^{n} \hat{p}_j(k) =$$

$$= e^{-n\sigma^2 k^2} \quad \div \quad p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-n\sigma^2 k^2} e^{-ikx} dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-n\sigma^2\left(k^2 + 2k\frac{ix}{2n\sigma^2} - \frac{x^2}{4n^2\sigma^2} + \frac{x^2}{4n^2\sigma^2}\right)} dk =$$

$$= \frac{e^{-\frac{x^2}{4n\sigma^2}}}{2\pi}\int_{-\infty}^{\infty} \exp\left[-n\sigma^2\left(k - \frac{ix}{2n\sigma^2}\right)^2\right] dk = \frac{e^{-\frac{x^2}{4n\sigma^2}}}{\sqrt{4\pi n\sigma^2}} \quad \Rightarrow \text{also Gauss, but}$$

$$\boxed{Var\{X\} \;\; \to \;\; Var\{X\} \times n} \tag{2.3}$$

Example 2. The same Gaussian, but in a different way:

$$p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left[e^{-\sigma^2 k^2}\right]^n dk = \quad \left\langle e^{-\xi} = 1 - \xi + \frac{\xi^2}{2!} - \dots, \quad -\infty < \xi < \infty \right\rangle \quad =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left[1 - \sigma^2 k^2 + \dots\right]^n dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} e^{n\log\left(1 - \sigma^2 k^2 + \dots\right)} dk =$$

$$= \langle \log(1-\xi) = -\left(\xi + \frac{\xi^2}{2} + \frac{\xi^3}{3} + \ldots\right), \quad -1 \le \xi < 1 \rangle \quad \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx-\sigma^2 k^2 n} dk \quad . \tag{2.4}$$

We got the same result while retaining only the second term in the expansion of the characteristic function.

Conclusion. Gaussian → Gaussian, $\boxed{Var\{X\} \quad \to \quad Var\{X\}\times n}$

Example 3. Laplace. $X_1, \ldots X_n$ i.i.d.

$$p_X(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}} \quad \div \quad \hat{p}(k) = \frac{1}{1+\sigma^2 k^2} \quad . \tag{2.5}$$

$$E[X] = 0 \quad , \quad Var[X] = E\left[\left(X - E[X]\right)^2\right] = \left\langle \left(x - <x>\right)^2 \right\rangle = 2\sigma^2 \; .$$

Let us calculate the PDF of the sum $\sum_{j=1}^{n} X_j$.

$$p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left[\frac{1}{1+\sigma^2 k^2}\right]^n dk \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left[1 - \sigma^2 k^2 + \ldots\right]^n dk =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} e^{n\ln\left(1-\sigma^2 k^2+\ldots\right)} dk \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx-n\sigma^2 k^2} dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-nb^2\left(k^2 + 2k\frac{ix}{2nb^2} - \frac{x^2}{4n^2b^4} + \frac{x^2}{4n^2b^4}\right)} dk =$$

$$= \frac{e^{-\frac{x^2}{4b^2 n}}}{2\pi}\int_{-\infty}^{\infty} \exp\left[-nb^2\left(k + \frac{ix}{2nb^2}\right)^2\right] dk = \frac{e^{-\frac{x^2}{4nb^2}}}{\sqrt{4\pi n b^2}} \quad . \tag{2.6}$$

Conclusion. Laplace → Gaussian, $\boxed{Var\{X\} \quad \to \quad Var\{X\}\times n}$

2.1.2. Central Limit Theorem

$X_1, \ldots, X_n$, , $p_X(x)$ with $Var\{X\} < \infty$

(i) the summands are i.i.d.

(ii) Var{X} finite

$$\hat{p}(k) \simeq 1 - \sigma^2 k^2 + o(k^2) \quad , \quad Var\{X\} = -\left.\frac{d^2\hat{p}(k)}{dk^2}\right|_{k=0} = 2\sigma^2 < \infty$$

$$p_n(x) \quad \to \quad \frac{1}{\sqrt{4\pi n\sigma^2}}\exp\left(-\frac{x^2}{4n\sigma^2}\right) \quad , \quad n \to \infty \; , \; \boxed{Var\{X\} \quad \to \quad Var\{X\}\times n}$$

The distribution of sum of *n* independent, identically distributed random variables possessing the mean *μ* and the variance *Var{X}* tends for large *n* to a Gaussian with the mean *μ×n* and the variance *Var{X}×n*.

2.1.3. Weakening (i) and (ii)

- Lindeberg Theorem

X1, …, Xn, … independent variables with Var{X1}, Var{X2}, …, Var{Xn}, …

$$p_n(x) \quad \to \quad \frac{1}{\sqrt{2\pi\left(\sum_{i=1}^{n} Var\{X_i\}\right)}} \exp\left(-\frac{x^2}{2\left(\sum_{i=1}^{n} Var\{X_i\}\right)}\right) \quad , \quad n \to \infty$$

$$\boxed{Var\{X\} \quad \to \quad \sum_{i=1}^{n} Var\{X_i\}} \quad .$$

- Borderline case

X_1, …, X_n,… i.i.d. with $p_X(x) \propto x^{-3}$

$$p_n(x) \quad \to \quad Gauss \quad with \quad Var\left\{\sum_{i=1}^{n} X_i\right\} \sim n \ln n \quad , \quad n \to \infty$$

Prefactor depends on the details of the distribution of *X*.

Paul Levy 1930s: Variance = ∞ ?

2.2. Generalized Central Limit Theorem. Simple example with symmetric distribution.

X_1, X_2, …, X_n, … i.i.d. $p_X(x)$, Var{*X*} infinite. Limit distribution ?

Intuitively:

$$\hat{p}(k) \simeq 1 - C_\alpha |k|^\alpha + o(k^2) \quad , \quad Var\{X\} = -\frac{d^2 \hat{p}(k)}{dk^2}\bigg|_{k=0} = \infty \ , \ 0 < \alpha < 2 \ .$$

Take

$$p_X(x) \quad \propto \quad \frac{A}{|x|^{1+\alpha}} \quad , \quad x \to \infty \quad , \quad 0<\alpha<2$$

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{ikx} p_X(x)dx = 2\int_0^{\infty} \cos(kx)p_X(x)dx = 1-2\int_0^{\infty}(1-\cos(kx))p_X(x)dx \approx$$

$$\approx 1-2A\int_0^{\infty}\frac{1-\cos(kx)}{x^{1+\alpha}}dx = 1-2A|k|^{\alpha}\int_0^{\infty}\frac{1-\cos(y)}{y^{1+\alpha}}dy .$$

Separately: $\int_0^{\infty}\frac{1-\cos(y)}{y^{1+\alpha}}dy = \langle u = 1-\cos y, dv = dy/y^{1+\alpha} \,||\, du = \sin y dy, v = -1/(\alpha y^{\alpha}) \rangle =$

$$= \frac{1}{\alpha}\int_0^{\infty}\frac{\sin y}{y^{\alpha}}dy = \frac{\pi}{2\Gamma(\alpha+1)\sin(\pi\alpha/2)} \quad .$$

Thus,

$$\hat{p}(k) \approx 1-\tilde{A}|k|^{\alpha}+\dots \quad , \tag{2.7}$$

where

$$\tilde{A} = \frac{\pi A}{\Gamma(\alpha+1)\sin(\pi\alpha/2)} \quad .$$

Now, in Eq.(2.7) the same trick as before with the Gaussian, namely, retain only these first two terms in the expansion of the CF $\hat{p}(k)$:

$$\hat{p}_n(k) = (\hat{p}(k))^n \approx \left(1-\tilde{A}|k|^{\alpha}\right)^n = e^{n\ln\left(1-\tilde{A}|k|^{\alpha}+\dots\right)} \approx e^{-\tilde{A}n|k|^{\alpha}} \quad . \tag{2.8}$$

Remembering the CF of the symmetric α-stable law:

$$\hat{l}_{\alpha,0}(k) = \exp\left(-\sigma^{\alpha}|k|^{\alpha}\right) \quad , \quad 0<\alpha<2 \quad ,$$

<u>GCLT, particular case</u>. We conclude that for i.i.d. with

$$\boxed{p_X(x) \quad \propto \quad \frac{A}{|x|^{1+\alpha}} \quad , \quad x \to \infty \quad , \quad 0<\alpha<2},$$

the PDF of $\sum_{i=1}^{n} X_i$

$$\boxed{p_n(x) \quad \to \quad l_{\alpha}(x;\sigma)} \quad .$$

where

$$\sigma^\alpha = \tilde{A}n = \frac{\pi A n}{\Gamma(\alpha+1)\sin(\pi\alpha/2)} = \frac{An}{C(\alpha)} \quad , \quad C(\alpha) = \frac{1}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1+\alpha) \quad .$$

Now, ATTENTION ! Remembering asymptotics of the α-stable PDF,

$$l_\alpha(x;\sigma) \approx C(\alpha)\frac{\sigma^\alpha}{|x|^{1+\alpha}} \quad , \quad x \to \pm\infty$$

The limit α-stable PDF has the asymptotics

$$l_\alpha\left(x;\sigma = \frac{An}{C(\alpha)}\right) \approx \frac{An}{|x|^{1+\alpha}} \quad , \quad x \to \pm\infty \quad .$$

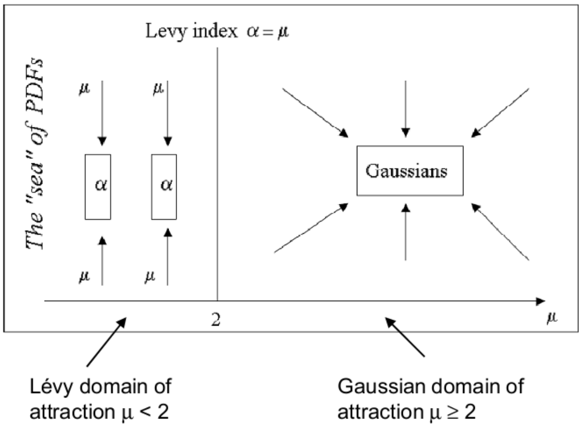
2.3. GCLT in words

Stable distributions (as the Gaussian law) are the limit ones for distributions of sums of random variables (have domain of attraction) ⇒ appear, when evolution of the system or the result of experiment is determined by the sum of a large number of random quantities

Gauss: attracts *f(x)* with finite variance:

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} x^2 f(x)dx < \infty$$

$$f(x) \propto |x|^{-1-\mu} \quad , \quad 0 < \mu < 2$$



(Lévy) stable: attract *f(x)* with infinite variance and the same asymptotic behavior

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \infty \quad , \quad f(x) \propto x^{-1-\alpha} \, , \quad 0 < \alpha < 2$$