Celestial Mechanics

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Inhaltsverzeichnis

Literature

- J. M. A. Danby (1992): Celestial Mechanics, *Willman & Bell*, Richmond.
- V. Szebehely (1962): Theory of orbits. The restricted problem of three bodies, *Academic Press*, London.
- D. Brouwer & G. M. Clemens (1961): Methods of celestial mechanics, *Academic Press*, London.
- C. D. Murray & S. F. Dermott (1999): Solar System Dynamics, *Cambridge Univ. Press*.x

1 Introduction

- **Classical Celestial Mechanics:** Description of the observed motion of *planets* (so-called “moving stars”). Since **Newton** this task has been upgraded on a strict mathematical fundament explaining the motion of gravitationally interacting point-masses via infinitesimal calculus (differential equations, also invented by **Newton**).

- **Modern Celestial Mechanics:** Extension of the field of applications of the theory to:

  - massive **cosmic objects** as: stellar systems like galaxies, clusters of them, cosmic disks (again galaxies, accretions disks around compact objects (white dwarfs, neutron stars, black holes), pre-planetary disks – the “nurseries” of planets, planetary rings etc.);
  - **Dynamical evolution/structure formation** of cosmic objects; which requires the consideration of large numbers of constituents[1], resp. their finite extent/size as well as non-gravitational interaction among them as there could be: physical collisions, electro-magnetic fields (charged and/or magnetized constituents);
  - Application of modern theories and methods – for instance **special - and general relativity**, thermodynamics/statistics and dynamics of **non-equilibribrium states** of (gravitationally interacting) ensembles of constituents – in order to quantify the evolution of systems/ensembles of constituents forming cosmic objects.
  - Calculation of trajectories of artificial satellites/spacecraft for their navigation, as used for many deep-space missions: *Pioneer, Voyager, Ulysses, Galileo, Cassini*, etc.
  - Important methods of the modern celestial mechanics: **Numerical simulations**.

The table below sketches early historical phases of *Celestial Mechanics* which are NOT subject of this lecture, but which deserve to be mentioned for completeness:

[1] with this term we denote the building blocks of an ensemble forming the cosmic object under consideration – examples: centimetre-sized granular (icy) particles ⇒ Saturn’s dense rings; planetesimals & planetary embryos ⇒ pre-planetary disks around young stars; stars ⇒ galaxies

Forscher	Lebensdaten	Entdeckungen
Aristoteles	384 - 322 v. Chr.	-Kugelgestalt der Erde
Aristarch	310 - 230 v. Chr.	-Ansatz eines **heliozentrischen Weltbildes** -Größe von Erde und Mond
Eratosthenes	284 - 192 v. Chr.	-Erdumfang
Hipparch	180 - 125 v. Chr.	- Entfernung Erde - Mond - Neigung der Ekliptik
Ptolemäus	87 - 165	-geozentrisches Weltbild
Kopernikus	1473 - 1543	**-heliozentrisches Weltbild**
Brahe	1546 - 1601	-Beobachtungen zur Prüfung des Kopernikanischen Weltbildes -“geoheliozentrisches“ Weltbild
Kepler	1571 - 1630	**-3 Keplerische Gesetze**
Galilei	1564 - 1642	-erstes optisches Teleskop -4 große Monde des Jupiter -Vorarbeiten zur Newtonschen Gravitationstheorie
Newton	1642 - 1727	**-Gravitationsgesetz** **-Bewegungsgesetze** **-Infinitesimalrechnung**
Halley	1656 - 1742	-Bahn des Kometen Halley
Titius Bode	1729 - 1796 1747 - 1826	-Titius-Bode-Reihe
Herschel	1738 - 1822	-Uranus (1781)
Piazzi Zach	1746 - 1826 1754 - 1832	-Ceres
Gauß	1777 - 1855	-Bahn von Ceres
Leverrier Adams	1811 - 1877 1819 - 1892	**-„inverse“ Störungstheorie** zur Suche nach transuranischen Planeten
Galle	1812 - 1910	-Neptun
Poincaré	1854 - 1912	**-Chaostheorie** -Dreikörperproblem
Einstein	1879 - 1955	**-Relativitätstheorie**

Historischer Abriß

- **Babylonier (1500 - 500 v. Chr.):** langjährige Venusbeobachtungen; Kenntnis der Phasen dieses Planeten; scheinbare Bahnbewegungen der Planeten - deren Epizyklen - Vorhersage von Sonnenfinsternissen

- **Aristoteles (384 -322 v. Chr.):** Folgerte die Kugelgestalt der Erde aus der Kreisform des Erdschattens bei Mondfinsternissen.
  Theorie der vier Elemente: Erde, Luft, Feuer und Wasser. Jede davon besitze seine eigene „Gravitation“ (entsprechend seiner „Schwere“). Deshalb bewege sich unter irdischen Bedingungen alles linear und geradlinig - den vier Elementen adäquat. Alle himmlischen Bewegungen sind jedoch, gefunden durch Beobachtungen, gekrümmt. Daraus schlußfolgert Aristoteles die Existenz eines fünften Elements - dem Äther, der für eine nicht geradlinige, sprich nichtirdische Bewegung verantwortlich sei.

- **Aristarch (310 - 230 v. Chr.):** Erste - allerdings unakzeptierte - Ansätze eines heliozentrischen Weltbildes der Bewegung der Sonne, der Erde und der Planeten. Bestimmte geometrisch die Größe von Himmelskörpern - Erde, Mond (mit 50% Fehler) - sowie die Entfernung Erde - Mond (ebenfalls ca. Faktor 2)

- **Eratosthenes (284 - 192 v. Chr.):** siehe Aristarch: Messung des Erdumfangs
  $\Longrightarrow$ Methode: man nehme 2 Orte gleicher geographischer Länge aber unterschiedlicher Breite
  $\Longrightarrow$ man beobachte Unterschiede in der Winkelhöhe (Parallaxe) eines Himmelskörpers (im konkreten Fall: der Sonne)
  $\Longrightarrow$ die Parallaxe sowie der bekannte Abstand zwischen beiden Orten, deren Basis, ergibt Hinweise auf die Erdkrümmung.

- **Hipparch (180 - 125 v. Chr.):** Geometrische Studien. Bestimmung der Entfernung Erde - Mond mit relativ hoher Genauigkeit (34 statt 30 Erdradien) über die Mondparallaxe. Er fand die Schiefe der Ekliptik sowie ihre Präzession (genauer die der Erdachse); stellte den ersten größeren Sternkatalog zusammen.

- **Ptolemäus (87 - 165 v. Chr.):** Begründer des geozentrischen Weltbildes (Werk: *Almagest*; Erde im Zentrum der Bewegung der Himmelskörper), welches bis zu Keplers Zeiten Gültigkeit behielt. Die Bewegung der Fixsterne und der Sonne ist damit exzellent zu beschreiben - Probleme bereiten die Planeten.
  Kunstgriff: Einführung der Epizyklen; das sind kleinere Kreise, deren Zentren sich wiederum auf exzentrischen Kreisen um die Erde bewegen.
  Damit ließ sich recht genau die Bewegung der damals bekannten Planeten vorhersagen.

- **Nicolaus Kopernikus (1473 - 1543):** Wegbereiter des heliozentrischen Weltbildes (Werk: *De revolutionibus orbium coelestium libri VI*; Sonne im Zentrum der Planetenbewegung). Einfache Kreisbahnen der Planeten um die Sonne lösten elegant das Problem der scheinbar komplizierten Epizyklenbewegung, die einfach durch die Projektion der Bewegungen auf die Himmelsphäre entsteht.

- **Tycho Brahe (1546 - 1601):** Erste akkurate Beobachtungen (Genauigkeit liegt bei Bogenminuten) um zu prüfen, ob das Ptolemäische oder das Kopernikanische Weltbild richtig ist. Entwickelte das sogenannte „geoheliozentrische“ Weltbild, bei dem die Sonne um die Erde kreist, jedoch alle anderen Planeten die Sonne umrunden. Begründer des ersten astronomischen Observatoriums. Unter anderem beobachtete er eine Supernova (1572) und einen Kometen, dessen Parallaxe er bestimmte und fand, daß dieser Himmelskörper wesentlich weiter entfernt ist als der Mond und daß Kometen keine atmosphärischen Erscheinungen sind.

- **Johannes Kepler (1571 - 1630):** Als Schüler Brahes formulierte er die drei berühmten Keplerschen Gesetze als Folge zahlreicher, für die damalige Zeit höchst präziser Beobachtungen, die auf dem Werk Brahes basieren. Unter anderem beobachtete er eine Nova (1604) und beschäftigte sich mit Optik.

- **Galileo Galilei (1564 - 1642):** Erfand das erste optische Teleskop mit dem er die vier nach ihm benannten großen Jupitermonde, als auch die längliche gestalt des Saturn entdeckte. Letztere stellte sich später als Ring des Planeten heraus. Erkannte, daß die Schwingungsperiode eines Pendels gegebener Länge nicht von der Schwingungsamplitude abhängt und daß die Trajektorien geworfener Körper unter der Erdgravitation Parabeln sind
  $\Longrightarrow$ wichtige Hinweise für Newtons Theorie

- **Sir Isaac Newton (1642 - 1727):** Begründer des „Goldenen“ Zeitalters der modernen Himmelsmechanik mit der Entwicklung der nach ihm benannten Bewgungsgesetze und des Gravitationsgesetzes. Die fundamentale Bedeutung seiner Arbeiten ist bis heute ungebrochen - bis auf wenige Ausnahmen basieren auch die modernen Methoden der Himmelsmechanik auf seinen Theorien. Sein Hauptwerk ist in der *„Principa Mathematica“* (1687) dargelegt. Erst Einsteins allgemeine Relativitätstheorie liefert genauere Ergebnisse, die allerdings nur nahe kompakter Objekte (Neutronensterne, weiße Zwerge, schwarze Löcher) relevant sind.

- **Edmund Halley (1656 - 1742):** Zeitgenosse und Freund Newtons. Berechnete erstmals systematisch Bahnelemente von Kometen auf der Grundlage der Theorie Newtons. Er sagte u.a. die Bahnperiode des nach ihm benannten Kometen voraus.

- **Johannes D. Titius (1729 - 1796); Johann E. Bode (1747 - 1826):** Fanden empirisches Gesetz der Abstände der Planeten von der Sonne;
  $r_n \approx 0.4 + 0.3 \cdot 2^n,\ n = -\infty, 0, 1, \cdots$; die Titius-Bode-Reihe. Sie postulierten die Existenz von Körpern an der Stelle $n = 3$ wo, wie wir heute wissen, der Asteroidengürtel liegt.

- **Friedrich Wilhelm Herschel (1738 - 1822):** Entdeckte den Planeten *Uranus* (1781), der sich in das Titius-Bodesche Schema unter dem Index $n = 6$ einfügt.

- **Guiseppe Piazzi (1746 -1826); Franz X. v. Zach (1754 - 1832):** Suche nach einem Planeten bei $n = 3$ der Titius-Bode-Reihe
  $\Longrightarrow$ Piazzi fand „vorläufig“ den Asteroiden *Ceres* (1800), der später (Silvesternacht 1801) nach theoretischen Bahnbestimmungen von Carl Friedrich Gauß (siehe unten) von Zach wiederentdeckt wurde.

- **Carl Friedrich Gauß (1777 - 1855):** Mathematiker; berechnete mit der Methode der kleinsten Quadrate mögliche Bahnellipsen von*Ceres*, die mit den Beobachtungen

von G. Piazzi verträglich waren.
Hauptzweck dieser Übung: den Asteroiden *Ceres*, der sich für mehr als ein Jahr der Beobachtung „entzog“, wiederzuentdecken (siehe auch oben: Zach)

- **1802 - 1804:** Die Entdeckung weiterer Planetoiden: *Pallas, Juno, Vesta;* deren radialen Abstände von der Sonne alle zu $n = 3$ der Titius-Bode-Reihe passen. Daraus folgerte man, daß sie Bruchstücke eines großen Planeten zwischen Mars und Jupiter gewesen sein könnten. Diese Hypothese ist allerdings sehr umstritten; genauso kann es sein, daß die Gravitationswirkung des Riesenplaneten Jupiter die Akkretation eines weiteren Planeten an dieser Stell verhindert.

- **Entdeckung Neptuns:** *Ausgangspunkt:* Die berechneten Bahnparameter der Uranus, die 1790 von Delambre (1749 - 1822) nach dem Vorbild der Bahnparameterbestimmung nach Gauß durchgeführt wurden, wichen mit der Zeit immer mehr von den beobachteten Örtern von Uranus ab. Neben anderen Ursachen wurde die Beschreibung der ständig anwachsenden Abweichungen durch die Gravitation eines transuranischen Planeten immer populärer ⇔ „inverse“ Störungstheorie zur Bestimmung der Bahn und Masse des unbekannten Planeten.

- **Jean J. Leverrier (1811 - 1877); John C. Adams (1819 - 1892):** Sie widmeten sich der o.g. Aufgabe der „inversen“ Störungstheorie mit dem Erfolg, daß beide unabhängig recht ähnliche Bahnparameter eines hypothetischen transuranischen Planeten vorlegten, die leider lange Zeit von der Fachwelt ignoriert wurden.

- **Johann J. Galle (1812 - 1910):** Begann am 18. September 1846 mit der systematischen Suche nach dem unbekannten Planeten, nachdem Leverrier ihm in einem Brief von seinen theoretischen Ergebnissen informierte. Galle entdeckte dann Neptun am 23. September 1846 in der von Leverrier und Adams angegebenen Gegend des Firmaments.

- **Henri Poincaré (1854 - 1912):** Zeigte anhand des Dreikörperproblems, daß die den Erfolgen der klassischen Himmelsmechanik geschuldete Vorstellung, alle Bewegungen im Kosmos seien beliebig genau bestimmbar, wenn man nur die Anfangsbedingungen genau genug kenne, nicht haltbar ist. Die Unvorhersagbarkeit ist in der Nichtlinearität und Komplexität himmelsmechanischer Vielkörperprobleme begründet.

- **Albert Einstein (1879 - 1955):** Entwickelte die spezielle und allgemeine Relativitätstheorie. Vor allem letztere ist bedeutsam für die Himmelsmechanik (kompakte Objekte: Schwarze Löcher, Neutronensterne, weiße Zwerge ⇔ Schwarzschild- & Kerrmetrik, Friedmann-Universum
  $\Longrightarrow$ es existiert kein homogenes stabiles Weltall; Periheldrehung der Merkurbahn etc.)
  Seine weitern wissenschaftlichen Großtaten: Photoeffekt; Theorie der Brownschen Bewegung

2 Repetition / Basic Mechanics

2.1 Curved Phase-Space Coordinates

Almost everything in celestial mechanics is about the phase-space coordinates of point-masses m_i: e.g. for a single mass point this is the location vector $\vec{r}$ and its corresponding momentum $\vec{p} = m\vec{v}$ with the velocity vector $\vec{v} = \mathrm{d}\vec{r}/\mathrm{d}t$. In cartesian coordinates we have for 1 point mass $\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ and for N-bodies we have the 3N-tupels of locations $\{x_i\} = (x_1, x_2, ..., x_{3N}$ and corresponding (canonical) momenta $\{p_i\} = (p_1, ..., p_{3N})$.

Generally one may express cartesian coordinates x_i in a $3N$-dimenional configuration space by other parameters q_j

$$\begin{aligned} x_i &= x_i(q_1, q_2, \cdots, q_N) \qquad (1) \\ \text{in D=3} \Longrightarrow x &= x(q_1, \cdots, q_3) \\ y &= y(q_1, \cdots, q_3) \\ z &= z(q_1, \cdots, q_3) \end{aligned}$$

For the moment we restrict to the configuration space of 1 particle $\Rightarrow$ i.e. three-dimensional $3D$

$$\vec{r} = x_i(q_1, ..., q_3)\, \vec{e}_i \qquad (2)$$

using sum convention of Einstein. The index i stands for the directions x, y and z. The vectors of directional derivatives with respect to the change of the parameter q_j read

$$g_j\ \vec{e}_{q_j} = \frac{\partial \vec{r}}{\partial q_j} \qquad (3)$$

with the scaling factors

$$g_j = \left|\frac{\partial \vec{r}}{\partial q_j}\right| = \sqrt{\sum_i \left(\frac{\partial x_i}{\partial q_j}\right)^2} \quad . \qquad (4)$$

The corresponding unit-vectors in the direction of changing q_j are given by

$$\vec{e}_{q_j} = \frac{1}{g_j}\frac{\partial \vec{r}}{\partial q_j} \quad . \qquad (5)$$

Knowing these vectors one may define infinitesimal surface- and volume-elements in the configuration space: here the suface element (as a cross product)

$$\mathrm{d}A_k\, \vec{e}_{q_k} = \frac{\partial \vec{r}}{\partial q_i} \times \frac{\partial \vec{r}}{\partial q_j}\, \mathrm{d}q_i \mathrm{d}q_j = g_i\ g_j\, \mathrm{d}q_i \mathrm{d}q_j\, \vec{e}_{q_k} \qquad (6)$$

as well as the volume element given by the scalar-triple product:

$$\mathrm{d}V = \mathrm{d}^3\vec{r} = \frac{\partial \vec{r}}{\partial q_k} \cdot \left\{\frac{\partial \vec{r}}{\partial q_i} \times \frac{\partial \vec{r}}{\partial q_j}\right\} \mathrm{d}q_i \mathrm{d}q_j \mathrm{d}q_k = g_i\ g_j\ g_k \mathrm{d}q_i \mathrm{d}q_j \mathrm{d}q_k \quad . \qquad (7)$$

As a simple example the cylinder-coordinates (ρ, φ, z) are given, which read

$$\begin{aligned} x &= \rho \sin\varphi \\ y &= \rho \cos\varphi \\ z &= z \end{aligned}$$

so that the position vector becomes

$$\begin{aligned} \vec{r} &= \rho(\cos\varphi \vec{e_x} + \sin\varphi\, \vec{e_y}) + z\, \vec{e_z} \\ &= \rho\, \vec{e_r} + z\, \vec{e_z} \ . \end{aligned} \tag{8}$$

Now one can derive the scaling factors and the corresponding unit vectors

$$g_\rho = 1 \quad ; \quad g_\varphi = \rho \quad ; \quad g_z = 1 \tag{9}$$

$$\vec{e}_\rho = \cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_y \quad ; \quad \vec{e}_\varphi = -\sin\varphi\, \vec{e}_x + \cos\varphi\, \vec{e}_y \quad ; \quad \vec{e}_z = \vec{e}_z \ . \tag{10}$$

A useful property of the resulting unit-vectors which can be mapped in each other by simple differentiation (as also in many other cases of curved coordinates)

$$\vec{e}_\varphi = \frac{\partial \vec{e}_\rho}{\partial \varphi} \quad ; \quad \vec{e}_\rho = -\frac{\partial^2 \vec{e}_\rho}{\partial \varphi^2} = -\frac{\partial \vec{e}_\varphi}{\partial \varphi} \ . \tag{11}$$

With these vectors the surface- and volume coordinates can be constructed:

$$\mathrm{d}A_\rho = \rho\ \mathrm{d}\varphi \mathrm{d}z \tag{12}$$

$$\mathrm{d}A_z = \rho\ \mathrm{d}\varphi \mathrm{d}\rho \tag{13}$$

$$\mathrm{d}V = \rho\ \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}z \ . \tag{14}$$

Further the velocity of that point mass in cylindrical coordinates reads

$$\vec{v} = \frac{\mathrm{d}\vec{\mathrm{r}}}{\mathrm{dt}} = \dot{\varrho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z \tag{15}$$

which is to be shown in a **homework**.

2.2 Newton's & Principle Mechanics

Newton's Axioms:

1. a body, NOT exposed to forces, remains at rest or moves straight-even motion: its momentum is unchanged $0 = \mathrm{d}(m\vec{v})/\mathrm{d}t$ with its mass m and velocity $\vec{v} = \dot{\vec{r}}$ as the time derivative of the location. The momentum is defined as $\vec{P} = m\vec{v}$.

2. a force $\vec{F}$ acting on the body, results in a corresponding change of the momentum:

$$\dot{\vec{P}} = \frac{\mathrm{d}(m\dot{\vec{r}})}{\mathrm{d}t} = m\ddot{\vec{r}} = m\vec{a} \tag{16}$$

3. for any force (interaction) exists an equal (in modulus) counter-force which acts in the opposite direction: *actio* = *reactio*:

$$\vec{F}_{ij} \;=\; -\vec{F}_{ji} \quad . \tag{17}$$

Here the indices i und j stand for two interactig mass points!

Again our simple cylindric coordinate example (ρ, φ, z) might serve as illustration: The inertia forces $\propto\; m\ddot{\vec{r}}$ (here just for 1 particle) needed for Newton's principles are directly obtained by differentiation

$$m\ddot{\vec{r}} \;=\; \left(\ddot{\rho} \,-\, \rho\,\dot{\varphi}^2\right)\;\vec{e}_\rho \,+\, \frac{1}{\rho}\,\frac{\mathrm{d}}{\mathrm{dt}}\;\left(\rho^2\dot{\varphi}\right)\;\vec{e}_\varphi \,+\, \ddot{z}\,\vec{e}_z \;=\; \vec{F} \quad . \tag{18}$$

The first term on the right hand side – in the brackets – contains inertia- and centrifugal forces, the middle term expresses the angular momentum conservation ($\left|\vec{L}\right| = \rho^2\dot{\varphi}$) for forces (in direction of $\vec{r}$). Differentiating out that middle term one obtains the Coriolis-force $\propto 2\dot{\rho}\dot{\varphi}$, as well as angular accelerations $\ddot{\varphi}$.

For the celestial mechanics a plane motion ($z = 0$) in a central potential (forces) is of interest

$$\Phi(\rho) \;=\; -\,\frac{\mu}{\rho} \tag{19}$$

expressed in polar coordinates ($z = 0$). Here we can test the principle mechanics: Lagrange-Eq. 2. kind as an expression of the Hamilton principle. As a **homework** one should derive the relevant differential equations by applying all methods: Newton 2., Lagrange and the Hamiltonian equations.

In celestial mechanics *interaction* is usually mediated via gravitational forces/interactions:

$$\vec{F}_{ij} \;=\; -\gamma\frac{m_i m_j}{r_{ij}^2}\,\frac{\vec{r}_{ij}}{r_{ij}} \tag{20}$$

with the potential (energy)

$$\Phi(r_{ij}) \;=\; -\gamma\,\frac{m_i m_j}{r_{ij}}\;, \tag{21}$$

where the vector of the relative distance between either bodies is denoted by $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ with the modulus r_{ij}. The gravity constant γ (often also labelled as G).

The theory of the calculation of gravitational potentials $\Phi(\vec{r})$ of spatially extended *mass distributions* $\varrho(\vec{r})$ in the frame of general relativity including the Newtonian limiting case l – *"flat"* space and small speeds $|v|\,/c \ll 1$ – is given in Section 3.

2.2.1 Hamilton Principle:

Here we want also to repeat parts of the principle mechanics. First, the *principle of smallest action* – loosely speaking, the mechanical system looks for the realization path of "lowest resistance" in the phase-space. The *Hamilton's principle* can be derived from two other axiomatic principle – the *d'Lambert Principle* and *Newton 2*, which states that re-active forces are always vectorielly perpendicular to the motion of the mass – a knowledge deduced from experience (experiments etc.). In this context, these principles are of the same importance and meaning for the theoretical mechanics like the the Newton-axioms. Even more – because it also considers re-active forces as well – in this sense the Hamilton principle is more general than Newton 2.

In order to develop the principle, the Lagrange- function must be defined, which is a difference between kinetic- $\mathcal{T}$ and potential energy Φ:

$$\tilde{L}(q_j, \dot{q}_j, t) \;=\; \mathcal{T}(\dot{q}_j) \;-\; \Phi(q_j) \quad , \tag{22}$$

where the action is given by the intergral

$$S \;=\; \int\limits_{t_0}^{t_1} \mathrm{d}t \; \tilde{L}(q_j, \dot{q}_j, t) \quad . \tag{23}$$

The quantities q_j and p_j are generalized canonical phase-space coordinates – standing for positions and momenta of the N particles, velocities are characterized by $\dot{q}_j$.

With this prerequisites the *Hamilton-variational Principle* can be formulated

$$\delta\, S \;=\; 0 \quad , \tag{24}$$

under the condition that only the position coordinates q_j are varied according to $\delta q_j \neq 0$ – except for the end-points at instants t_0 and t_1, i.e. $\delta q_j(t_0) \;=\; \delta q_j(t_1) = 0$.

Performing mathematically the variation and integrating partially once one is lead to the Lagrange-equation of II. kind:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}_j} \;-\; \frac{\partial \tilde{L}}{\partial q_j} \;=\; 0 \quad . \tag{25}$$

For example, in case of polar (cylinder) coordinates (ρ, φ) he Lagrange-function $\tilde{L}(q_\nu, \dot{q}_\nu)$ is constructed with the aid of $\vec{v} \;=\; \dot{\varrho}\vec{e}_\rho \;+\; \rho\dot{\varphi}\vec{e}_\varphi$ (only the plane problem, $z = 0$)

$$\tilde{L} \;=\; \frac{m}{2}\,\vec{v}^2 + \frac{\mu}{\rho} \;=\; \frac{m}{2}\left(\dot{\rho}^2 \;+\; \rho^2\dot{\varphi}^2\right) \;+\; \frac{\mu}{\rho} \quad , \tag{26}$$

so that one can write down the components of the equations of motion by applying Eq. (25). The ρ-component of those eqs. of motion read:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\rho}) \;-\; \frac{\partial}{\partial \rho}\left(\frac{m}{2}\rho^2\varphi^2 \;+\; \frac{\mu}{\rho}\right) \;=\; 0 \tag{27}$$

so one directly obtains

$$m\left[\ddot{\rho} \;-\; \rho\dot{\varphi}^2\right] \;=\; -\frac{\mu}{\rho^2} \quad . \tag{28}$$

The φ-component direct offers the angular momentum conservation

$$m\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho^2\;\dot{\varphi}\right] \;=\; \frac{\mathrm{d}}{\mathrm{d}t}L \;=\; 0 \tag{29}$$

where one can directly conclude the conservation $L = \rho^2\dot{\varphi} = \text{konstant}$. These terms can also be identified in Newton 2 (18), if one writes for the central gravity force $\vec{F} = -(\mu/\rho^2)\vec{e}_\rho$.

Via a *Legendre-transformation* we change the independent variables from q_ν and $\dot{q}_\nu$ to the canonical phase-space variables q_ν and p_ν Schließlich seien in diesem Rahmen noch die Hamiltonschen Gleichungen erwähnt, deren Konzept für die Störungsrechnungen im Verlaufe der Vorlesungen wichtig werden. Mit der Legendre-Transformation

$$H(p_j, q_j, t) \;=\; \sum_k \; p_k\, \dot{q}_k \;-\; \tilde{L}(q_j, \dot{q}_j, t) \quad , \tag{30}$$

wobei die kanonischen Impulse als $p_j = \partial L/\partial \dot{q}_j$ definiert sind, gewinnt man durch Bildung des totalen Differenzials unter Verwendung von Gl. (25) und mittels Koeffizientenvergleich die Krone der theoretischen Mechanik – die Hamiltonschen Gleichungen:

$$\dot{q}_j \;=\; \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j \;=\; -\frac{\partial H}{\partial q_j} \quad . \tag{31}$$

2.2.2 Conservative N-particle systems ⇔ Newton

Given is a system of $i \in (1, N)$ particles which is exposed to outer forces ($\vec{F}^{(a)}$) - as well as conservative particle interactions ($\vec{F}_{ij}$) act between the particles. The N body-system is completely characterized by the $3N$ location q_ν and $3N$ momentum coordinates p_ν.

Using the *2.* Newton axiom the equations of motion read

$$m_i\, \ddot{\vec{r}}_i \;=\; \vec{F}^{(a)}_i + \sum_{j;j\neq i} \vec{F}_{ij} \quad . \tag{32}$$

In the following we will use the convention: greek indices, $\alpha \in (1, 3N)$ always denote the single phase space coordinates, while arabic indices $(i, j, ...)$ denote values assigning certain particles i. For instance the location coordinates of particle k read $\vec{r}_k(q_\alpha, ...) = \vec{r}_k(q_{3(k-1)+1}, ..., q_{3(k-1)+3})$. Coordinates and metrics as well as their transformations follow in Section 2.1.

Conservation of Momentum, Angular Momentum and Energy:

With the *Second Newtonian Axiom* (16) & (32) and the *Third Axiom*

$$\vec{F}_{ij} \;=\; -\vec{F}_{ji} \tag{33}$$

one can directly formulate conservation laws resp. integrals of motion. Summing Eq. (32) over all particles one obtains

$$\sum_i m_i\, \ddot{\vec{r}}_i \;=\; \sum_i \vec{F}^{(a)}_i + \underbrace{\sum_{ij;j\neq i} \vec{F}_{ij}}_{\equiv\, 0} \quad , \tag{34}$$

where the vanishing of the last term is guaranteed by the axiom *"actio = re-actio"*, so that one can state

$$\begin{aligned} Sum &= \sum_{ij;j\neq i} \vec{F}_{ij} = \sum_{ji;j\neq i} \vec{F}_{ji} \\ &= \frac{1}{2} \sum_{ij;j\neq i} \left(\vec{F}_{ij} + \underbrace{\vec{F}_{ji}}_{=-\vec{F}_{ij}} \right) \equiv 0 \end{aligned}$$

so that Eq. (34) simplifies to

$$\sum_i m_i \ddot{\vec{r}}_i = \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_i m_i \dot{\vec{r}}_i \right] = \dot{\vec{P}} = \sum_i \vec{F}_i^{(a)} = \vec{F}_A \quad . \tag{35}$$

It becomes directly evident, that the *total momentum* of the system $\vec{P} = \sum m_i \dot{\vec{r}}_i$ is determined by the sum of external forces $\vec{F}_A = \sum \vec{F}_i^{(a)}$. If no external forces act on the system $\vec{F}_A = 0$, the *total momentum* becomes an integral of motion

$$\dot{\vec{P}} = 0 \quad \text{, and thus,} \quad \vec{P} = \text{konstant} \quad . \tag{36}$$

It has to be noted that the above case only hold, if the masses do NOT change during the motion, i.e. $\dot{m}_i = 0$ and with it $\dot{M} = 0$. If the masses are changing (rocket equations, Ziolkowsky) the problem needs an extra consideration.

A further conserved value – integral of motion – constitutes the center of mass

$$\vec{r}_0 = \frac{\sum\limits_i m_i \vec{r}_i}{\sum\limits_i m_i} \tag{37}$$

where the second derivative of Eq. (37)

$$M \ddot{\vec{r}}_0 = \vec{F}_A \tag{38}$$

(**to be calculated in a homework**) leads to the conservation of the *center of mass* $\dot{\vec{r}}_0 = constant$ for the case $\vec{F}_A = 0$. **Law of Center of Mass:** *for $\vec{F}_A = 0$ it moves on straight-even trajectory – the total mass $M = \sum m_i$, concentrated in $\vec{r}_0$, travels according the* **First Newton Axiom**.

In order to derive the next integral of motion – the *conservation of angular momentum* – we vectorially multiply Eq. (32) with $\vec{r}_i \times$ and sum again over all particles i so that we obtain:

$$\sum_i m_i \vec{r}_i \times \ddot{\vec{r}}_i = \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i \right] = \dot{\vec{L}} = \vec{M} + \underbrace{\sum_{ij;j\neq i} \vec{r}_i \times \vec{F}_{ij}}_{\equiv 0} \quad , \tag{39}$$

so that changes of the total angular momentum $\vec{L} = \sum \vec{r}_i \times \dot{\vec{r}}_i$ are dominated by the torques of outer forces $\vec{M} = \sum \vec{r}_i \times \vec{F}_i^{(a)}$. Again, the absence of outer forces $\vec{F}_A = 0$ guarantees the *conservation of Angular Momentum* (**homework**)

$$\dot{\vec{L}} = 0 \quad \text{und damit} \quad \vec{L} = \text{konstant} \quad . \tag{40}$$

Up to now we have 9 intergrals of motion for closed systems: **Center of mass, Momentum & Angular Momentum**

Another integral of motion can be found in the case of conservative forces: $\vec{F}_i(\vec{r}_i) = -\nabla_i \Phi(\vec{r}_1, ..., \vec{r}_N)$ with the potential Φ – in this case the energy is conserved too, so that we have summa summarum 10 *integrals of motion*. Such potentials can also be formulated for internal forces, for instance in the case of central internal forces: $\vec{F}_{ij}(|\vec{r}_i - \vec{r}_j|)$, so that oen can write

$$\Phi_i \;=\; \Phi_i^{(a)} + \sum_{j \neq i} \Phi_{ij} \quad , \tag{41}$$

so that the total potential constitutes the total potential energy of the system and can be written

$$\Phi \;=\; \sum_i \Phi_i^{(a)} + \frac{1}{2} \sum_{ij;j\neq i} \Phi_{ij} \;=\; \Phi^{(a)} \,+\, \Phi^{(i)} \; . \tag{42}$$

In case of the interactions – for Celestial Mechanics usually gravity – the forces have to follow the property

$$\vec{F}_{ij} = \left| \vec{F}_{ij}(|\vec{r}|) \right| \, \frac{\vec{r}_{ij}}{|\vec{r}_{ij}|} \quad . \tag{43}$$

The expansion into center-of-mass- and relativ coordinates will be of further advantage in the some reductions of the description of the system.

Spatial differentiation of Eq. (42) , i.e. deriving the corresponding forces (here only the interaction contribution, outer forces are simply trivial), yields:

$$\begin{aligned} -\frac{\partial}{\partial \vec{r}_k} \Phi^{(i)} \;&=\; -\frac{\partial}{\partial \vec{r}_k} \frac{1}{2} \sum_{ij;j\neq i} \Phi_{ij} \;=\; -\,\frac{1}{2} \left[\sum_j \frac{\partial \Phi_{kj}}{\partial \vec{r}_k} + \sum_i \frac{\partial \Phi_{ik}}{\partial \vec{r}_k} \right] \;= \\ &=\; -\,\frac{1}{2} \sum_l \left(\frac{\partial \Phi_{kl}}{\partial \vec{r}_k} + \frac{\partial \Phi_{lk}}{\partial \vec{r}_k} \right) \;=\; -\sum_l \frac{\partial \Phi_{lk}}{\partial \vec{r}_k} \quad . \end{aligned} \tag{44}$$

Here we have used that $\Phi_{lk} = \Phi_{kl}$, because potentials only depend on absolute values of interparticle distances. The right-hand-side is nothing but the sum of the forces which act on the particle k excerted by all other particles l. Newton 3, *actio = reactio*, is also contained having in mind that $\vec{r}_i - \vec{r}_j = -(\vec{r}_j - \vec{r}_i)$ holds and that internal derivative (chain-rule) of $\Phi_{ij}(|\vec{r}_i - \vec{r}_j|)$ causes a exchange of the sign when interchanging particle indices l with k (**This can easily be checked by directly differentiating potential of 2 or 3 particles – homework**).

In order to show the *energy conservation* in case of conservative forces we have to scalarly multiply Eq. (32) with $\dot{\vec{r}}_i \cdot$, sum over all particles and obtain

$$\sum_i m_i \, \dot{\vec{r}}_i \cdot \ddot{\vec{r}}_i \;=\; \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\left\{ \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} \right\}}_{\equiv\, T} \;=\; -\dot{\vec{r}}_i \cdot \frac{\partial}{\partial \vec{r}_i} \Phi \;=\; -\,\frac{\mathrm{d}}{\mathrm{d}t} \Phi \tag{45}$$

$$\mathcal{T} \,+\, \Phi \;=\; E \quad , \tag{46}$$

where E is the energy of the system. With this we have 10 integrals of motion of conservative celestial particle systems.

2.2.3 Kurzabriß ART/Relativistik/Metrik

Um gleich den Weg zu ebnen für eventuelle Abstecher in die *allg.* Relativitätstheorie – Himmelsmechanik für „Privilegierte" – definieren wir zunächst Abstände (noch 3D)

$$\mathrm{d}\vec{r}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{47}$$
$$\mathrm{d}\vec{r}^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2 \tag{48}$$
$$\mathrm{d}\vec{r}^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2 \tag{49}$$

zunächst für kartesische -, Zylinder- und Kugelkoordinaten.

Die *metrische Fundamentalform* kann wie folgt aufgeschrieben werden (*Einstein Summen* in diesem Kapitel)

$$\mathrm{d}s^2 = g_{\alpha\beta}(x^\nu)\mathrm{d}x^\alpha\mathrm{d}x^\beta \tag{50}$$

mit der für die allgemeine Relativitätstheorie grundlegenden Größe – der *Metrik* oder auch dem symmetrischen *metrischen Tensor* $g_{\alpha\beta} = g_{\beta\alpha}$. Der Name deutet schon darauf, dass man damit grundlegende Abstände und Winkel messen kann. Hat der Tensor nur diagonale Komponenten $g_{\alpha\beta} \propto \delta_{\alpha\beta}$ nennt man die Metrik orthogonal. So die Determinante nicht verschwindet, existiert mit $g^{\mu\nu}$ ein inverser Tensor, d.h. es gilt $g_{\alpha\beta}g^{\beta\nu} = g^\nu_\alpha = \delta^\nu_\alpha \Rightarrow$ Einheitsmatrix bzw. Einheitstensor (δ - Kronecker Symbol). Für die obigen Ausdrcke (47)-(49) im Ortsraum (3D) liefert die Verjüngung des Tensors/Spur $g_\alpha{}^\nu \rightarrow g^a{}_a = 3$; für die vierdimensionale Raumzeit ergibt sich $g^a{}_a = 4$ (siehe unten).

Der metrische Tensor $g_{\alpha\beta}$, die *Metrik,* wird – erweitert um die 4. Dimension ct im *Riemann'schen Raum* – die entscheidende Größe bei der *Allgemeinen Relativitätstheorie*, der *Einsteinschen Graviationstheorie* spielen. Bevor wir uns der Physik zuwenden, möchte ich einige Bemerkungen zum Rechnen mit Vierervektoren machen.

Die quadratische Form (50) im 4-dimensionalen Riemann'schen Raum enthält die Zeitkomponente $c\mathrm{d}t$, so dass für den „einfachen" des flachen (Minkowski) Raums wird

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\mathrm{d}t^2 \quad . \tag{51}$$

Die flache Metrik $g_{\alpha\beta} \approx \eta_{\alpha\beta}$ lautet in Matrix-Form

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \tag{52}$$

im Allgemeinen gilt Gl. (50), wobei der Index 4 die zusätzlichen zeitlichen Komponenten bezeichnet.

In Gl. (52) fällt auf, dass die zeitliche Komponente hier eine negatives Vorzeichen[2] zeigt. Hier wollen wir nur einige Rechenregeln für Vierervektoren andeuten. Es seien $\vec{A}^{(4)} = A_i$ und $\vec{B}^{(4)} = B_i$ Vierervektoren — z.B. Ortsvektor $x_i = (\vec{r}, -ct)$, Vierergeschwindigkeit $u_i = \mathrm{d}x_i/\mathrm{d}\tau = \gamma(\vec{v}, -c)$, bzw. $x^i = (\vec{r}, ct)$ und $u^i = \mathrm{d}x^i/\mathrm{d}\tau = \gamma(\vec{v}, c)$ mit Eigenzeit-Differenzial $\mathrm{d}\tau = \mathrm{d}t'/\gamma$ und $\gamma = (1 - v^2/c^2)^{-1/2}$ — wobei hier schon die

$$\text{kovarianten} \;\; A_i \;=\; (\vec{A}, -A_4) \tag{53}$$

und

$$\text{kontravarianten} \;\; A^i \;=\; (\vec{A}, A_4) \tag{54}$$

Darstellungen angegeben sind. Mit beiden werden Beträge $A^2 = A_i A^i$ (Einstein-Summen) und Skalarprodukte gebildet $\vec{A}^{(4)} \cdot \vec{B}^{(4)} = A_j B^j$ oder Tensoren verjüngt $T^{\alpha a}_{\ \ a\beta} = T^{\alpha}_{\ \beta}$.

Mittels der Metrik $g_{\alpha\beta}$ kann man kovariante und kontravariante Darstellungen von Tensoren, Vektoren ineinander überführen: $A_i = g_{i\alpha} A^\alpha$; bzw. $A^k = g^{kj} A_j$ und Tensoren z.B. $T_i^j = g_{il} T^{lj}$. Bei Hebungen bzw. Senkungen von Indizes ist zu beachten, dass ein Vorzeichenwechsel beim zeitlichen Index (4) stattfindet (bei anderen alternativen Darstellungen/Konventionen betrifft das die räumlichen Komponenten, Indizes 1, 2, 3). Daraus folgt sofort, dass die Verjüngung (Spur) des (orthogonalen) metrischen Tensors $g_{\alpha\beta} \Rightarrow g_a^a = 4$ ist.

Zurück zur Gravitationstheorie: Die Masseverteilungen werden die Krümmungen des Raumes und auch die Bewegungen der Massenpunkte determinieren. In der Nichtrelativistischen Theorie halten wir an der Unveränderbarkeit der *Raum-Zeit* fest – und eingezwängt in dieses „Korsett“ bemühen wir die *Poisson-Gleichung* für das Gravitationspotenzial $\Phi(\vec{r})$, welches ebenfalls durch die Massenverteilung bestimmt ist, und dessen Gradienten $-\nabla\Phi$ dann in die Bewegungsgleichungen eingehen. Die *Allgemeine Relativitätstheorie* reduziert dieses Problem der Bewegung im Gravitationsfeldern auf eine reine geometrische Aufgabe. Dazu am Ende des Kapitels mehr.

Hier wollen wir zunächst nochmal auf die *Lagrange-Gleichung II.* in der Metrik-Formulierung (zunächst im Ortsraum) kommen – mit dem Ergebnis, dass wir sehen werden: Massenpunkte bewegen sich auf *Geodäten* – kürzesten Strecken in der Raum-Zeit bei gegebener Metrik (i.d. 4D Raumzeit). Mit der Definition des infinitesimalen Abstandsquadrats können wir sofort einen Zusammenhang zur Dynamik und Mechanik herstellen:

$$v^2 \;=\; \left(\frac{\mathrm{ds}}{\mathrm{dt}}\right)^2 \;=\; g_{\alpha\beta}\frac{\mathrm{dx}^\alpha}{\mathrm{dt}}\frac{\mathrm{dx}^\beta}{\mathrm{dt}} \;=\; g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \tag{55}$$

und so die Lagrange-Fkt. für zunächst ein freies Teilchen aufschreiben:

$$\tilde{L} \;=\; \frac{m}{2}\, g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \tag{56}$$

womit wir in die Lagrange-Gl. II gehen. Wir verabreden folgende Schreibweise für die partielle Ableitung[3] einer Funktion

$$\frac{\partial F}{\partial x^\nu} \;=\; F_{,\nu}$$

[2] es gibt Darstellungen, mit Vorzeichenwechsel in den räumlichen Komponenten

[3] Auf weitere Rechenregeln wie partielle- und kovariante Ableitungen verweisen wir auf tiefer gehende Literatur

womit wir unter Beachtung der Einsteinsummen und Produktregel erhalten:

$$\frac{\partial \tilde{L}}{\partial \dot{x}^\nu} \;=\; \underbrace{m\, g_{\alpha\nu} \dot{x}^\alpha}_{Summe+Produktregel} \quad ; \quad \frac{\partial \tilde{L}}{\partial x^\nu} \;=\; \tilde{L}_{,\nu} \;=\; \frac{m}{2} g_{\alpha\beta,\nu}\, \dot{x}^\alpha \dot{x}^\beta \quad . \tag{57}$$

Das frisch eingesetzt in Lagrange II und auch Kettenregel beachtet (ACHTUNG: Abhängigkeit: $g_{\alpha\beta}(x^\mu)$) erhält man

$$g_{\alpha\nu}\, \ddot{x}^\alpha \;+\; g_{\alpha\nu,\beta}\, \dot{x}^\alpha \dot{x}^\beta \;-\; \frac{1}{2} g_{\alpha\beta,\nu}\, \dot{x}^\alpha \dot{x}^\beta \;=\; 0 \quad .$$

Kettenregel beachten, Abhängigkeiten bei den Ableitungen etc. Nun nutzen wir die Identität aus (!Einstein Summen!):

$$g_{\alpha\nu,\beta}\, \dot{x}^\alpha \dot{x}^\beta \;=\; g_{\beta\nu,\alpha}\, \dot{x}^\alpha \dot{x}^\beta$$

so dass man schreiben darf

$$g_{\alpha\nu,\beta}\, \dot{x}^\alpha \dot{x}^\beta \;=\; \frac{1}{2} \left\{ g_{\alpha\nu,\beta}\, \dot{x}^\alpha \dot{x}^\beta \;+\; g_{\beta\nu,\alpha}\, \dot{x}^\alpha \dot{x}^\beta \right\}$$

und man erhält schließlich die Bewegungsgleichung eines Massepunktes – hier noch im Ortsraum,

$$\ddot{x}^\mu \;+\; \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \;=\; 0 \quad , \tag{58}$$

mit den *Christoffel*-Symbolen

$$\Gamma^\mu_{\alpha\beta} \;=\; \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right) \quad . \tag{59}$$

Zu gleichen Schlussfolgerungen gelangt man in der 4D-Raum-Zeit, wenn man fordert, dass sich Massepunkte auf *Geodäten* in der 4D-Raum-Zeit bewegen müssen

$$S \;=\; \int_a^b \mathrm{d}s \;=\; \text{Minimum} \quad , \tag{60}$$

wobei man unter Beachtung von Gl. (51) und Multiplikation mit der (quasi-klassischen) Konstanten mc erhält

$$mc\, ds = \left\{ -mc^2 \cdot \sqrt{1 - \frac{v^2}{c^2}} \right\} \mathrm{d}t \tag{61}$$

womit man den Ausdruck in der geschweiften Klammer als Lagrange-Funktion $\tilde{L}$ identifiziert (im Fall $|v/c| \ll 1$ ergibt die 1. Ordnung der Taylor-Entwicklung den bekannten Newton'schen Ausdruck - abgesehen von der Ruheenergie mc^2).

Die Variationsaufgabe auf eine der Gl. (58) äquivalente Form führt

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} \;+\; \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \;=\; 0, \tag{62}$$

wobei λ ein freier Kurvenparameter ist, der für freie Teilchen proportional der Zeit τ aber auch der Wegstrecke s ist (oft findet man statt $\mathrm{d}\lambda$ auch $\mathrm{d}\tau$ oder $\mathrm{d}s$). Mit den Gln. (58) & (62) haben wir Geodäten als Bahnkurven gefunden, die auch noch bei Anwesenheit von Massen/Gravitation die Trajektorien darstellen - da die Massen (Energie) Verteilung $\varrho(\vec{r})$ die Quellen für Modifikationen (Krümmungen) der Metrik $g_{\alpha\beta}$ darstellen.

Einen ersten Vorgeschmack auf den Einfluss der Gravitation kann man anhand der Gleichungen (61)-(62) aufzeigen. Dazu erweitern wir die genäherte Lagrange-Funktion (Taylor-Entwicklung der geschweiften Klammer in Gl. (61)) wie folgt:

$$\tilde{L} \approx -mc\left[-c + \frac{1}{2}\frac{v^2}{c} - \frac{\Phi}{c}\right] \tag{63}$$

um das Newton'sche Potenzial, so das wir für die infinitesimale Viererwegstrecke auch schreiben können

$$\mathrm{d}s = \left(-c + \frac{1}{c}\frac{v^2}{2} - \frac{\phi}{c}\right)\mathrm{d}t \quad . \tag{64}$$

Quadrieren wir diesen Ausdruck, gewinnen wir

$$\begin{aligned} \mathrm{d}s^2 &= \left\{-c^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - 2\Phi + O(c^{-2})\right\}\mathrm{d}t^2 \\ &= \mathrm{d}r^2 - \left[1 + \frac{2\Phi}{c^2}\right]c^2\mathrm{d}t^2 \ , \end{aligned} \tag{65}$$

wobei wir Terme der Ordnung $O(c^n)$, $\forall\, n \geq 2$ vernachlässigt haben. Somit können wir festhalten, dass wir in der linearen Näherung – d.h. geringe Gravitation und daraus folgende geringe Abweichungen von der flachen Metrik $\eta_{\alpha\beta}$ – schreiben können:

$$g_{44} = -\left\{1 + \frac{2\Phi}{c^2}\right\} \quad . \tag{66}$$

Als nächstes werden wir die wichtigsten Annahmen, die zur Entwicklung der Einstein-Gleichung der *Allgemeinen Relativitätstheorie/Gravitationstheorie* führten, skizzieren:

1. *Allgemeines Relativitätsprinzip:* $\Rightarrow$ alle Bezugsysteme (nicht nur geradlinig gleichförmig bewegte wie bei der *Speziellen RT*) sind physikalisch gleichberechtigt – man kann es auch kurz umreißen mit der Kurzformel: *Schwere Masse* $=$ *Träge Masse.*

2. Die Quellen der gravitativen Wirkungen ist die *Verteilung der Masse/Energie* $\Rightarrow$ charakterisiert durch den *Energie-Impuls Tensor* $\hat{T}(\varrho)$

3. Die Quellen bewirken eine Krümmung der Raum-Zeit, d.h. Modifikation der Metrik $g_{\alpha\beta}$. Das *allgemeine Relativitätsprinzip* (siehe Punkt 1.) gebietet, dass die Gleichungen der *Allgemeneinen Relativitätstheorie* Tensorgleichungen (2.ter Stufe in der Raum-Zeit) sein müssen. Wie alle Feldgleichungen in der Physik sollten es partielle Differentialgleichungen bis maximals Ableitungen 2. Ordnung sein.

4. Bei genügend schwachen Gravitationsfeldern (dazu zählen die von Planeten, „normalen“ Sternen)[4] sollen die allgemeinen Feldgleichungen in die Newton'sche Poisson-Gleichung übergehen:

$$\frac{\partial^2 \Phi}{\partial \vec{r}^2} \;=\; 4\pi\gamma\varrho(\vec{r}) \quad .$$

Mit diesen physikalischen Requisiten können wir nun die Einstein'schen Gleichungen formulieren

$$R_{\alpha\beta} \,-\, \frac{1}{2}\, R\, g_{\alpha\beta} \;=\; C_E\, T_{\alpha\beta} \tag{67}$$

wobei der Ricci-Tensor durch Verjüngung $R^a_{\alpha a\beta}$ (Spurbildung) des Krümmungstensors

$$R^a_{\alpha b\beta} \;=\; \Gamma^a_{\alpha\beta,b} \,-\, \Gamma^a_{\alpha b,\beta} \,+\, \Gamma^a_{mb}\Gamma^m_{\alpha\beta} \,-\, \Gamma^a_{m\beta}\Gamma^m_{\alpha b} \tag{68}$$

entsteht – die Christoffelsymbole $\Gamma^i_{\;jk}$ sind mit Gl. (59) definiert.

Der Tensor $R_{\alpha\beta}$ charakterisiert die Krümmung der Raum-Zeit, ist *nichtlinear* in der Metrik g_{ij} und enthält deren 2. Ableitungen, wie oben gefordert. Seitens der Differenzialgeometrie wäre noch viel mehr über dessen Eigenschaften zu sagen – vor allem die Gleichungen für konkrete Probleme formulieren bzw. lösen möchte. Für unsere knappe Skizze der Physik der *Allgemeinen Relativitätstheorie* soll das Obige zunächst genügen.

Newton'scher Grenzfall

Hier ist das Ziel, die 1. Näherung der Metrik $g_{\alpha\beta}$ als Abweichung vom flachen Raum $\eta : \alpha\beta$, verursacht durch ein schwaches Gravitationsfeld, über den Ansatz

$$g_{mn} \;=\; \eta_{mn} \,+\, \delta_{mn} \tag{69}$$

zu formulieren. Bei Abwesenheit von e.-m. Feldern und gegebener Masseverteilung $\varrho(\vec{r})$ und bei nicht-relativistischen Geschwindigkeiten $|v/c| \ll 1$ enthält der Energie-Impuls Tensor – in den sonst neben der Energie-Massen-Dichte auch noch die Flüsse von Energie (z. B. Poynting-Vektor $\vec{S}$) u. Impuls (Spannungstensoren $\hat{\sigma}$) eingehen – in dem Fall nur

$$T_{44} = c^2 \varrho(\vec{r})$$

(die restlichen Flußkomponenten sind von der Ordnung v/c und damit vernachlässigbar).

Unter Beachtung dieser Annahmen bilden wir die Spur der Gl. (67) – zur Vereinfachung der Feldgleichungen – und erhalten zunächst für den Krümmungsskalar

$$-R \;=\; C_E\, T^a_a \;=\; C_E\, T \;=\; C_E\, c^2 \varrho(\vec{r}) \tag{70}$$

und den ersetzt in Gl. (67) gibt

$$R_{mn} \;=\; C_E \left(T_{mn} \,-\, \frac{1}{2}\eta_{mn} T \right) \quad . \tag{71}$$

[4] Allgemein-relativistische Effekte kommen nur bei kompakten Objekten – schwarze Löcher, Neutronensternen, weiße Zwerge – zum tragen.

Nun ist für unsere linearisierte Version der Einstein-Gleichungen in Newton-Näherung von den 10 Gleichungen nur eine von Interesse (**! Übung:** η_{44} =? und T^a_a = ?)

$$R_{44} \;=\; C_E\left(T_{44} \;-\; \frac{1}{2}\eta_{44}T\right) \;=\; \frac{C_E}{2}\,c^2\,\varrho(\vec{r}) \quad . \tag{72}$$

Um die Komponente des Ricci-Tensors zu bestimmen, greifen wir auf die Definition des Krümmungstensors (68) zurück, vernachlässigen quadratische Terme in den Christoffel-Symbolen – und verjüngen das erhaltene. Zudem ist zu beachten, dass Ableitungen nach x_4 den Faktor c^{-1} liefern und somit vernachlässigbar werden, womit man folgende Vereinfachungskette gewinnt:

$$\begin{aligned} R^a_{\alpha b\beta} \;&=\; \Gamma^a_{\alpha\beta,b} \;-\; \Gamma^a_{\alpha b,\beta} = \frac{1}{2}\,\eta^{as}\left\{\delta_{s\beta,\alpha b} + \delta_{\alpha b,s\beta} - \delta_{\alpha\beta,bs} - \delta_{bs,\alpha\beta}\right\} \\ R_{44} \;=\; R^a_{4a4} \;&=\; \frac{1}{2}\eta^{as}\left[\delta_{s4,4a} + \delta_{4a,s4} - \delta_{44,as} - \delta_{as,44}\right] \;=\; -\frac{1}{2}\eta^{as}\delta_{44,as} \; . \end{aligned}$$

Da wir die Zeitableitungen (Index 4) in den Termen auf der rechten Seite vernachlässigen können – sprich die Indizes a und s nur über räumliche zu erstrecken sind, folgt unmittelbar

$$R_{44} \;=\; -\frac{1}{2}\Delta\delta_{44} = \frac{C_E}{2}\,c^2\,\varrho(\vec{r}) \quad . \tag{73}$$

Und es kommt noch besser, wenn man Gl. (66) und die Linearisierung (69) zu Grunde legt, wird man direkt auf die Poisson-Gleichung (89) geführt, d.h. es gilt:

$$\Phi \;=\; -\frac{c^2}{2}\delta_{44} \quad . \tag{74}$$

Eine direkte Bestätigung unsere Vermutung (66) bringt die Analyse der Geodätengleichung (62) (prima **Übung**) – d.h. o.g. Näherungen angewandt auf die Geodätengleichung – die

$$\ddot{\vec{r}} \;=\; -\,\nabla\Phi \tag{75}$$

Newtonschen Bewegungsgleichung.

2.2.4 Rotating Frames

The Eqs. of motion (27)-(29) can alternatively derived by quite an elegant method:

First of all, a rotation about the z-axis with constant distance $\rho = const.$ can formally be described with time-derivative ($n = \dot{\varphi}$) of the location vector $\vec{r} = \rho\vec{e}_\rho$:

$$\begin{aligned} \left.\frac{\partial\vec{r}}{\partial t}\right|_{Rotation} \;&=\; \rho\,\frac{d}{dt}(\cos\varphi\vec{e_x} + \sin\varphi\vec{e_y}) \\ &=\; \rho\,\dot{\varphi}(-\sin\varphi\vec{e_x} + \cos\varphi\vec{e_y}) \\ &=\; \rho\,n\vec{e_\varphi}\;, \end{aligned}$$

which can also be expressed as a vector-product:

$$\begin{aligned}
\vec{n} \times \vec{r} &= \begin{vmatrix} \vec{e_r} & \vec{e_\varphi} & \vec{e_z} \\ 0 & 0 & n \\ \rho & 0 & 0 \end{vmatrix} \\
&= \vec{e_\varphi} \rho n \quad .
\end{aligned}$$

The comparison directly gives $\frac{d\vec{r}}{dt} = \vec{n} \times \vec{r} = n\rho\vec{e_\varphi}$ – allows one for changes of the modulus of the vector (in the above example for ρ), one has to write

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}}\Big|_{CM} + \vec{n} \times \vec{r} \quad , \tag{76}$$

where the index (CM) indicates the derivative in the co-moving frame.

Generally, for a arbitrary vector in a accelating frame, which rotates with the angular velocity $\vec{n}$:

$$\boxed{\frac{d}{dt}\vec{A} = \dot{\vec{A}}\Big|_{CM} + \vec{n} \times \vec{A}}$$

so that one obtains for the spatial accelaration:

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= \dot{\vec{r}} + \vec{n} \times \vec{r} \\
\Longrightarrow \frac{d^2\vec{r}}{dt^2} &= \left(\frac{d}{dt}\Big|_{CM} + \vec{n}\times \right) \left(\dot{\vec{r}} + \vec{n} \times \vec{r} \right) \\
&= \ddot{\vec{r}} + 2\vec{n} \times \dot{\vec{r}} + \vec{n} \times (\vec{n} \times \vec{r}) + \dot{\vec{n}} \times \vec{r}
\end{aligned}$$

Coriolis-Accelaration: $2\vec{n} \times \dot{\vec{r}}$
Centrifugal-Accelaration: $\vec{n} \times (\vec{n} \times \vec{r})$
Angular Accelaration: $\dot{\vec{n}} \times \vec{r}$
Especially one obtains $\vec{n} \perp \vec{r}$, and $\vec{n} = const.$:

$$\begin{aligned}
\frac{d^2\vec{r}}{dt^2} &= \ddot{\vec{r}} + 2\vec{n} \times \dot{\vec{r}} - \vec{r}n^2 \\
m\ddot{\vec{r}_i} &= \sum_a \vec{F_a} + \sum_j \vec{F_{ij}} + \vec{F_I} \\
\vec{F} &= -2\vec{n} \times \dot{\vec{r}} + \vec{r}n^2
\end{aligned}$$

For a test-particle in a point mass gravity-field and for a circular motion $\dot{r} = 0$, i.e. a Keplerian circular orbit, one obtains:

$$\Longrightarrow -n^2\vec{r} = \sum \vec{F} = -\frac{\mu}{r^3}\vec{r}$$

A fixed point mass in a rotating coordinate system experiences a centrifugal forces so that for the Kepler-circle mean motion follows: $n^2 = \frac{\mu}{r^3}$, with $\mu = \gamma M_{ZK}$, where M_{ZK} is the mass of the central body, and γ denotes gravity constant.

2.3 Coordinate Transforms

In this Section we will favor the matrix formulation as an abbreviation. Given are two coordinate systems, characterized by their unit-vectors $\vec{e}_1, ..., \vec{e}_3$ and $\vec{e}_1{}', ..., \vec{e}_3{}'$ as bases – or writing the bases as abbreviation $\{\vec{e}_i\}$ or $\{\vec{e}_i{}'\}$, respectively. The latter are connected by a (unitary) transformation, characterized by a 3x3 matrix/tensor $\hat{\mathcal{U}}$

$$\begin{pmatrix} \vec{e}_1{}' \\ \vec{e}_2{}' \\ \vec{e}_3{}' \end{pmatrix} = \hat{\mathcal{U}} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} = \left\{ \begin{matrix} u_{11} \cdots u_{13} \\ \ddots \\ u_{31} \cdots u_{33} \end{matrix} \right\} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} \tag{77}$$

with the unitary (orthogonal) transformation matrix

$$\hat{\mathcal{U}} = \left[\hat{\mathcal{U}}^{-1}\right]^T \quad ; \quad \hat{\mathcal{U}}^{-1} \cdot \hat{\mathcal{U}} \ = \ \hat{I} \ ; \ \det \hat{\mathcal{U}} \ = \ 1 \quad .$$

Vectors, for instance $\vec{A}$, are represented as matrices like $\vec{A} \to A^T = (A_1, A_2, A_3)$ related to the base frame $\vec{e}_i$ – analoguously we will write for $\vec{A}\,' \to (A\,')^T = (A_1{}', A_2{}', A_3{}')$. The same reads (using Einstein's convention):

$$\vec{A}\,' \ = \ \underbrace{\sum_i A_i{}'\, \vec{e}_i{}'}_{(A_i{}')^T\{\vec{e}_i{}'\}} \ = \ (A\,')^T\, \hat{\mathcal{U}}\, \{\vec{e}_i\} \ = \ A^T \underbrace{\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}}_{\{\vec{e}_i\}} \ = \ \sum_j A_j\, \vec{e}_j \ = \ \vec{A}$$

$$(A\,')^T\ \hat{\mathcal{U}}\ \{\vec{e}_i\} \ = \ A^T\ \{\vec{e}_i\}$$

so that a comparison of coefficients directly gives:

$$A^T \ = \ (A\,')^T\, \hat{\mathcal{U}} \ , \ \ A^T \ = \hat{\mathcal{U}}^T\ A\,' \tag{78}$$

or respectively

$$A\,' \ = \ \left(\hat{\mathcal{U}}^{-1}\right)^T\ A^T \ = \hat{\mathcal{U}}\ A^T\ . \tag{79}$$

Fazit: Bases of a vector transform covariant, whereas coordinates transform contravariant with respect to their bases.
Again we use polar-coordinates to demonstrate an example of a rotation of the co-ordinate system around the z-axis, i.e. a rotation by the angle φ:

$$\hat{\mathcal{U}} \ = \ \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{80}$$

It is simple to show that the relation

$$\hat{\mathcal{U}}^T\, \hat{\mathcal{U}} \ = \ \hat{I} \quad \text{und} \quad \hat{\mathcal{U}}^T \ = \ \hat{\mathcal{U}}^{-1} \quad .$$

indeed holds.

An important application of a subsequent series of such rotation matrices is related to Kepler-orbits (solution of the planar (2D) two-body problem, see Section 4). The two-body-motion, for instance an ellipse (used here as an example), is two-dimensional - meaning, 4 parameters suffice to characterize the ellipse: the semi-major axis a, the eccentricity e, the longitude of pericenter ϖ, and the instant of pericenter passage t_p. That means, for an ellipse-shape, characterized by a and e, one has 1 degree of freedom ϖ, the longitude of pericenter, to change its position in the plane.
In three dimensions it can occur that the ellipse is not located in a plane given by the used frame – which is the general case. For instance, in the Solar-system dynamics the X-Y axes of a chosen inertial frame are usually chosen to lie in the ecliptic, the plane defined by the Earth-orbit – the Z axis is then defined by $\vec{e}_Z = \vec{e}_X \times \vec{e}_Y$. Generally the orbits of Solar-system bodies do NOT lie in the ecliptic-plane but are inclined by an angle $\imath$, so that the orbit-ellipse pierces through the ecliptic at two points, the ascending- and descending nodes. The line connecting both points is defining a direction whose angle with the chosen X-axis is denoted as Ω.
One may transform the frame, which is related to the orbital ellipse, by three rotations: starting with the planar coordinate system $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, where the direction $\vec{e}_x$ coincides with the apsidal line (directing to the pericenter of the orbit), unit vector $\vec{e}_y$ is perpendicular to the former and lies in the orbital plane too, and the vertical, out of orbital-plane direction is $\vec{e}_z = \vec{e}_x \times \vec{e}_y$. Now, we can transform the orbit defined coordinate system $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ to the inertial frame $(\vec{e}_X, \vec{e}_Y, \vec{e}_Z)$ via three rotations

1. rotation $\hat{\mathcal{U}}_\varpi$ about $\vec{e}_z$ with an angle $-\varpi$
2. rotation $\hat{\mathcal{U}}_\imath$ about the new X'-axis with an angle $-\imath$
3. rotation $\hat{\mathcal{U}}_\Omega$ about the Z-axis with an angle Ω

Man erhält (**Übung**)

$$\begin{pmatrix} \vec{e}_X \\ \vec{e}_Y \\ \vec{e}_Z \end{pmatrix} = \hat{\mathcal{U}}_\Omega \, \hat{\mathcal{U}}_\imath \, \hat{\mathcal{U}}_\varpi \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} \quad , \tag{81}$$

with the rotation matrices:

$$\hat{\mathcal{U}}_\varpi = \begin{pmatrix} \cos\varpi & -\sin\varpi & 0 \\ \sin\varpi & \cos\varpi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{82}$$

$$\hat{\mathcal{U}}_\imath = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\imath & -\sin\imath \\ 0 & \sin\imath & \cos\imath \end{pmatrix} \tag{83}$$

$$\tag{84}$$

$$\hat{\mathcal{U}}_\Omega = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{85}$$

Now the 6 parameters necessary in order to describe a Kepler-ellipse in 3D are: the already defined ones: a, e, t_0 and ϖ, as well as the inclination $\imath$ and the longitude of the ascending (descending) node Ω.

3 Gravitational Potential

Heuristics:
Starting with the gravity of a point mass m

$$\Phi(r) \;=\; -\gamma\,\frac{m}{r} \tag{86}$$

which is determined from the distance r of a test particle to that mass – we can apply this to a mass element $\mathrm{d}m = \mathrm{d}V\varrho$ in order to quantify the gravity of a continious mass distribution $\varrho(\vec{r})$, then one obtains

$$\mathrm{d}\Phi \;=\; -\gamma\,\frac{\mathrm{d}m(\vec{r}')}{|\vec{r}' \;-\; \vec{r}|} \;=\; -\gamma\,\frac{\mathrm{d}V\,\varrho(\vec{r}')}{|\vec{r}' \;-\; \vec{r}|} \tag{87}$$

where the distance of the volume- resp. mass element $\mathrm{d}m = \varrho\mathrm{d}V$ to a given point at $\vec{r}$ (usually the center of mass of the distribution) is given by $|\vec{r}\,' \;-\; \vec{r}|$. Integration of relation (87) over the spatial range B of the source-body gives finally the Green-integral

$$\Phi(\vec{r}) \;=\; -\gamma \int\limits_B \mathrm{d}^3\vec{r}'\,\frac{\varrho(\vec{r}')}{|\vec{r}' \;-\; \vec{r}|} \tag{88}$$

which is the solution of the *Poisson-Equation*

$$\nabla\cdot[\nabla\Phi] \;=\; \Delta\Phi(\vec{r}) \;=\; 4\pi\gamma\,\varrho(\vec{r}) \quad . \tag{89}$$

This important classical gravity-potential formula can be also derived in analogy to that of an electro-static potential, which we will demonstrate briefly.

The sources of the gravity is their mass distribution $\varrho\,(\vec{r})$ – as we alreaday concluded in the general-relativistic description via the metric g_{ik} above, or a plausible analog concerns the electric field $\vec{E}$ (not to mess with the Energy E) with its sources contained in charge density $\rho(\vec{r})$ related via $\nabla\cdot\vec{E} \propto \rho$ – so that we can analogously state

$$\nabla\cdot\vec{F} \;=\; C\,\gamma\,\varrho\,(\vec{r}) \;, \tag{90}$$

with the gravity constant γ and a further, not yet determined additional constat C. In order to calculate the latter, we assume simply spherical geometry and integrate (using spherical coordinates) Eq. (90) and obtain

$$\begin{aligned}\int\limits_B \mathrm{d}^3\vec{r}\,\nabla\cdot\vec{F}(r) \;&=\; \oint\limits_{\partial B} \mathrm{d}\vec{A}_r\cdot\vec{F}(r) \\ &=\; 4\,\pi\,r^2\,F(r) \;=\; C\,\gamma\,M\end{aligned} \tag{91}$$

On the other hand, phenomenology and experimental experience tells us that the force reads

$$F(r) \;=\; -\frac{\gamma\,M}{r^2} \quad \text{i.e.} \quad \vec{F}(\vec{r}) \;=\; -\frac{\gamma\,M}{r^2}\,\frac{\vec{r}}{r} \tag{92}$$

Comparison of the coefficients in Eqs. (90) - (92) directly provides us with the constant we are seeking for

$$C \;=\; 4\,\pi \quad . \tag{93}$$

This gravitational force is, of course, a conservative one (**homework**), which requires:

$$\oint \vec{F}\cdot \mathrm{d}\vec{s} \;=\; \int\!\!\int_B \left(\nabla\times\vec{F}\right)\,\mathrm{d}\vec{A} \;=\; 0 \;\;\text{(Integral-law of Stokes).}$$

The independence of the integral on the path requires that the integrand has to vanish:

$$\begin{aligned} \nabla\times\vec{F} &= 0 \\ \Longrightarrow \vec{F} &= -\nabla\Phi \end{aligned} \tag{94}$$

so that we obtain again the Poisson-equation

$$\boxed{\Delta\Phi(\vec{r}) \;=\; 4\,\pi\,\gamma\,\varrho(\vec{r})} \tag{95}$$

whose solution for given boundary conditions determines the gravitational potential.

The general solution of the Poisson-equation (95) can be constructed by a Greens-function $G(\vec{r}',\vec{r}) \;=\; 1/\left|\vec{r}-\vec{r}'\right|$

$$\begin{aligned} \Phi\left(\vec{r}\right) &= -\gamma\int_B \frac{\varrho\left(\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|}\mathrm{d}V' \\ \Longrightarrow \nabla\Phi &= \gamma\int_B \frac{\varrho\left(\vec{r}'\right)\left(\vec{r}-\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|^3}\mathrm{d}^3\vec{r}' \end{aligned} \tag{96}$$

The divergency of Eq. (96) has to yield again the Poisson-Eq. (89) resp. (95), as the following check demonstrates:

$$\Delta\Phi \;=\; \gamma\frac{\partial}{\partial\vec{r}}\cdot\int_B \frac{\varrho\left(\vec{r}'\right)\left(\vec{r}-\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|^3}\mathrm{d}V' \quad .$$

In course of further manipulation of the intergral, two assumption are used: the mass distribution is assumed to be homogeneous $\varrho(\vec{r}) \;=\; const.$, and further we consider only a small Volume K_d occupied with matter and calculate the intergral in the outside region. With this and using the distance vector $\vec{d} = \vec{r}' - \vec{r}$ and Gauß' integral law we obtain:

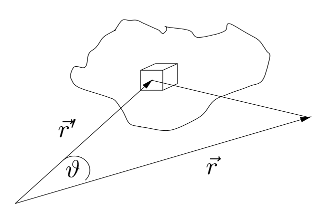


Abbildung 1:

$$\Delta\Phi = \gamma\varrho(\vec{r}) \int\limits_{K_d} \frac{\partial}{\partial\vec{r}} \cdot \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \mathrm{d}V' = -\gamma\varrho(\vec{r}) \int\limits_{K_d} \frac{\partial}{\partial\vec{r}'} \cdot \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \mathrm{d}V' \tag{97}$$

$$= \gamma\varrho(\vec{r}) \oint\limits_{\partial K_d} \frac{\vec{r}'-\vec{r}}{|\vec{r}'-\vec{r}|^3} \mathrm{d}\vec{A}$$

$$= \gamma\varrho(\vec{r}) \oint\limits_{\partial K_d} \frac{d^2 \sin\vartheta}{d^3}\, d\,\vec{e}_d \cdot \vec{e}_d \,\mathrm{d}\varphi\,\mathrm{d}\vartheta = \gamma\varrho(\vec{r}) \underbrace{\oint\limits_{\partial K_d} \sin\vartheta \mathrm{d}\varphi \mathrm{d}\vartheta}_{4\pi}$$

$$\Longrightarrow \qquad \Delta\Phi = 4\pi\gamma\varrho(\vec{r}) \tag{98}$$

One should note that for $\left|\vec{\delta}\right| > 0$ follows (**homework**)

$$\nabla' \cdot \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = -\frac{3}{d^3} + \frac{3\vec{d}\cdot\vec{d}}{d^5} \to \begin{cases} 0 \text{ for } |d| \neq 0 \\ \infty \text{ for } |d| = 0 \end{cases} = \delta(\vec{r}-\vec{r}') \tag{99}$$

which constitutes Dirac's Delta function $\delta(x)$.
The derivation in Eq. (99) illustrates that contributions of the integral come just from the close environment at $\left|\vec{d}\right| \to 0$ so that one may write $\varrho(\vec{r}) = \varrho(\vec{r}')$ and is allowed to draw the density ϱ out of the integral and to assume spherical symmetry. With definition of the Green function $G(\vec{r}', \vec{r})$ this directly leads to

$$\Delta G(\vec{r}', \vec{r}) = \delta(\vec{r}-\vec{r}') \tag{100}$$

the formula to determine that function. Below we gonna sketch the Green-solution approach.

3.1 Sketch of the Green-solutions

Green's method: given is the inhomogeneous partial differential equation

$$\hat{O}(\hat{\vec{p}}, \vec{r}, t)\, \psi = A(\vec{r}) \tag{101}$$

and we state/postulate the existence of a function $G(\vec{r}|\vec{r}')$ satisfying the equation:

$$\hat{O}\, G(\vec{r}|\vec{r}') = \delta(\vec{r}-\vec{r}') \tag{102}$$

Given this a special solution of the inhomogeneous Eq. (102) can be written as follows:

$$\psi(\vec{r}, t) = \int \mathrm{d}^3\vec{r}\, G(\vec{r}|\vec{r}')\, A(\vec{r}) \quad . \tag{103}$$

Exactly in this sense the integral (96) can be understood as the solution of the Poisson Eq. (95).

In a next step, we want to offer a simple and plausible one-dimensional presentation of the above relation (102). A partial differential equation is given by

$$\hat{\mathcal{D}}\, y(x) \;=\; f(x) \tag{104}$$

which we try to solve with the Ansatz

$$y(x) \;=\; \int \mathrm{d}x'\, G(x,x')\; f(x') \quad , \tag{105}$$

which we directly plug in Eq. (104) so that one obtains

$$\hat{\mathcal{D}}\, y(x) \;=\; \int \mathrm{d}x'\, \hat{\mathcal{D}}\, G(x,x')\; f(x') \;=\; f(x) \quad . \tag{106}$$

The right equality and the rules of δ-distributions require directly that the following relation holds:

$$\hat{\mathcal{D}}\, G(x,x') \;=\; \delta(x-x') \;\; , \tag{107}$$

illustrating that the Greens function has to fulfil the Eq. (102)
Consequently, one has to find the Greens-function as a solution of a differential-equation with a δ-inhomogeneity – which is, for instance, realized by piecewise integration or other methods – some of the latter are sketches below.

3.1.1 Greens Function Calculations

Again we consider an inhomogeneous partial differential eq. (λ is NOT an Eigen-value !) with the hermitian, differential operator $\hat{\mathcal{D}}(\partial_{\vec{r}}, \partial^2_{\vec{r}}, ...)$

$$\hat{\mathcal{D}} y(\vec{r}) - \lambda y(\vec{r}) \;=\; f(\vec{r}) \tag{108}$$

(Examples $\Rightarrow$ see Lectures).

1. Expansion with Eigen-functions of $\hat{\mathcal{D}}$:

The Eigen-value problem, we are interested, in is given by

$$\hat{\mathcal{D}}\;|n\rangle \;=\; \lambda_n\;|n\rangle \;\;;\text{ or };\;\; \hat{\mathcal{D}}\;\varphi_n(\vec{r}) \;=\; \lambda_n\;\varphi_n(\vec{r}) \tag{109}$$

where the Eigen-functions $|n\rangle$ respective $\varphi_n(\vec{r})$ form a complete, ortho-normal base in a Hilbert-space, which obey the relations

$$\begin{aligned} \text{ortho-normality:} &\quad \langle n|k\rangle \;=\; \delta_{nk} \quad ;\text{ or }; \quad \int \mathrm{d}^3\vec{r}\;\varphi_n^*(\vec{r})\;\varphi_k(\vec{r}) \;=\; \delta_{nk} \\ \text{completeness:} &\quad \sum_n \varphi_n^*(\vec{r}\,')\,\varphi_n(\vec{r}) \;=\; \delta(\vec{r}-\vec{r}\,') \quad . \end{aligned} \tag{110}$$

Such a complete base allows to expand any analytical function, as for instance, those ones needed in Eq. (108)

$$y(\vec{r}) = \sum_n c_n |n\rangle \quad \& \quad f(\vec{r}) = \sum_n \langle n|f\rangle\, |n\rangle\ , \tag{111}$$

and plugged this in the original problem (108) one obtains by coefficient-comparison and with the completeness[5] condition (**Homework**)

$$c_n = \frac{\langle n|f\rangle}{\lambda_n - \lambda} \tag{112}$$

$$y(\vec{r}) = \int_B \mathrm{d}^3\vec{r}'\, G(\vec{r},\vec{r}')f(\vec{r}') \ ; \text{ with } ; \quad G(\vec{r},\vec{r}') = \sum_n \frac{\varphi_n^*(\vec{r}')\,\varphi_n(\vec{r})}{\lambda_n - \lambda} \tag{113}$$

where again the Green-function has to obey the differential equation

$$\hat{\mathcal{D}}G(\vec{r},\vec{r}') - \lambda G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}') \quad . \tag{114}$$

2. Spektral Method:
Fourier transform the original Eq. (114) gives directly

$$\left\{\tilde{\mathcal{D}}(\imath\vec{k}, -k^2 ...) - \lambda\right\} \tilde{G}(\vec{k}) = 1 \tag{115}$$

where we have used the the translational invariance of the original part. diff. Eq. (108) with respect of the dependence $\vec{x} = \vec{r} - \vec{r}'$. The result of the Fourier-transform is an algebraic equation for the Fourier-transform of the Greens-function which can simply be re-arranged to give algebraic expression for $\tilde{G}(\vec{k})$ in the Fourier-space. A back-transform gives the Greens-function we seeked for:

$$G(\vec{r},\vec{r}') = y(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{k}\, \tilde{G}(\vec{k})e^{\imath\vec{k}\cdot\vec{x}} \quad . \tag{116}$$

3.1.2 Calculation of Gravitational Fields

The Greens-function $G(\vec{r};\vec{r}') \propto 1/|\vec{r}-\vec{r}'|$ comprises the key of the calculation of the gravitational fields of given mass-density distributions $\varrho(\vec{r})$. One has to distinguish between solutions of the spatial region inside the mass-covered region and that one outside. The corresponding equations are the Poisson- and the Laplace equation:

$$\Delta\Phi = 4\pi\gamma\varrho(\vec{r}) \quad ; \quad \Delta\Phi = 0 \quad . \tag{117}$$

[5]The completeness-relation is easily shown (here in 1-D) by expanding the δ-function $\delta(x-x') = \sum_n a_n(x')\varphi(x)$ with the coefficients $a_n(x') = \langle n|\delta\rangle = \int \mathrm{d}x\, \varphi^*(x)\delta(x-x') = \varphi^*(x')$. Inserting this coefficient in the δ-expansion directly gives the above completeness condition (110).

The solution of these equations are given by the general solution (88). Because many celestial bodies have spherical form in first approximation so that spherical coordinates seem to be suited for the analysis of this integral with the Greens-function $\propto 1/|\vec{r}-\vec{r}'|$. Deviations of the spherical symmetry are described by the multi-pole expansion where base of it is the expansion of the Greens-function according to Legendre polynomials, whose generating function is the Gravity-Greens function for the outer case of $|r| > r'$ and $z = r'/r < 1$:

$$
\begin{aligned}
|\vec{r}' - \vec{r}|^{-1} &= \left(r^2 + r'^2 - 2rr'\cos\vartheta\right)^{-1/2} = \frac{1}{r}\left(1 + z^2 - 2zt\right)^{-1/2} \\
&= \frac{1}{r}\left(P_0(t) + P_1(x)\, z + P_2(x)\, z^2 + \cdots\right) \\
\Longrightarrow \quad \Phi &= -\gamma \int \frac{\varrho(\vec{r}')}{r} \left(\sum_0^\infty P_i z^i\right) \mathrm{d}^3\vec{r}' \qquad (118)
\end{aligned}
$$

The integral can be evaluated by using spherical coordinates where the limits of integration are given by the borders of the celestial body under consideration, where, for instance, R_p is its equatorial radius of a flattened planet or star. Then with the definition (96) follows for the potential in the 'òutet space" (outside the range B):

$$
\Phi(\vec{r}) = -\frac{\gamma M}{r}\left(1 - \sum_{n=1}^{\infty} J_{2n}\left(\frac{R_p}{r}\right)^{2n} P_{2n}(\cos\vartheta)\right) \qquad (119)
$$

For simplicity, we assume in the following a homogeneous density $\varrho(\vec{r}') = \varrho_0$. The independence of φ reflects the axial-symmetry of the potential (rotational ellipsoid) as observed for many celestial bodies (most planets of the Solar system) $\mu = \gamma M$ - so that we obtain for the first terms of the expansion:

$$
\begin{aligned}
\text{0. moment:} \quad \Phi &= -\frac{\mu}{r} \quad \text{(point-mass)} \\
\text{1. moment:} \quad \Phi &= \frac{\mu}{r} J_2 \left(\frac{R_p}{r}\right)^2 P_2(\cos\vartheta) \quad \text{(quadrupol-moment)} \\
&\vdots \\
\Longrightarrow \text{grav. Potential:} \quad \Phi &= -\frac{\mu}{r}\left(1 - J_2\left(\frac{R_P}{r}\right)^2 P_2(\cos\vartheta) \pm \cdots\right)
\end{aligned}
$$

4 The Two-Body Problem (TBP)

4.1 Unperturbed TBP

The two-body problem (TBP) describes the motion of two bodies under their mutual gravitational interaction. The corresponding ordinary differential equations of second order according *Newton II* read

$$m_i\ddot{\vec{r_i}} \;=\; -\gamma\frac{m_i m_j}{|\vec{r_i}-\vec{r_j}|^3}\,(\vec{r_i}-\vec{r_j}) = \vec{F}_{ij} \quad . \tag{120}$$

Individuelly written the two equations of motion (2. order) read

$$m_1\ddot{\vec{r_1}} \;=\; -\gamma m_1 m_2\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} \tag{121}$$

$$m_2\ddot{\vec{r_2}} \;=\; -\gamma m_1 m_2\frac{\vec{r_2}-\vec{r_1}}{|\vec{r_1}-\vec{r_2}|^3} \quad . \tag{122}$$

The TBP can be reduced to the Kepler-problem of two bodies by expanding their motion into that of the center of mass

$$\vec{r_s} \;=\; \frac{\sum\limits_i m_i\,\vec{r_i}}{\sum\limits_i m_i} \quad , \tag{123}$$

and that of the relative motion/coordinates of either bodies

$$\vec{r} \;=\; \vec{r_1} - \vec{r_2} \quad . \tag{124}$$

The reduction is then carried out by multiplying the Eqs. (121) & (122) "over cross" with m_2 bzw.m_1 so that one obtains

$$m_1 m_2\ddot{\vec{r_1}} \;=\; -\gamma m_1 m_2^2\frac{\vec{r}}{r^3} \tag{125}$$

$$m_1 m_2\ddot{\vec{r_2}} \;=\; \gamma m_1^2 m_2\frac{\vec{r}}{r^3} \tag{126}$$

– and subtracting Eqs. (121) *minus* (122)one arrives at the equation:

$$\ddot{\vec{r}} \;=\; -\gamma(m_1+m_2)\frac{\vec{r}}{r^3} \;=\; -\mu\frac{\vec{r}}{r^3} \;=\; \vec{F} \quad . \tag{127}$$

where the constant gravitational mass parameter $\mu \;=\; \gamma\,(m_1+m_2)$ has been introduced. Further one may define effective potential with the angular momentum (**homework**: by using polar coordinates or Lagrange Eq. II or the Hamiltonian Eqs.):

$$\Phi_{eff}\,(r) \;=\; \frac{L^2}{2r^2} + \phi\,(r) \tag{128}$$

$$\vec{L} \;=\; \vec{r}\times\vec{v} \quad , \tag{129}$$

Without loss of generality one may assume a plane motion $z=0$ so that with polar coordinates $\vec{r} = r\,\vec{e_r} = r(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j})$ the equation of motion reads:

$$\ddot{r} - \dot{\varphi}^2 r + \frac{\mu}{r^2} \;=\; 0 \qquad \text{(r-component)} \quad . \tag{130}$$

The angular momentum conservation follows from the fact, that the gravity $\vec{F} \propto \vec{r}$ constitutes a central force giving $\vec{r} \times \ddot{\vec{r}} = 0$, (because of: $\vec{r} \times \vec{r} \, = \, 0$), and one may write:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \dot{\vec{r}}) \;=\; 2r\dot{r}\dot{\varphi}\vec{e_z} = \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) = 0 \qquad (\text{conservation of } \vec{L}) \quad , \tag{131}$$

with

$$\left|\vec{L}\right| \;=\; \left|\vec{r} \times \dot{\vec{r}}\right| \;=\; r^2\,\dot{\varphi} \;=\; const. \tag{132}$$

so that one may write for the r-component of the eqn. of motion can be formulated

$$\begin{aligned} \ddot{r} \;&=\; \frac{L^2}{r^3} - \frac{\mu}{r^2} \\ &=\; -\frac{\partial \Phi_{eff}}{\partial r} \quad . \end{aligned} \tag{133}$$

The effective potential is obtained with Eq. (128) where gravitational potential reads

$$\Phi(r) \;=\; -\frac{\mu}{r} \quad . \tag{134}$$

4.1.1 Solution of TBP

With the angular momentum conservation (131), and $\frac{\mathrm{d}}{\mathrm{d}t} \to \dot{\varphi}\frac{\mathrm{d}}{\mathrm{d}\varphi}$ one obtains

$$\begin{aligned} \ddot{r} \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Lr'}{r^2}\right) = \frac{L}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{L}{r^2}r'\right) \\ &=\; \frac{L}{r^2}\left(-\frac{2L}{r^3}r'^2 + \frac{L}{r^2}r''\right) \\ \Longrightarrow r'' \;&=\; \frac{2r'^2}{r} + r - \mu\left(\frac{r}{L}\right)^2 \end{aligned} \tag{135}$$

with $r' \;=\; \mathrm{d}r/\mathrm{d}\varphi$. Next we transform variables via $r = 1/x$, and as a result one is led to the equation of an harmonic oscillator (136), which remains to be solved

$$\begin{aligned} r' &= -\frac{x'}{x^2} \\ r'' &= 2\frac{x'^2}{x^3} - \frac{x''}{x^2} \\ \Longrightarrow x'' + x - \frac{\mu}{L^2} \;&=\; 0 \quad . \end{aligned} \tag{136}$$

With a further translation transformation $y \,=\, x \,-\, \mu/L^2$ we arrive at a differential Eq. with constant coefficients

$$y'' \;+\; y \;=\; 0 \tag{137}$$

with $x = \tilde{A}\cos\left(\varphi - \varpi\right)$ and ϖ as the pericenter length, can easily be solved, where one identifies cone-sections as trajectories:

$$\begin{aligned} x \;&=\; \frac{1}{r} = A\cos\varphi + \frac{\mu}{L^2} \\ \Longrightarrow r \;&=\; \frac{L^2/\mu}{1 + AL^2/\mu\;\cos f} \quad \text{with} \qquad f = \varphi - \varpi \quad . \end{aligned} \tag{138}$$

With the definition of the eccentricity $e = AL^2/\mu$ one obtains

$$\boxed{r \;=\; \frac{Ae}{1\,+\,e\;\cos f} \;=\; \frac{p}{1\,+\,e\;\cos f}} \tag{139}$$

with the *semi-latus rectum*

$$p(e) \;=\; \frac{L^2}{\mu} \tag{140}$$

where the dependence on the eccentricity e is determined by the type of the orbit: $p \,=\, a(1-e^2)$ for an ellipse ($e < 1$), $p \,=\, a(e^2-1)$ for a hyperbola ($e > 1$) – and last but not least for a parabola one obtains $p = q(1+e) = 2q$ (q is the distance of the pericenter from the focus, $e \to 1$, see below).

Orbital Energy

The quantification of the orbital energy is most easily calculated at the pericenter r_{min}, wich can be expressed as:

$$\dot{\vec{r}} \;=\; \dot{r}\vec{e_r} + r\dot{\varphi}\vec{e_\varphi} \tag{141}$$

$$r \;=\; \frac{L^2/\mu}{1+e\cos f} \quad \overbrace{\Longrightarrow}^{\cos f \to 1} \quad r_{min} = \frac{L^2/\mu}{1+e} \tag{142}$$

where at the pericenter the radial speed is $\dot{r} = 0$, and thus one obtains

$$\begin{aligned}
E \;&=\; \left(\frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} - \frac{\mu}{r}\right)\Bigg|_{PZ} \;=\; \frac{L^2}{2r_{min}^2} - \frac{\mu}{r_{min}} \\
&=\; \frac{L^2\mu^2\,(1+e)^2}{2L^4} - \frac{\mu^2\,(1+e)}{L^2} \;=\; \frac{\mu^2\,(1+e)}{2L^2}\,(e-1)
\end{aligned}$$

$$\Longrightarrow \boxed{E \;=\; \frac{\mu^2}{2L^2}\left(e^2-1\right)} \tag{143}$$

Homework: Calculate the energy by using the general radius (142), i.e. dropping the pericenter condition and using the speed $\dot{r}$ (helping hint: $\dot{\varphi} \,=\, \dot{f} \,=\, L^2/\mu$ because $f = \varphi - \varpi$).

$$\begin{aligned}
e \;<\; 1\text{: Ellipse} \;&\Longrightarrow\; E \;=\; -\frac{\mu}{2a} \;<\; 0\,(e=0\text{: Kreis}) \\
e \;>\; 1\text{: Hyperbel} \;&\Longrightarrow\; E \;=\; \frac{\mu}{2a} \;>\; 0 \\
e. \;=\; 1\text{: Parabel} \;&\Longrightarrow\; E \;=\; 0
\end{aligned}$$

Angular Momentum

In this Subsection all considerations are referred to a bound, elliptical orbit – generalized relations can easily derived and are given below. The Keplerian theorem of areas (angular momentum conservation) can be written as follows – when averaged over an orbital period T

$$L \;=\; r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} \qquad \Rightarrow \qquad L\ \mathrm{d}t \;=\; r^2\ \mathrm{d}\varphi \tag{144}$$

$$L\,T \;=\; \int\limits_0^{2\pi} \mathrm{d}\varphi\, r^2(\varphi) \;=\; 2A \tag{145}$$

with area of an ellipse

$$A \;=\; \pi ab \quad \text{und} \quad b^2 \;=\; a^2\left(1-e^2\right)$$

where a and b denote either semi-axes: the semi-major and the semi-minor axis, respectively.

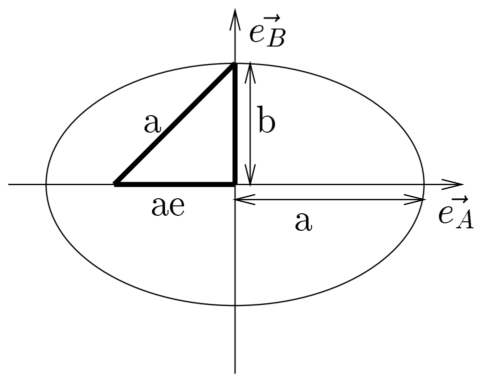


Abbildung 2: Characteristic triangle of an ellipse.

With the area of the parallelogram (per time) $L = |\vec{r} \times \dot{\vec{r}}| = 2\dot{A} \;=\; r^2\dot{\varphi}$ and with the mean motion $n = 2\pi/T$ we obtain for an ellipse :

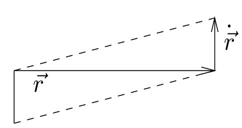


Abbildung 3:

$$2A = 2\;\pi\;ab \;=\; \int_0^T L\,\mathrm{d}T \;=\; TL \;\Longrightarrow\; n(a) \;=\; \frac{L}{ab} \;=\; \frac{1}{a^2\sqrt{1-e^2}}$$

and with it for the angular momentum

$$\Longrightarrow \boxed{L \;=\; n\,a^2\sqrt{1-e^2} \;=\; \sqrt{\mu a\,(1-e^2)}\ \ .} \tag{146}$$

and another expression for the mean motion n

$$n(a) \;=\; \sqrt{\frac{\mu}{a^3}} \tag{147}$$

with the radius

$$r = \frac{a\,(1\ -\ e^2)}{1\ +\ e\cos f}\quad . \tag{148}$$

In the case of a hyperbola we may write with $e\ >\ 1$ (because of $E > 0$)

$$\Longrightarrow\ \boxed{L\ =\ n\,a^2\sqrt{e^2-1}\ =\ \sqrt{\mu a\,(e^2\ -\ 1)}\ \ .} \tag{149}$$

Time dependence – Kepler Equation

For the moment we want to derive general expressions for the time-dependence of the orbital motion based upon the angular momentum conservation, and then conclude for a geometrical foundation of the obtained relation. To this aim we integrate Eq. (144) where we keep the time t and the corresponding angle φ free to obtain

$$t\ -t_0\ =\ \frac{L^3}{\mu^2}\int\limits_0^{\varphi(t)} \mathrm{d}f\ \ (1+e\ \cos f)^{-2}\quad . \tag{150}$$

In order to solve the integral in a closed form, we apply the following transformations

$$\tau\ =\ \tan\frac{f}{2}\quad ,\quad \cos f\ =\ \frac{1-\tau^2}{1+\tau^2}\quad ,\quad \mathrm{d}f\ =\ \frac{2}{1+\tau^2}\,\mathrm{d}\tau \tag{151}$$

with the result that the integrand of integral (150) becomes a rational function to give:

$$t\ -t_0\ =\ \frac{2L^3}{\mu^2}\int\limits_0^{\tau(t)} \mathrm{d}\tau\ \frac{1+\tau^2}{[(1+e)+(1-e)\tau^2]^2}\quad . \tag{152}$$

This integral has to be analyzed for all three types of Keplerian orbits: ellipses, hyperbolas, and parabolas. The latter case is the simplest one, where even a explicit time solution can be obtained – which cannot be achieved in the other two cases.

(A) Parabel:
For $e = 1$ the equation (152) can be integrated directly

$$t\ -\ t_0\ =\ \frac{L^3}{2\mu^2}\int\limits_0^{\tau(t)} \mathrm{d}\tau\ \left(1+\tau^2\right)\ =\ \frac{L^3}{2\mu^2}\left[\tau(t)+\frac{\tau(t)^3}{3}\right]\quad . \tag{153}$$

Introducing the parabolic mean anomaly

$$\Lambda\ =\ \frac{\mu^2}{L^3}\ (t-t_0) \tag{154}$$

one obtains the *Barkers*-equation

$$\tau^3 \ + \ 3\tau \ - \ 6\Lambda \ = \ 0 \quad , \tag{155}$$

which can be solved with the aid of Cardan's formulas

$$\tau(t) \ = \ \sqrt[3]{3\Lambda + \sqrt{1+9\Lambda^2}} + \sqrt[3]{3\Lambda - \sqrt{1+9\Lambda^2}} \tag{156}$$

$$\varphi(t) \ = \ 2\arctan\tau(t) \quad . \tag{157}$$

With this, the trajectory $\vec{r}(t) \ = \ r\,(\varphi(t)) \ \vec{e}_r$ can be explicitly determined, so that the parameter-expression of the parabola follows with $p = 2q$ and $e = 1$ as

$$r(f) \ = \ \frac{2q}{1+\cos f} \ = \ \frac{q}{\cos^2\frac{f}{2}} \quad . \tag{158}$$

In this context it is worth noting to leave a few words about the *Semilatus Rectum* $p = a(1-e)(1+e)$ of the parabula! To this aim we start with the ellipse and try for the limit $e \to 1$ where the energy of an ellipse is $E = -\mu/2a \ = \ \mathcal{T} + \Phi(r) \to 0$, because *ad infinitum*, i.e. $r \ \to \infty$ follows that the test-body rests $v = 0$ so that the kinetic- and potential energy both vanish $\mathcal{T} = mv^2/s = \Phi = -\mu/r = 0$. This also means that $a \to \infty$. On the other hand the factor $(1-e) \to 0$ vanishes, where the product, the parabolic pericenter-distance $q = a(1-e) \ < \ \infty$ is finite as the *Semilatus Rectum* $p = a(1-e)(1+e)$ too.
The normal mathematical form of a parabola is obtained by choosing a coordinate system, whose origin has a distance of $p = 2q$ from the focus (location of the central body). With polar coordinates we thus have $x = p - r\cos f$ and $y = r\sin f$. Eliminating the cosine $\cos f = (p-x)/r$ and insert this in Eq. (158) we obtain:

$$r \ = \ \frac{pr}{p+r-x} \quad \Rightarrow \quad r \ = \ x \quad . \tag{159}$$

This is nothing but an expression for the locus definition of the parabola: *The perpendicular distance of a point of the parabola from the guiding line (our y-axis) x equals its distance from the focus r.* With the definition of the latter $r = \sqrt{(p-x)^2 + y^2} \ = \ x$ one obtains the normal form of a parabola

$$y^2 \ = \ 4q(x-q) \quad . \tag{160}$$

For the ellipse and hyperbola this does not work so easy – those cases are treated below.

(B) Ellipse:

In this case we have $e < 1$, so that we obtain with the transformation

$$\tau \ = \ \sqrt{\frac{1+e}{1-e}}\,\kappa \ = \ \frac{v_p}{an(a)}\,\kappa \tag{161}$$

the integral

$$\frac{\mu^2}{L^3}\,(t\ - t_0) \ = \ \frac{2}{\sqrt{(1-e^2)^3}} \int\limits_0^{\kappa(t)} \mathrm{d}\kappa\, \frac{(1-e)+(1+e)\kappa^2}{(1+\kappa^2)^2} \quad . \tag{162}$$

Just another transformation – providing the definition of the *eccentric anomaly* $\tilde{E}$ –

$$\kappa \ = \ \tan\frac{\tilde{E}}{2} \qquad \Leftrightarrow \qquad \mathrm{d}\tilde{E} \ = \ \frac{2}{1+\kappa^2}\,\mathrm{d}\kappa \tag{163}$$

gives (**nice homework**)

$$\frac{\mu^2}{L^3}\,(t\ - t_0) \ = \ \frac{2}{\sqrt{(1-e^2)^3}} \int\limits_0^{\tilde{E}(t)} \mathrm{d}\tilde{E}\,\left(1 - e\cos\tilde{E}\right) \ = \ \frac{2}{\sqrt{(1-e^2)^3}}\,\left(\tilde{E} - e\sin\tilde{E}\right) \ . \tag{164}$$

Re-arranging simply above equation leads to the **famous Kepler-equation**:

$$\boxed{M \ = \ n(a)\,(t-t_0) \ = \ \left(\tilde{E} - e\sin\tilde{E}\right) \quad .} \tag{165}$$

The corresponding parametric representation of the distance from the central body (or center of mass) now becomes

$$\boxed{r(\tilde{E}) \ = \ a\,\left(1 - e\cos\tilde{E}\right)} \tag{166}$$

as we will prove geometrically right below.

Geometrical interpretation (ellipse):

For an ellipse and a circle we have the following relations:

$$a^2 \ = \ b^2 \ + \ a^2e^2 \ \Longrightarrow \ b \ = \ a\sqrt{1-e^2} \tag{167}$$

$$\frac{x^2}{a^2} \ + \ \frac{y^2}{b^2} \ = \ 1 \qquad \text{Ellipse} \tag{168}$$

$$x^2 \ + \ y'^2 \ = \ a^2 \qquad \text{Kreis} \tag{169}$$

$$\Longrightarrow \ \frac{b}{a} \ = \ \frac{y}{y'} \tag{170}$$

By inspecting Figure 4 one may easily identify the following geometrical identities:

$$r^2(\tilde{E}) \ = \ (\overline{DF})^2 + (\overline{DP})^2$$

$$(\overline{DF}) = r\cos f \ = \ -r\cos(\pi - f) = (\overline{CD}) - (\overline{CF}) = a\left(\cos\tilde{E} - e\right) \tag{171}$$

$$(\overline{DP}) = r\sin f \ = \ y = \frac{b}{a}y' = b\sin\tilde{E} = a\sqrt{1-e^2}\,\sin\tilde{E} \tag{172}$$

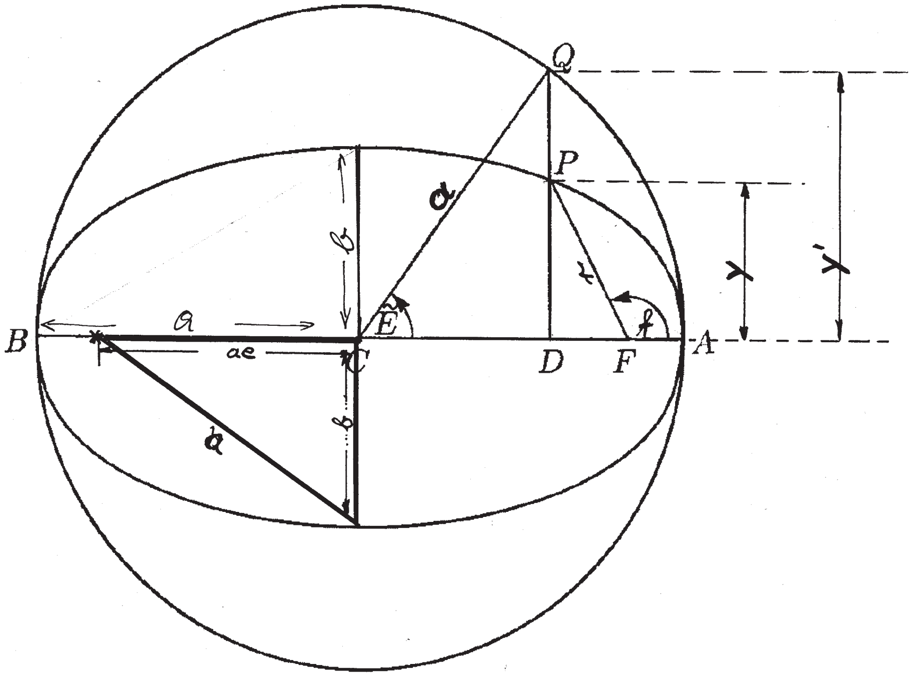


Abbildung 4: Parameters of the description of the ellipse with respect to its center and illustration of the geometrical interpretation of the *Kepler*-equation and the radius-function $r(\tilde{E})$.

so that we may write

$$\Longrightarrow \boxed{r\left(\tilde{E}\right) \;=\; a\left(1 - e\cos\tilde{E}\right)} \quad . \tag{173}$$

The *Kepler*-equation of an ellipse can also confirmed by the geometrical relations (171) - (172) and by using the angular momentum conservation (144)

$$L = r^2\dot{\varphi} = r^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} \quad \Leftrightarrow \quad \mathrm{d}t \;=\; \frac{r^2}{L}\,\mathrm{d}\varphi \quad . \tag{174}$$

and the following formula manipulations. First, one calculates the fraction

$$\tan f \;=\; \frac{(172)}{(171)} \;=\; \frac{\sqrt{1-e^2}\sin\tilde{E}}{(\cos\tilde{E} - e)}$$

and differentiates the result to obtain (reminding that $f = \varphi - \varpi$)

$$\mathrm{d}\tan f \;=\; \frac{\mathrm{d}\varphi}{\cos^2 f} \;=\; \frac{\sqrt{1-e^2}(1 - e\cos\tilde{E})}{(\cos\tilde{E} - e)^2}\,\mathrm{d}\tilde{E} \quad . \tag{175}$$

Re-arranging Eq. (171) one can get

$$\begin{aligned} \mathrm{d}\varphi \;&=\; \cos^2 f\,\frac{\sqrt{1-e^2}(1 - e\cos\tilde{E})}{(\cos\tilde{E} - e)^2}\,\mathrm{d}\tilde{E} \\ &\overset{(171)}{=}\; \frac{a^2}{r^2}\sqrt{1-e^2}(1 - e\cos\tilde{E})\,\mathrm{d}\tilde{E} \end{aligned} \tag{176}$$

so that one finally can write with Eq. (174)

$$\begin{aligned} L\,\mathrm{d}t &= \sqrt{\mu a(1-e^2)}\,\mathrm{d}t = a^2\sqrt{1-e^2}\,(1-e\cos\tilde{E})\,\mathrm{d}\tilde{E} \\ n(a)\,\mathrm{d}t &= (1-e\cos\tilde{E})\,\mathrm{d}\tilde{E} \end{aligned} \tag{177}$$

Integration of this equation finally reproduces the *Kepler*-equation of an ellipse (165) so that for an ellipse we can summarize

$$\begin{aligned} n(a)\,(t-t_0) &= \tilde{E} - e\sin\tilde{E} \\ r &= a\left(1-e\cos\tilde{E}\right) \quad . \end{aligned}$$

(C) Hyperbola:

For hyperbolas we have: $e>1$. For the evaluation of the integral in Eq. (152) we choose the following trafo

$$\tau(t) = \sqrt{\frac{e+1}{e-1}}\,\kappa(t) \quad , \quad \mathrm{d}\tau = \sqrt{\frac{e+1}{e-1}}\,\mathrm{d}\kappa \quad , \tag{178}$$

and with it one obtains for the integral

$$\frac{\mu^2}{L^3}\,(t-t_0) = \frac{2}{\sqrt{(e^2-1)^3}} \int_0^{\kappa(t)} \mathrm{d}\kappa\,\frac{(1+e)+(e-1)\kappa^2}{(\kappa^2-1)^2} \quad . \tag{179}$$

With a further transformation we introduce of the *hyperbolic anomaly* $\tilde{U}$

$$\tanh\frac{\tilde{U}}{2} = \kappa \quad , \quad \mathrm{d}\tilde{U} = \frac{2}{1-\kappa^2}\,\mathrm{d}\kappa \tag{180}$$

which finally leads to the

$$M = n(a)\,(t-t_0) = e\sinh\tilde{U} - \tilde{U} \tag{181}$$

$$r = a\left(e\cosh\tilde{U} - 1\right) \tag{182}$$

Kepler-equation for hyperbolas. For the moment Eq. (182) has deduced from an analogy with the elliptical case, which can be geometrically confirmed.

A geometrical interpretation of a hyperbola is more difficult than that one of the ellipse. The sketch (4.1.1) should illustrate the introduction of the hyperbolic anomaly $\tilde{H}$ which serves as an analogue of the eccentric anomaly for the ellipse. The latter fact is, unfortunately, not so easy to demonstrate as in the elliptical case.
At first some geometrical definitions

$$\begin{aligned} x &= r\cos f \\ a &= \tilde{y}\cos\tilde{H} \implies \tilde{y} = \frac{a}{\cos\tilde{H}} \\ \implies ae &= x+\tilde{y} = r\cos f + a\sec\tilde{H} \end{aligned}$$

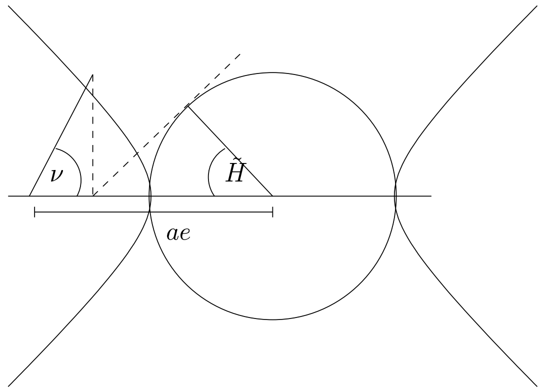


Abbildung 5: Geometrical description of a hyperbola with a single parameter $\tilde{H}$

With this the following relation between the true anomaly f of the hyperbola trajectory and the parameter $\tilde{H}$:

$$\cos f \quad = \quad \frac{a(e - \sec\tilde{H})}{r} \qquad \text{with} \qquad \sec\tilde{H} = \frac{1}{\cos\tilde{H}} \quad , \tag{183}$$

and this inserted in Eq. (139) yields the representation of the hyperbola trajectory as a function of the parameter $\tilde{H}$:

$$r \quad = \quad a(e\sec\tilde{H} - 1) \quad , \tag{184}$$

where a comparison with Gl. (182) suggests the identity

$$\sec\tilde{H} \quad = \quad \cosh\tilde{U} \quad , \tag{185}$$

whose meaning can be made plausible in the *excercise- seminars* or a **home work** (math repetitions: relations between hyperbolic- and trigonometric functions – as a result one finally obtains the Eqs. (181) - (182).

The results of orbit determination are summarized in the following table:

Ellipse	Hyperbola	Parabola
$n\,(t-t_0) = \left(\tilde{E} - e\sin\tilde{E}\right)$	$n\,(t-t_0) = \left(e\sinh\tilde{U} - \tilde{U}\right)$	
$r = a\left(1 - e\cos\tilde{E}\right)$	$r = a\left(e\cosh\tilde{U} - 1\right)$	
$\tau = \sqrt{\frac{1+e}{1-e}}\kappa,\ \kappa = \tan\left(\frac{\tilde{E}}{2}\right)$	$\tau = \sqrt{\frac{1+e}{1-e}}\kappa,\ \kappa = \tanh\left(\frac{\tilde{U}}{2}\right)$	$\tau^3 + 3\tau = 6\frac{\mu^2}{L^3}\,(t-t_0)$
$\tau = \tan\left(\frac{\varphi}{2}\right)$	$\tau = \tanh\left(\frac{\varphi}{2}\right)$	$\tau = \tan\left(\frac{\varphi}{2}\right)$

Base of the above Equations
$L = r^2\dot{\varphi}$, $\quad r = \frac{L^2/\mu}{1+e\cos f}$, $\quad \int r^2 \mathrm{d}\varphi = L\,(t-t_0)$

4.1.2 Solution of the Kepler-Equation (Ellipse)

Despite of its rather simple form, the Kepler-equation (165) is not easy to solve because $\tilde{E}(M)$ cannot be explicitly formulated in closed form. Here we want to present some approximate solutions for the bound, elliptic motion.
One possibility offers the

Iterative Method:

$$\tilde{E}_{i+1} \; = \; M \, + \, e \; \sin \tilde{E}_i \tag{186}$$

with $i \in N$. As a beginning of the iteration we choose

$$\tilde{E}_0 \; = \; M \quad . \tag{187}$$

With the addition theorems for - and expansion of the trigonometric functions

$$\begin{aligned} \sin(x+y) \; &= \; \sin x \cos y \, + \, \sin y \cos x \\ \sin x \; \approx \; x \; - \; \frac{1}{6} x^3 \; + \; O(x^5) \;\; &; \;\; \cos x \; \approx \; 1 \; - \; \frac{1}{2} x^2 \; + \; O(x^4) \end{aligned}$$

one easily arrive at the three first iterative solutions

$$\tilde{E}_1 \; = \; M \; + \; e \sin M \tag{188}$$

$$\begin{aligned} \tilde{E}_2 \; &= \; M \; + \; e \; \sin(M + e \sin M) \; \approx \; M \; + \; e \sin M \cos(e \sin M) \; + \; e^2 \sin M \cos M \\ &\approx \; M \; + \; e \sin M \; + \; \frac{1}{2} e^2 \sin 2M \; + \; O(e^3) \end{aligned} \tag{189}$$

$$\tilde{E}_3 \; \approx \; M \; + \; \left(e - \frac{1}{8} e^3 \right) \sin M \; + \; \frac{1}{2} e^2 \sin 2M \; + \; \frac{3}{8} e^3 \sin 3M \; + \; O(e^4) \quad . \tag{190}$$

With a generalization via complete induction one arrives at the a Fourier series

$$\tilde{E} \; - \; M \; = \; \sum_{k=1}^{\infty} b_k(e) \; \sin kM \quad . \tag{191}$$

Unfortunately, this series diverges for $e > 0.66$ so that only numerical methods (Newtons approach etc. - see below) then serve the purpose.
Next we will present series expansions for elliptical orbits, which concern the most frequent problems:

Elliptical Expansions:

Very often eccentricities and inclinations are rather small $e; i \ll 1$ – for example in planetary rings the corresponding values are $\langle e \rangle; \langle i \rangle < 10^{-6}$. In such cases it is advantageous to Fourier-expand the Kepler-equation

$$\tilde{E} - M \; = \; e \sin \tilde{E} \tag{192}$$

with respect such small quantities . The right hand side of this equation is an uneven function in $\tilde{E}$ so that it makes sense to expand it in sine-functions in order construct the relation between $\tilde{E}$ and M:

$$e \sin \tilde{E} \; = \; \sum_{k=1}^{\infty} b_k(e) \sin kM \quad , \tag{193}$$

where the coefficients can be identified as

$$\begin{aligned}
b_k(e) \; &= \; \frac{2}{\pi} \int\limits_0^{\pi} \mathrm{d}M \; e \sin \tilde{E} \; \sin kM \\
&= \; \frac{2}{k\pi} \underbrace{\Big[e \sin \tilde{E} \cos kM \Big]_{\pi}^{0}}_{\equiv 0} + \frac{2}{k\pi} \int\limits_0^{\pi} \mathrm{d}M \; \left(\frac{\mathrm{d}\tilde{E}}{\mathrm{d}M} \; - \; 1 \right) \; \cos kM \\
&= \; \frac{2}{k\pi} \int\limits_0^{\pi} \mathrm{d}\tilde{E} \; \cos kM \; - \; \underbrace{\frac{2}{k\pi} \int\limits_0^{\pi} \mathrm{d}M \; \cos kM}_{\equiv 0} \quad .
\end{aligned} \tag{194}$$

In the first integral last line of Eq. (194) we replace the argument of the mean anomaly again by using the Kepler-equation $kM = k(\tilde{E} - e \sin \tilde{E})$ so that we obtain

$$b_k(e) \; = \; \frac{2}{k\pi} \int\limits_0^{\pi} \mathrm{d}\tilde{E} \; \cos \Big[k(\tilde{E} - e \sin \tilde{E}) \Big] \; = \; \frac{2}{k} \, J_k(ke) \quad . \tag{195}$$

The integral is one representation of the Bessel-functions of the first kind

$$J_k(x) \; = \; \frac{1}{k!} \; \left(\frac{x}{2} \right)^k \; \sum_{\nu=0}^{\infty} (-1)^{\nu} \; \frac{(x/2)^{2\nu}}{\nu!(k+1)(k+2)...(k+\nu)} \quad , \tag{196}$$

which describe the eccentricity dependence of the coefficients $b_k(e)$.
With all these presumptions we can write the approximate solution of the Kepler-equation for the elliptic motion

$$\tilde{E} \; = \; M \, + \, 2 \, \sum_{k=1}^{\infty} \frac{J_k(ke) \sin kM}{k} \tag{197}$$

$$\approx \; M \, + \, e \sin M \, + \, \frac{e^2}{2} \sin 2M \, + \, e^3 \left(\frac{3}{8} \sin 3M \; - \; \frac{1}{8} \sin M \right) \, + \, O(e^4) \quad , \tag{198}$$

which we have obtained in this order already via iteration.
In the following we will give some expressions, important for the derivation of the perturbation functions (- potentials), which all base upon the above elliptical expansion:

$$\frac{r}{a} \; = \; 1 \, + \, \frac{e^2}{2} \; - \; 2e \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{\mathrm{d}}{\mathrm{d}e} \; J_k(ke) \cos kM \tag{199}$$

$$\approx 1 \;-\; e\cos M \;+\; \frac{e^2}{2}\,(1-\cos 2M) \;+\; O(e^3) \quad ,$$

$$\left(\frac{a}{r}\right)^3 \;=\; 1 \;+\; 3e\cos M \;+\; e^2\left(\frac{3}{2} \;+\; \frac{9}{2}\cos 2M\right) \;+\; O(e^3) \quad , \tag{200}$$

$$\cos\tilde{E} \;=\; -\frac{1}{2}e \;+\; 2\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\mathrm{d}}{\mathrm{d}e}\; J_k(ke)\cos kM \tag{201}$$

$$\approx\; \cos M \;+\; \frac{e}{2}(\cos 2M - 1) \;+\; \frac{3e^2}{8}(\cos 3M - \cos M) \;+\; O(e^3)$$

$$\sin f \;=\; 2\sqrt{1-e^2}\;\sum_{k=1}^{\infty}\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}e}\; J_k(ke)\sin kM \tag{202}$$

$$\approx\; \sin M \;+\; e\sin 2M \;+\; e^2\left(\frac{9}{8}\sin 3M \;-\; \frac{7}{8}\sin M\right) \;+\; O(e^3)$$

$$\cos f \;=\; -e \;+\; \frac{2(1-e^2)}{e}\;\sum_{k=1}^{\infty} J_k(ke)\cos kM \tag{203}$$

$$\approx\; \cos M \;+\; e(\cos 2M - 1) \;+\; \frac{9e^2}{8}(\cos 3M - \cos M) \;+\; O(e^3)$$

$$f \;-\; M \;=\; \sum_{k=1}^{\infty}\frac{e^k}{k!}\;\frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}}\left\{2\sin M \;-\; \frac{3}{4}e\sin 2M \;+ \ldots\right\} \tag{204}$$

$$\approx\; 2e\sin M \;+\; \frac{5}{4}e^2\sin 2M \;+\; e^3\left(\frac{13}{12}\sin 3M \;-\; \frac{1}{4}\sin M\right) \;+\; O(e^4) \quad .$$

Apart from the above analytical methods, numerical schemes are widely used, as for instance, the *Newton iteration*-scheme which returns the *Zeros* of the $\tilde{E}_0$ function

$$F(\tilde{E}) \;=\; \tilde{E} \;-\; e\sin\tilde{E} \;-\; M \;=\; 0 \quad , \tag{205}$$

as a solution. **Homework:** programming the equation (205) in C, FORTRAN or IDL or Python.

Gyrocentric approximation:

Goal of this expansion is to express the elliptic motion in the frame of a gyro-center – also called guiding center. The latter is represented with a circular orbit with the same semi-major axis a as that of the ellipse (meaning also the same orbital period $T(a)$). This motion is also called epi-cyclic and is usually taken for small eccentricities $e \ll 1$ in the connection of the perturbation theory. Next we will proof that the trajectory obtained in the guiding center frame is also an ellipse.

Mathematical expression of the fact that the orbital period of the ellipse and the guiding center circle is obtained the circular force balance

$$\left(\dot{M}\right)^2 \; a \;=\; n^2 a \;=\; \frac{\mu}{a^2} \quad , \tag{206}$$

with the common semi-major axis a.
We put our origo of the frame (x, y) on a (reference) point on the circle whose mean motion is $M(t)$ with the unit vector $\vec{e}_x = \cos M\; \vec{i} + \sin M\; \vec{j}$ pointing always to that origin of the co-rotating frame. The location of our test-particle moving on an ellipse is given by $\vec{R} = R\vec{e}_R$ und $X = R\cos f$; $Y = R\sin f$. The unit vectors $\vec{i}$ and $\vec{j}$ are the X and Y directions of our inertial system. The elliptic particle motion is given by

$$\vec{R} \;=\; R\,\vec{e}_R \;=\; \frac{a(1-e^2)}{1+e\cos f}\;\vec{e}_R \tag{207}$$

$$\vec{e}_R \;=\; \cos f\;\vec{i} \;+\; \sin f\;\vec{j} \quad . \tag{208}$$

It is plausible that for small $e \ll 1$ the anomalies M and f are close to each other and the particle fulfills a closed trajectory around the gyro-center at $\vec{A} = a\vec{e}_x$. Now the difference $f - M$ (see expansions (204)) becomes the argument of the trajectory in the co-rotating coordinate system, which we want to derive now based upon the difference vector $\vec{R} - \vec{A}$, respective upon its components which read

$$x \;=\; R\;\cos(f - M) \;-\; a \quad ; \quad y \;=\; R\sin(f - M) \tag{209}$$

with

$$R^2 \;=\; X^2 + Y^2 \quad \text{und} \quad r^2 \;=\; x^2 + y^2 \quad ,$$

where r measures the distance to the gyro-center. The coordinate x oints in the radial direction $\vec{e}_x$ of the moving frame, y points perpendicular to it.
Using the lowest expansion (204) for the difference angle $f - M \approx 2e\sin M$ we re-write the arguments of the trigonometric functions in Eqs. (gyro1) and further we expand the sin-function one obtains:

$$\sin(f - M) \;\approx\; \sin(2e\sin M) \;\approx\; 2e\sin M \quad\Rightarrow\quad y \;\approx\; 2ae\sin M \quad . \tag{210}$$

With $\cos(f - M) \approx 1$ and $R \approx a(1 - e\cos f) \approx a(1 - e\cos M)$ we obtain

$$x \;=\; -ae\cos M \quad . \tag{211}$$

Finally, with the Eqs. (210) and (211) one may identify the epicyclic motion clearly as an ellipse

$$\frac{x^2}{(ea)^2} + \frac{y^2}{(2ea)^2} \;=\; 1 \tag{212}$$

with the semi-major axis $(2ae)$ in azimuthal direction and in the radial one with the semi-minor axis (ae) where term of the order $O(e^2)$ are neglected.

Orbital Vectors:

The Two-Body Problem can also be characterized with the aid of 2 vectors: the *angular momentum*, including its conservation, and the *Runge-Lenz-vector* . The latter will be derived and discussed in this Section. To this aim we represent the location vector $\vec{r}$ with by using orbit-vectors $\vec{A} = a\ \vec{e_a}$ and $\vec{B} = b\ \vec{e_b}$. Both point in the direction of the corresponding semi-axes: $(\vec{e_a}; \vec{e_b})$ of the ellipse and can be expressed with the relations (172) and (171) as:

$$\vec{r} = \vec{A}\left(\cos\tilde{E} - e\right) + \vec{B}\sin\tilde{E} \tag{213}$$

$$\text{with} \quad |\vec{r}| = a\left(1 - e\cos\tilde{E}\right)$$

$$\text{and} \quad \vec{A} = a\vec{e_a},\ \vec{B} = b\vec{e_b} = a\sqrt{1-e^2}\vec{e_b} \quad . \tag{214}$$

Differentiating the vector (213) with respect to time t we obtain the orbit velocity

$$\dot{\vec{r}} = \left(-\vec{A}\sin\tilde{E} + \vec{B}\cos\tilde{E}\right)\frac{\mathrm{d}\tilde{E}}{\mathrm{d}t} \quad . \tag{215}$$

Differentiating the Kepler equation (165) one arrives at $n\mathrm{d}t = (1 - e\cos\tilde{E})\mathrm{d}\tilde{E}$ so that one may write $\dot{\tilde{E}} = n(1 - e\cos\tilde{E})^{-1}$ which inserted in Eq. (215 gives for the velocity

$$\dot{\vec{r}} = \frac{\left(-\vec{A}\sin\tilde{E} + \vec{B}\cos\tilde{E}\right)n}{\left(1 - e\cos\tilde{E}\right)} \tag{216}$$

Having the the location vector $\vec{r}$ and the velocity $\vec{v} = \dot{\vec{r}}$ we can again formulate the angular momentum

$$\begin{aligned}\vec{L} &= \left[\vec{A}\left(\cos\tilde{E} - e\right) + \vec{B}\sin\tilde{E}\right] \times \left[\frac{\vec{B}\cos\tilde{E} - \vec{A}\sin\tilde{E}}{1 - e\cos\tilde{E}}\right] n \\ &= \frac{\vec{A}\times\vec{B}\left(1 - e\cos\vec{E}\right)n}{1 - e\cos\tilde{E}} = n\,\vec{A}\times\vec{B}\end{aligned} \tag{217}$$

In the following we compare the angular momentum (217), obtained with the orbit vectors, with that one (146) derived from the solution of the TBP, which, of course, gives with the orbit vectors (214) the same:

$$\left|\vec{L}\right| = na^2\sqrt{1-e^2} = \sqrt{\mu a\left(1-e^2\right)}$$

Now we introduce a new vector, which will turn out to be another additional integral of motion, the Runge-Lenz vector:

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}}\times\vec{L}\right) - \frac{\vec{r}}{r} \quad . \tag{218}$$

The meaning of this vector becomes clear after a few modifications of the above formula, e.g. expanding the double vector product $\dot{\vec{r}} \times \vec{L} = \dot{\vec{r}} \times (\vec{r} \times \dot{\vec{r}})$ gives

$$\vec{e} = \frac{1}{\mu}\left\{\dot{\vec{r}} \times \left(\vec{r} \times \dot{\vec{r}}\right)\right\} - \frac{\vec{r}}{r} = \frac{1}{\mu}\left\{\vec{r}\,\dot{\vec{r}}^2 - \dot{\vec{r}}(\dot{\vec{r}} \cdot \vec{r})\right\} - \frac{\vec{r}}{r} \,. \tag{219}$$

A geometric interpretation emerges if one inserts the expressions for $r, \dot{r}$, for the angular momentum $\vec{L}$ and energy E following from the solution of the TBP – Eqs. (139)-(140):

$$\begin{aligned}
\vec{r} &= r\vec{e_r} \;; \quad \dot{\vec{r}} = r\dot{\varphi}\vec{e_\varphi} + \dot{r}\vec{e_r} \;; \quad \vec{L} = r^2\dot{\varphi}\vec{e_z} \\
\Longrightarrow \vec{e} &= \frac{r^3\dot{\varphi}^2\vec{e_r}}{\mu} - \frac{\dot{r}r^2\dot{\varphi}\vec{e_\varphi}}{\mu} - \vec{e_r} = \frac{L^2\vec{e_r}}{\mu r} - \frac{\dot{r}r^2\dot{\varphi}\vec{e_\varphi}}{\mu r} - \vec{e_r} \\
&= \left(\frac{L^2}{\mu r} - 1\right)\vec{e_r} - \frac{\dot{r}L}{\mu}\vec{e_\varphi} \quad .
\end{aligned} \tag{220}$$

Squaring the Eq. (220) yields

$$e^2 = 1 + \frac{2L^2E}{\mu^2} \tag{221}$$

which is exactly the same as Eq. (143) – just re-arranges in favor of e^2. This value e is practically the *eccentricity vecor* $\vec{e} = e\vec{e_a}$ pointing from the focus to the pericenter. The same is obtained if one uses Eqs. (216) and (217) for evaluation of the Runge-Lenz vector (218) as:

$$\begin{aligned}
\vec{e} &= \frac{\left[-\vec{A}\sin\tilde{E} + \vec{B}\cos\tilde{E}\right] \times n^2\left[\vec{A} \times \vec{B}\right]}{\mu\left(1 - e\cos\tilde{E}\right)} - \frac{\vec{r}}{r} \\
&= \frac{n^2\left[-\vec{A} \times \left(\vec{A} \times \vec{B}\right)\sin\tilde{E} + \vec{B} \times \left(\vec{A} \times \vec{B}\right)\cos\tilde{E}\right]}{\mu\left(1 - e\cos\tilde{E}\right)} - \frac{\vec{A}\left(\cos\tilde{E} - e\right) + \vec{B}\sin\tilde{E}}{a\left(1 - e\cos\tilde{E}\right)} \\
&= \frac{n^2\left[a^2\vec{B}\sin\tilde{E} + b^2\vec{A}\cos\tilde{E}\right]}{\mu\left(1 - e\cos\tilde{E}\right)} - \frac{\vec{A}\left(\cos\tilde{E} - e\right) + \vec{B}\sin\tilde{E}}{a\left(1 - e\cos\tilde{E}\right)} \\
&\overset{\mu=n^2a^3}{=} \frac{\vec{B}\sin\tilde{E} + (1 - e^2)\vec{A}\cos\tilde{E} - \vec{A}\left(\cos\tilde{E} - e\right) - \vec{B}\sin\tilde{E}}{a\left(1 - e\cos\tilde{E}\right)} \\
&= \frac{\vec{A}\left(e - e^2\cos\tilde{E}\right)}{a\left(1 - e\cos\tilde{E}\right)} = \frac{e}{a}\vec{A} \;=\; e\;\vec{e_a}
\end{aligned} \tag{222}$$

These expression make clear that $\vec{e}$ is another integral of motion, because e is constant in time and the direction $\vec{e_a}$ can also not change!

With the **orbit vectors** and the **Runge-Lenz vector** the orbit of the TBP is completely described as:

$$\boxed{\vec{L} \;=\; n\vec{A} \times \vec{B} \;, \quad \vec{e} \;=\; e\frac{\vec{A}}{a}} \tag{223}$$

The Runge-Lenz vector points in the direction of the pericenter in the orbital plane whereas the direction of the angular momentum $\vec{L} \propto \vec{A} \times \vec{B}$ defines the orbital plane in the 3D configuration space – the moduli of the Orbit-vectors characterize the type of the orbit (Ellipse, Hyperbola, Parabola)!

4.1.3 Three-dimensional Orbitelements

In case of perturbations become important, that an inertial system is chosen
If one deals with the unperturbed *TBP*, the following relations do hold:

$$\dot{\vec{L}} = \dot{\vec{e}} = \vec{e} \cdot \vec{L} = 0 \quad , \tag{224}$$

ensuring that orbit-vectors, as integrals of motion, do not change in time, and that the angular momentum $\vec{L}$ is perpendicular on to the orbital plane.
In the 3D-configuration space (e.g. cartesian coords.: x, y, z), the *angular momentum* $\vec{L} = n\vec{A} \times \vec{B}$, i.e. the *orbit vectors*, and the *Runge-Lenz vector* $\vec{e}$ can be expressed as follows: One projects vector $\vec{L}$ in the x-y plane so that the either vectors can written in terms of the inclination i and the longitude of the ascending node Ω:

$$\vec{L} = L \begin{pmatrix} \sin i \sin \Omega \\ -\sin i \cos \Omega \\ \cos i \end{pmatrix} \tag{225}$$

$$\vec{e_Q} = \cos \Omega \vec{e_x} + \sin \Omega \vec{e_y} = \frac{\vec{e_z} \times \vec{L}}{\left| \vec{e_z} \times \vec{L} \right|} \qquad \text{ascending node} \tag{226}$$

$$\cos i = \frac{\vec{e_z} \cdot \vec{L}}{L} \qquad \text{inclination} \tag{227}$$

$$\cos \omega = \frac{\vec{e} \cdot \vec{e_Q}}{e} \qquad \text{pericenter} \quad . \tag{228}$$

With these trigonometric relations plus Eq. (223) and the related orbital elements $a, e, i, \tilde{\omega} = \Omega + \omega, \Omega$ and t_0 the motion of a test particle in a point-mass central field is fully described.

a - große Halbachse (Energie)
e - Exzentrität (Drehimpuls)
i - Inklination (z-Komponente von $\vec{L}$)
ω - Länge zum Perizentrum
Ω - Knotenlänge (Lage von $\vec{L}$ in $\vec{e_x}$ und $\vec{e_y}$)

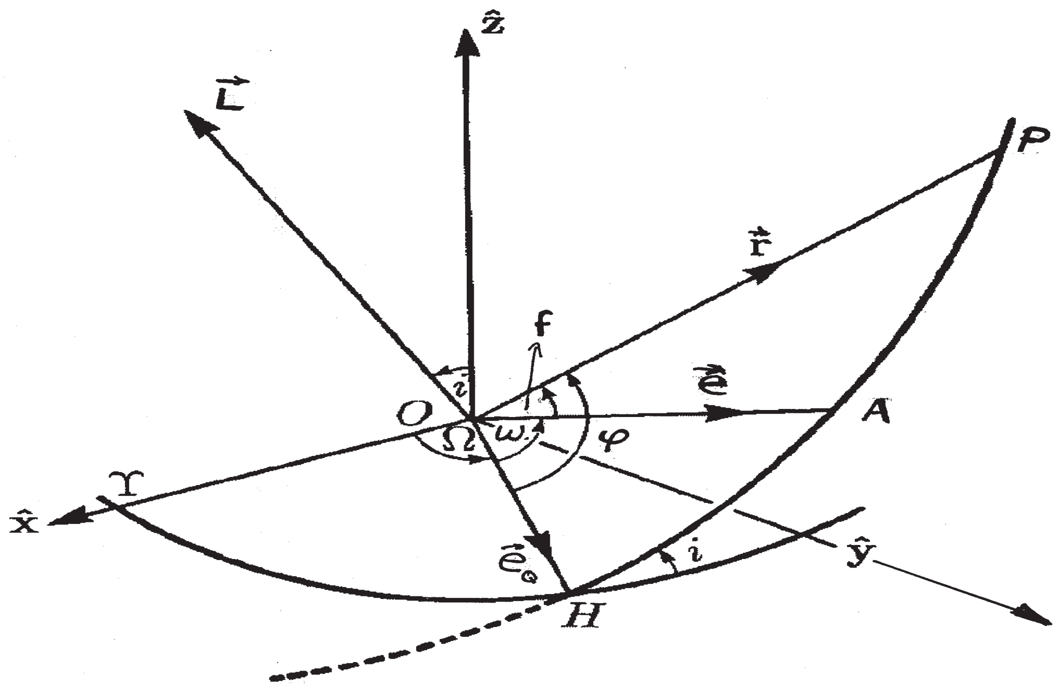


Abbildung 6: Lage der dreidimensionalen Orbitelemente

4.2 Perturbed TBP

The perturbation theory of the Kepler-solution is based upon the following equation:

$$\begin{aligned} \ddot{\vec{r}} &= -\frac{\mu}{r^3}\vec{r} - \nabla\Phi'(\vec{r}) \\ &= -\frac{\mu}{r^3}\vec{r} + \vec{F}_p \quad , \end{aligned} \tag{229}$$

with the perturbation potential Φ'. In a Hamiltonian formulation one may write

$$H(\vec{p},\vec{r}) \;=\; H_0(\vec{p}_0,\vec{r}_0) \,+\, H'(\vec{p}',\vec{r}) \;=\; H_0 \,-\, \nabla\Phi'(\vec{r}) \tag{230}$$

where H_0 is the unperturbed Hamiltonian of the TBP and primed values denote the changes. The perturbations are small so that we require the condition $||\Phi'/H_0|| \ll 1$ resulting in only moderate (resp. small) modifications of the orbit: $\vec{r}(t)$ and $\vec{p}(t) \propto \dot{\vec{r}}$. Thus, in this Section we will drop the primes because the unperturbed quantities are doubtlessly characterized by the index 0.
In order to solve Eq. (229) we use the method of the variation of constants $\vec{c} = \{a, e, \tilde{\omega}, i, \Omega, t_0\}$ – a 6-tupel or 6D preudo-vector combining all integrals of motion of the TBP, where we allow for $\partial_t\vec{c} \neq 0$ (in the following Eqs., Einstein-sums are used):

$$\begin{aligned} \vec{r} &= \vec{r}(c_1,\ldots,c_6,t) = \vec{r}(\vec{c},t) \\ \dot{\vec{r}} &= \vec{v}(\vec{c},t) \;= \\ &= \vec{v} = \frac{\partial\vec{r}}{\partial t} + \frac{\partial\vec{r}}{\partial\vec{c}}\cdot\frac{\partial\vec{c}}{\partial t} = \frac{\partial\vec{r}}{\partial t} + \left[\frac{\partial x_i}{\partial c_j}\vec{e}_i\vec{e}_j\right]\cdot\frac{\partial c_k}{\partial t}\vec{e}_k \end{aligned}$$

$$= \frac{\partial \vec{r}}{\partial t} + \frac{\partial x_i}{\partial c_j}\frac{\partial c_j}{\partial t}\vec{e}_i = \frac{\partial \vec{r}}{\partial t} + \underbrace{\frac{\partial \vec{r}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t}}_{\equiv 0} \tag{231}$$

$$\Longrightarrow \ddot{\vec{r}} = \dot{\vec{v}} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{c}}{\partial t} \cdot \frac{\partial \vec{v}}{\partial \vec{c}} \quad . \tag{232}$$

It should be noted that a restricting condition is used here

$$\frac{\partial \vec{r}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t} = 0 \tag{233}$$

so that the velocity and acceleration of the perturbed particle can be written

$$\vec{v} = \frac{\partial \vec{r}}{\partial t} \tag{234}$$

$$\dot{\vec{v}} = \frac{\partial^2 \vec{r}}{\partial t^2} + \frac{\partial \vec{c}}{\partial t} \cdot \frac{\partial \vec{v}}{\partial \vec{c}} \tag{235}$$

giving for the perturbed Kepler-equation:

$$\underbrace{\frac{\partial^2 \vec{r}}{\partial t^2} + \frac{\mu}{r^3}\vec{r}}_{\text{TBP} \Rightarrow 0} + \frac{\partial \vec{v}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t} = -\nabla\Phi'(\vec{r}) \quad . \tag{236}$$

The sum of the first two terms vanishes because it fulfills the equation of motion of the TBP – the rest will lead to the variation of the constants.
How could condition (233) be understood? For each component $[x_i \in (x, y, z)]$ of the location $\vec{r}$ we have a scalar-product between the 6D gradient $\partial_{\vec{c}} x_i = \nabla_{\vec{c}} x_i$ and the 6D pseudo-vector of the time-changes of the integrals of motion $\dot{\vec{c}} = \partial_t \vec{c}$, which equates to zero:

$$\nabla_{\vec{c}}\, x_i \cdot \frac{\partial \vec{c}}{\partial t} = 0 \quad .$$

First this means, that any summand of the form $\partial_{c_j} x_i \dot{c}_j \rightarrow 0$ is a rather small value so that even its sum can safely be neglected. This is because each factor, $\left|\partial_{c_j} x_i\right|$ and $|\dot{c}_j|$, itself is supposed to be rather small so that their product is tiny in the sense of second order which is going to be neglected here.

In the end the condition (233) means: the point $\vec{r} = \vec{r}_0$ of the unperturbed TBP-trajectory (cone-section, ellipse ...) is not changed $\left(\partial_t \vec{r}\right|_0 = 0)$ by the perturbation $(-\nabla\Phi')$ but only causes mild changes of the velocity $(\partial_t \vec{v} \neq 0)$ at the otherwise Keplerian trajectory.
In this sense at the fixed locus $\vec{r} = \vec{r}_0$ we expand the orbit to characterize dynamical changes caused by $-\nabla\Phi'$, while the character/type of the orbit as a cone section is still conserved (meaning an ellipse stays an ellipse). At the same time the integrals of motion $\vec{c}$ change only slowly (creeping change) with respect to orbital dynamical time scales – e.g. the orbital period $T \propto n^{-1}$. With all these assumptions the perturbed equations of motion read:

$$\frac{\partial \vec{v}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t} \;=\; -\nabla\Phi'(\vec{r}) \;=\; \nabla\mathcal{R} \tag{237}$$

$$\frac{\partial \vec{r}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t} \;=\; 0 \;\;, \tag{238}$$

with the parameter vector $\vec{c} \;=\; \{a, e, i, \tilde{\omega}, \Omega, t_0\}$. Here we have introduced the perturbation function $\mathcal{R} = -\nabla\Phi'$ which is frequently used in Astrophysics – especially in the anglo-american scientific community.
This inhomogeneous system of equations has to be solved under the assumption, that the constant expansion coefficients taken at the given expansion point $\vec{r}$, being $\frac{\partial \vec{v}_0}{\partial \vec{c}}$ and $\frac{\partial \vec{r}_0}{\partial \vec{c}}$, are taken from the unperturbed Kepler-solution, as argued in the paragraph above Eqs.(238).
As a next step, we manipulate the system:

$$\begin{aligned} \frac{\partial \vec{r}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t} &= 0 && \Big|\left(-\frac{\partial \vec{v}}{\partial c_i}\right) \\ \frac{\partial \vec{v}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t} &= \frac{\partial \mathcal{R}}{\partial \vec{r}} && \Big|\left(\frac{\partial \vec{r}}{\partial c_i}\right) \end{aligned}$$

and add the resulting equations, so that we obtain for the right hand side

$$-\frac{\partial \Phi'}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial c_i} \;=\; \frac{\partial \mathcal{R}}{\partial c_i} \quad .$$

so that can write for the system

$$\left(\frac{\partial \vec{v}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t}\right) \cdot \frac{\partial \vec{r}}{\partial c_i} \;-\; \left(\frac{\partial \vec{r}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t}\right) \cdot \frac{\partial \vec{v}}{\partial c_i} \;=\; \frac{\partial \mathcal{R}}{\partial c_i} \tag{239}$$

$$\frac{\partial \vec{r}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t} \;=\; \frac{\partial x_i}{\partial c_j}\frac{\partial c_j}{\partial t}\vec{e}_i \qquad \text{(Einstein sum)} \quad .$$

Here is the moment to repeat the pseudo-vectors (for brevity we use Einstein-sum in cartesian coords), the phase space variables are the location $\vec{r} = x_l \vec{e}_l$ and the velocity (momentum) $\vec{v} = v_l \vec{e}_l$; both 3D with $l \in (1,3)$; whereas the 6D parameter-vector reads $\vec{c} = c_k \vec{e}_k$. With these definitions, we can write for the derivatives involved and their combinations appearing in Eq. (239):

$$\frac{\partial \vec{v}}{\partial \vec{c}} \;=\; \frac{\partial v_l}{\partial c_j}\,\vec{e}_l\vec{e}_j \qquad ; \qquad \frac{\partial \vec{c}}{\partial t} \;=\; \frac{\partial c_k}{\partial t}\,\vec{e}_k$$

$$\left[\frac{\partial \vec{v}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t}\right] \cdot \frac{\partial \vec{r}}{\partial c_i} \;=\; \left[\frac{\partial x_l}{\partial c_i}\frac{\partial v_l}{\partial c_j}\right]\,\frac{\partial c_j}{\partial t}$$

$$-\left[\frac{\partial \vec{r}}{\partial \vec{c}} \cdot \frac{\partial \vec{c}}{\partial t}\right] \cdot \frac{\partial \vec{v}}{\partial c_i} \;=\; -\left[\frac{\partial v_l}{\partial c_i}\frac{\partial x_l}{\partial c_j}\right]\,\frac{\partial c_j}{\partial t}$$

Inserting these expressions in Eq. (239) and rearranging efficiently one arrives at the system of perturbative differential equations

$$\begin{pmatrix} [c_1, c_1] & \cdots & [c_1, c_6] \\ \vdots & & \vdots \\ [c_6, c_1] & \cdots & [c_6, c_6] \end{pmatrix} \begin{pmatrix} \frac{\partial c_1}{\partial t} \\ \vdots \\ \frac{\partial c_6}{\partial t} \end{pmatrix} = \hat{\mathcal{C}}\,\dot{\vec{c}} = \begin{pmatrix} \frac{\partial \mathcal{R}}{\partial c_1} \\ \vdots \\ \frac{\partial \mathcal{R}}{\partial c_6} \end{pmatrix} = \frac{\partial \mathcal{R}}{\partial \vec{c}} \quad , \tag{240}$$

with Lagrange-brackets, its definitions and symmetry-properties (**Homework**):

$$[c_i, c_j] \;=\; \sum_{k=1}^{3} \left(\frac{\partial x_k}{\partial c_i} \frac{\partial \dot{x}_k}{\partial c_j} - \frac{\partial x_k}{\partial c_j} \frac{\partial \dot{x}_k}{\partial c_i} \right) \tag{241}$$

$$[c_i, c_i] \;=\; 0 \;,\; [c_i, c_j] \;=\; -\,[c_j, c_i] \quad . \tag{242}$$

With the relations (242) the above matrix $\hat{\mathcal{C}}$ simplifies to 15 instead 36 coefficients which are non-zero. Furthermore, the Lagrange-brackets are in time independent (resulting the assumption that orbital changes are expanded starting from a fixed location $\vec{r}$).

$$\frac{\partial}{\partial t}[c_i, c_j] \;=\; 0 \quad .$$

Inverting the Lagrange-coefficient matrix to yield $\hat{\mathcal{C}}^{-1}$, which obeys the identity matrix $\hat{I} \;=\; \hat{\mathcal{C}}^{-1}\hat{\mathcal{C}}$, and applying it to the system (240) yields the inverted system

$$\boxed{\dot{\vec{c}} \;=\; \hat{\mathcal{C}}^{-1}\, \frac{\partial \mathcal{R}}{\partial \vec{c}} \quad .} \tag{243}$$

Here is the first hurdle to derive the perturbation function $\mathcal{R} \;=\; -\Phi'$ respective the perturbation potential, the latter being an usual quantity astrophysics and celestial mechanics with roots back to the historical beginning of the scientific discipline. The calculation of the Lagrange-brackets $[c_i, c_j]$ and of the function $\mathcal{R}$ are subject of the subsequent Subsections.

4.2.1 Lagrange-brackets

Very briefly – first, one has to start from a definite coord-system whose x-y plane falls on top of the orbital plane

$$x \;=\; r\,\cos f \quad \text{und} \quad y \;=\; r \sin f \tag{244}$$

which has to be transformed in an inertial (3D) coord-system $\{X, Y, Z\}$ via the relations (81)-(85). With these transformation we arrive at the new coordinates expressed by the orbital elements and the true anomaly as

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \;=\; r(a, e, M) \begin{pmatrix} \cos\Omega\cos(\omega + f) - \sin\Omega\sin(\omega + f)\cos\imath \\ \sin\Omega\cos(\omega + f) + \cos\Omega\sin(\omega + f)\cos\imath \\ \sin(\omega + f)\sin\imath \end{pmatrix} \quad . \tag{245}$$

With this, and having in mind $\vec{r} = X_i \vec{e}_i$, and $\vec{v} = \dot{X}_i \vec{e}_i$ (Einstein-Sums), we are able to calculate the Lagrange-brackets (241).
In the following we will rather directly give the equations referring to recipe given in the last two subsections.

4.2.2 Lagrange-Perturbation Equations

With the knowledge of the perturbation function $\mathcal{R}$, the general perturbation equation (243), and further definitions made by Lagrange as

$$\lambda \;=\; M \;+\; \tilde{\omega} \;=\; n(t - t_0) \;+\; \tilde{\omega} \;=\; nt \;+\; \epsilon \tag{246}$$

with λ the mean longitude, M mean anomaly, $\tilde{\omega}$, the longitude of the peric-center, and ϵ the mean longitude of epoch, the moment of the peri-center passage is finally t_0.
With all this the Lagrange perturbation equations read:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial \mathcal{R}}{\partial \epsilon} \tag{247}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{\sqrt{1-e^2}}{nea^2}\left(1-\sqrt{1-e^2}\right)\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{nea^2}\frac{\partial \mathcal{R}}{\partial \tilde{\omega}} \tag{248}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{2}{na}\frac{\partial \mathcal{R}}{\partial a} + \frac{\sqrt{1-e^2}\left(1-\sqrt{1-e^2}\right)}{nea^2}\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan \imath/2}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial \imath} \tag{249}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin \imath}\frac{\partial \mathcal{R}}{\partial \imath} \tag{250}$$

$$\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{nea^2}\frac{\partial \mathcal{R}}{\partial e} + \frac{\tan \imath/2}{na^2\sqrt{1-e^2}}\frac{\partial \mathcal{R}}{\partial \imath} \tag{251}$$

$$\frac{\mathrm{d}\imath}{\mathrm{d}t} = -\frac{\tan \imath/2}{na^2\sqrt{1-e^2}}\left(\frac{\partial \mathcal{R}}{\partial \epsilon} + \frac{\partial \mathcal{R}}{\partial \tilde{\omega}}\right) - \frac{1}{na^2\sqrt{1-e^2}\sin \imath}\frac{\partial \mathcal{R}}{\partial \Omega} \quad . \tag{252}$$

The main challenge in formulating the problem is finally the derivation of the perturbation function $\mathcal{R}(\vec{c})$.

4.2.3 Perturbation function - Examples

In context of the gravitational potentials and Greens solutions we have already debated the perturbation potential which is caused by an oblate planet/central body

$$\mathcal{R} \;=\; -\frac{\mu}{r}\sum_{n=1}^{\infty} J_{2n}\left(\frac{R_p}{r}\right)^{2n} P_{2n}(\cos\theta) \quad . \tag{253}$$

The terms containing r^{-2n}, can expressed by the already given elliptic expansions (199) and (200) as a function of the mean anomaly M. Further it simply hold $\cos\theta = \sin\imath$.

Another important example is the perturbation caused by a third small (or quite distant) point mass m' at the location $\vec{r}'$:

$$\mathcal{R} \;=\; \frac{\gamma m'}{|\vec{r}' - \vec{r}|} \;-\; \gamma m' \, \frac{\vec{r}\cdot\vec{r}'}{r'^3} \quad . \tag{254}$$

In this function, the locations $\vec{r}(\vec{c})$ or respective $\vec{r}'(\vec{c}')$ must be expressed by the respective orbital elements of either masses ($\vec{c}$ und $\vec{c}'$), which is a rather difficult task. Special functions are needed, as for instance the Legendre polynomials whose generating function, $\propto 1/\,|\vec{r}' - \vec{r}|$, appears in Eq. (254) as

$$\frac{1}{|\vec{r}' - \vec{r}|} \;=\; \frac{1}{r'}\left[1 + \left(\frac{r}{r'}\right)^2 - 2\frac{r}{r'}\cos\Theta\right]^{-1/2} \quad . \tag{255}$$

The angle Θ is measured between both vectors $\vec{r}$ und $\vec{r}'$ which cannot simply expressed by the parameters $\vec{c}$ but can also be related to either inclinations $\imath$ and $\imath'$.
Generally the perturbing body's orbit is inclined to the orbital plane of the both prime-bodies – latter the allocate the bodies forming the TBP. The formulation of the function $\mathcal{R}$ can become quite involved. Detailed examples are discussed in the monography *Murray und Dermott* „Solar System Dynamics“ — for instance the derivation of $\mathcal{R}$ for a planar motion of the primaries M & m as well as the perturber m' in different configurations.
The basic principle, however, remains always the same: first, one determines the angle Θ from the geometry (e.g. planar motion: $\Theta = (f' + \tilde{\omega}') - (f + \tilde{\omega})$) giving $\cos\Theta$. With the addition theorems of the trigonometric functions and using results of elliptical expansions (199) - (204), for instance, to express the true anomaly $f + \tilde{\omega}$ as a function of the mean anomaly M.

4.2.4 Gauß Equations and Orbit Vectors

A more general way to derive the derivatives $\partial_t\vec{c}$ even for *non-conservative forces* — gas- or plasma drag forces which cannot be described by a potential or a Hamiltonian.
For the derivation we will use the orbit vectors (214) - (218), which are the *Runge-Lenz vector* and *angular momentum*

$$\begin{aligned} \vec{e_0} \;&=\; \frac{e_0}{a_0}\vec{A}_0 \;\;,\;\; \vec{L}_0 \;=\; \Omega_0\,\vec{A}_0 \times \vec{B}_0 \\ \dot{\vec{L}}_0 \;&=\; 0 \;\;;\;\; \dot{\vec{e_0}} \;=\; 0 \\ &\Longrightarrow \;\text{basic Kepler-solution labeled with index 0.} \end{aligned} \tag{256}$$

In order to calculate timely changes of the above TBP integrals, the perturbing force is vectorially expanded

$$\vec{F}(\vec{r}) \;=\; R\,\vec{e}_r \,+\, T\vec{e}_T \,+\, N\vec{e}_L \quad , \tag{257}$$

where the components and unit vectors are: first the radial direction (from the most massive primary M focus to m) $\vec{e}_r/r$, and the two others are defined as

$$\vec{e}_L = \frac{\vec{L}}{L} \quad ; \quad \vec{e}_T = \frac{\vec{L}\times\vec{r}}{rL} = \vec{e}_L \times \vec{e}_r \quad , \tag{258}$$

where further geometric relations of these unit vectors read

$$\vec{e}_r = \cos f\ \vec{e}_a + \sin f\ \vec{e}_b \quad ; \quad \vec{e}_T = \frac{\partial \vec{e}_r}{\partial f} \quad ; \quad \vec{e}_L = \vec{e}_r \times \vec{e}_T \tag{259}$$

$$\vec{e}_a = \cos\omega\ \vec{e}_Q + \sin\omega\ \vec{e}_\perp \quad ; \quad \vec{e}_b = \frac{\partial \vec{e}_a}{\partial \omega} \quad ; \quad \vec{e}_\perp = \vec{e}_L \times \vec{e}_Q \quad . \tag{260}$$

Only the perturbing force $\vec{F}$ can cause time changes of the otherwise conservatived quantities $\vec{L}_0$ and $\vec{e}_0$. Under usage of Eq. (229) one obtains owing differentiation of both vectors

$$\begin{aligned}
\dot{\vec{L}} &= \vec{r}\times\ddot{\vec{r}} = \vec{r}\times\left(-\frac{\mu}{r^3}\vec{r}+\vec{F}\right) = \vec{r}\times\vec{F} \\
\dot{\vec{L}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\mu a\,(1-e^2)}\frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\mu a\,(1-e^2)}\ \vec{e_L}\right) \\
\Longrightarrow \vec{r}\times\vec{F} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\mu a\,(1-e^2)}\ \vec{e_L}\right)
\end{aligned} \tag{261}$$

Analogously one obtains via differentiation of the Runge-Lenz vector following

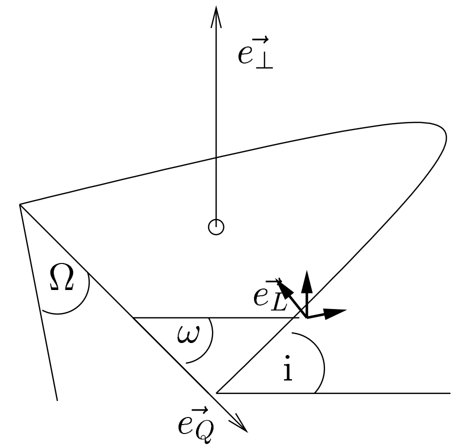
$$\begin{aligned}
\mu\dot{\vec{e}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vec{r}}\times\vec{L}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\frac{\vec{r}}{r}\right) = \ddot{\vec{r}}\times\vec{L}+\dot{\vec{r}}\times\dot{\vec{L}} - \mu\frac{\dot{\vec{r}}}{r}+\mu\frac{\vec{r}\dot{r}}{r^2} \\
&= \left(-\frac{\mu}{r^3}\vec{r}+\vec{F}\right)\times\left(\vec{r}\times\dot{\vec{r}}\right) + \dot{\vec{r}}\times\dot{\vec{L}} - \mu\frac{\dot{\vec{r}}}{r}+\mu\frac{\vec{r}\dot{r}}{r^2} \\
&= -\frac{\mu}{r^3}\left(\vec{r}\left(\vec{r}\cdot\dot{\vec{r}}\right) - \dot{\vec{r}}r^2\right) + \vec{F}\times\vec{L}+\dot{\vec{r}}\times\dot{\vec{L}} - \mu\frac{\dot{\vec{r}}}{r}+\mu\frac{\vec{r}\dot{r}}{r^2} \\
&= -\frac{\mu}{r^3}\vec{r}\dot{r}r + \frac{\mu}{r}\dot{\vec{r}} - \mu\frac{\dot{\vec{r}}}{r} + \vec{F}\times\vec{L}+\dot{\vec{r}}\times\left(\vec{r}\times\vec{F}\right)+\mu\frac{\vec{r}\dot{r}}{r^2} \\
\dot{e}\vec{e}_a + e\dot{\vec{e}}_a &= \mu^{-1}\,\vec{F}\times\vec{L}+\dot{\vec{r}}\times\dot{\vec{L}} \quad .
\end{aligned} \tag{262}$$

The generating equations of the perturbed orbit are described by Eqs. (261) and (262). In a next step one has to perform the time derivatives in Eqs. (261) and (262) for a given perturbation $\vec{F}$ leading to the desired changes $\frac{\partial a}{\partial t}$, $\frac{\partial e}{\partial t}$, $\frac{\partial \omega}{\partial t}$, $\frac{\partial \Omega}{\partial t}$, and $\frac{\partial \imath}{\partial t}$.

Energy & angular Momentum

In order to arrive at the time changes of the orbital elements $\dot{a}$, $\dot{e}$ etc., we will offer another, physically quite plausible and alternative possibility – the performance law:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \vec{F}\cdot\dot{\vec{r}} \tag{263}$$



$$\text{again with: } \vec{F} \;=\; R\vec{e_r} + T\vec{e_T} + N\vec{e_L}\ , \quad \text{and the unit vectors} \quad \vec{e_r} = \frac{\vec{r}}{r}$$

$$\vec{e}_T \;=\; \frac{\vec{L}}{L} \times \frac{\vec{r}}{r} \;=\; \vec{e}_L \times \vec{e}_r.\ ,\ \vec{e}_L = \frac{\vec{L}}{L} \quad .$$

With the force-components $R = \vec{F}\cdot\vec{e_r}$, $T = \vec{F}\cdot\vec{e_T}$, and $N = \vec{F}\cdot\vec{e_L}$, and having in mind that the unperturbed orbit lies in the plane (i.e. $\vec{e_L} \times \dot{\vec{r}} = 0$), one obtains for the energy change from Eq. (263):

$$\dot{E} \;=\; \vec{F}\cdot v_r\vec{e_r} \,+\, \vec{F}\cdot v_T\vec{e_T} \tag{264}$$

Note that we start with the perturbation evaluation at the unperturbed orbital plane defined by the vectors $\vec{r}$ and the velocity

$$\vec{v} \;=\; \dot{\vec{r}} \;=\; v_R\vec{e_r} \,+\, v_T\vec{e_T} \tag{265}$$

as defined/required by the perturbation theory condition (233). With the TBP-definition $E = -\mu/(2a)$ and its derivative $\dot{E} \;=\; \frac{\mu}{2a^2}\dot{a} \;=\; Rv_r \,+\, Tv_T$ one obtains:

$$\boxed{\frac{\mu}{2a^2}\dot{a} \;=\; \dot{r}\ R \,+\, r\dot{\varphi}\ T} \tag{266}$$

Now one just uses the relations obtained by the TBP-solution, i.e. Eq. (148) and its time derivative, we obtain the relations

$$r \;=\; \frac{p}{1 + e\cos f} \quad \text{with} \quad p = \frac{L^2}{\mu}$$

$$\dot{r} \;=\; \frac{ep\sin f}{(1+e\cos f)^2}\dot{\varphi} \;=\; e\sin f\,\underbrace{\frac{p^2}{(1+e\cos f)^2}}_{=r^2}\frac{\dot{\varphi}}{p}$$

$$\stackrel{(131)}{=} \;\; e\sin f \cdot \frac{L}{p}$$

and finally arrive at the radial speed

$$\boxed{\dot{r} \;=\; \frac{\mu}{L}\, e \sin f \quad .} \tag{267}$$

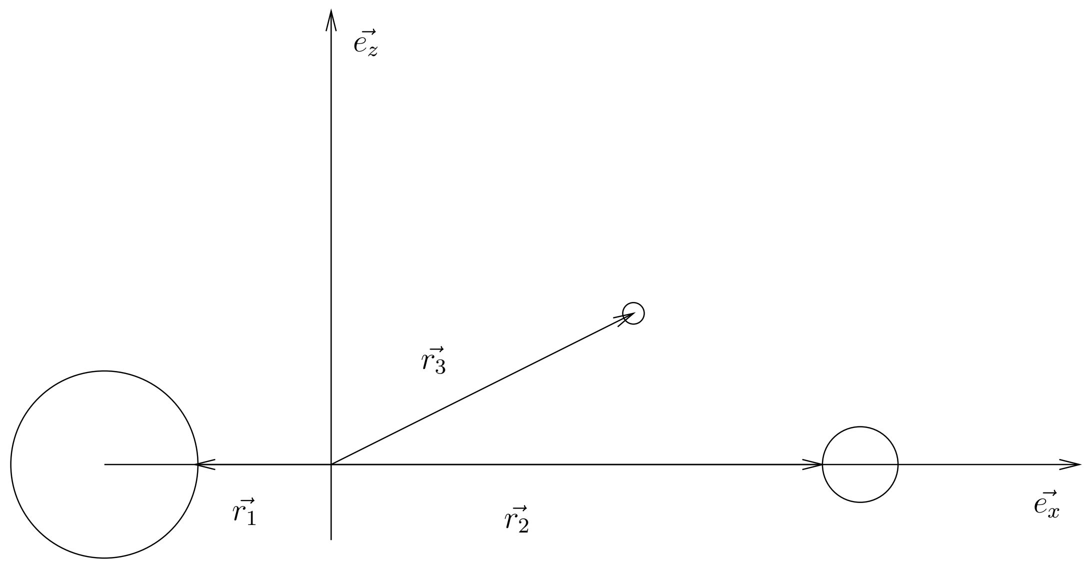
Inserting this expression and $\dot{\varphi} = L/r^2$ in Eq.(266 one obtains the Gauß equation for $\dot{a}$ below, where also the other variations of the orbital elements are presented:

$$\boxed{\begin{aligned}
\dot{a} &= \tfrac{2a^{3/2}}{\sqrt{\mu(1-e^2)}} \left[(e\sin f)\, R + T\,(1+e\cos f)\right] \\
\dot{e} &= \sqrt{\tfrac{a(1-e^2)}{\mu}} \left[R\sin f + T(\cos\tilde{E} + \cos f)\right] \\
\dot{\omega} &= \sqrt{\tfrac{a(1-e^2)}{\mu}} \left(\tfrac{1}{e}\right) \left[-R\cos f + T\sin f \left(\tfrac{2+e\cos f}{1+e\cos f}\right)\right] - \dot{\Omega}\cos i \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \sqrt{\tfrac{a(1-e^2)}{\mu}} \tfrac{N\cos\varphi}{1+e\cos f} \\
\dot{\Omega} &= \sqrt{\tfrac{a(1-e^2)}{\mu}} \tfrac{N\sin\varphi}{\sin i(1+e\cos f)}
\end{aligned}} \tag{268}$$

5 The Three-Body-Problem

The three-body problem (3BP) describes the motion of three bodies under their mutual gravitational interaction. Using a coordinate frame whose origo is fixed at the center of mass of the three massesn $\sum m_i \vec{r_i} = 0$, $i \in (1,3)$, the equation of motion read:

$$m_i \ddot{\vec{r_i}} \;=\; -\sum_{j,j\neq i} \gamma \frac{m_i m_j}{|\vec{r_i}-\vec{r_j}|^3}(\vec{r_i}-\vec{r_j}) \; i = 1,2,3 \tag{269}$$



Generally the three-body-problem is not solvable in a closed form – in order to proceed analytically simplifications are necessary, which will be discussed in the next subsections.

5.1 The Restricted Three Body Problem - RTB

The major conditions of the RTB problem are assumptions about the masses — we introduce $m_1 = M$, $m_2 = m$, and with $M \gg m \gg m_3$. The both massive bodies, M and m, are called primaries whose motion is not influenced by the third one m_3, so that it can be described by the two body problem (TBP). The equations motion of the TBP, describing the motion of the primaries $(i; j \in (1,2))$, read:

$$\ddot{\vec{r_i}} \;=\; -\gamma \frac{m_j}{|\vec{r_i}-\vec{r_j}|^3}(\vec{r_i}-\vec{r_j}) - \gamma \frac{m_3}{|\vec{r_i}-\vec{r_3}|^3}(\vec{r_i}-\vec{r_3}) \tag{270}$$

$$\;=\; -\gamma \left(\frac{m_j}{|\vec{r_i}-\vec{r_j}|^3}(\vec{r_i}-\vec{r_j}) + O(\frac{m_3}{m_i}) \right) \quad . \tag{271}$$

Introducing the relative coordinates between the primaries, i.e. the distance-vector $\vec{D} = \vec{r_2} - \vec{r_1}$, the TBP equation (Subsection 4) reads

$$\ddot{\vec{D}} \;=\; -\gamma(M+m)\frac{\vec{D}}{D^3} \;=\; -\mu \frac{\vec{D}}{D^3} \quad , \tag{272}$$

with the solution for an ellipse

$$D \;=\; \frac{a_p(1-e_p^2)}{1+e_p\cos f} \quad , \tag{273}$$

with the orbital elliptical parameters indexed with p.
The motion of the third body (m_3) is described in a frame co-rotating with the primaries to give the differential equation (containing also the inertia forces: Coriolis- & centrifugal forces)

$$\ddot{\vec r} + \dot{\vec\Omega}\times\vec r + 2\vec\Omega\times\dot{\vec r} + \vec\Omega\times(\vec\Omega\times\vec r) \;=\; -\nabla\Phi_G \quad , \tag{274}$$

where we have dropped the index 3 of the location vector of the third body, for simplicity. Next we choose a normalization according to the mean motion n and semi-major-axis a_p of the primaries according to

$$a_p^3\,\vec n^2 \;=\; \gamma\,(M+m) \;=\; \gamma M(1+\mu_*) \;=\; \mu \;=\; 1 \;\;;\;\; a_p \;=\; n = 1 \quad , \tag{275}$$

with the mass-ratio $\mu_* = m/M$ of the primaries. According Eq. (275) we may write $\gamma M \;=\; 1/(1+\mu_*)\,;\, \gamma m \;=\; \gamma\mu_* M \;=\; \mu_*/(1+\mu_*)$. With these definition the potentials involved read

$$\begin{aligned}\Phi_G \;=\; -\gamma\left(\frac{M}{|\vec r-\vec r_1|}+\frac{m}{|\vec r-\vec r_2|}\right) \;&=\; \frac{-1}{(1+\mu_*)}\left(\frac{1}{|\vec r-\vec r_1|}+\frac{\mu_*}{|\vec r-\vec r_2|}\right)\\ &=\; -\frac{r_2^*}{\vec r-\vec r_1} - -\frac{r_1^*}{\vec r-\vec r_2} \quad .\end{aligned} \tag{276}$$

Here we have introduced the mass parameters $r_1^* = \mu_*/(1+\mu_*)$, and $r_2^* = 1/(1+\mu_*)$ whose sum is simultaneously the normalized distance between either primaries $1 = r_1^* + r_2^*$. The following relations hold:

$$\begin{aligned}\vec r_1 \;=\; x_1\,\vec e_x \;=\; -r_1^*\,D\,\vec e_x \quad &; \quad \vec r_2 \;=\; x_2\,\vec e_x \;=\; r_2^*\,D\,\vec e_x\\ x_1 \;=\; -r_1^*\,D \quad &; \quad x_2 \;=\; r_2^*\,D \qquad (277)\\ x_2 - x_1 \;=\; D \quad &; \quad r_1^* + r_2^* \;=\; 1 \quad . \qquad (278)\end{aligned}$$

In other words, these normalized distances $r_{1/2}^*$ in principle stand for the gravity-parameters $\gamma M \to r_2^*$, and $\gamma m \to r_1^*$. This also means, that all state values in the RTB-problem are of the order $O(1)$, when using the normalizations (275) - (278) – in other words herewith "astronomically large" numbers will be avoided[6].

In the following, we shift the origin of the frame in the center of the planet (central body), because it's easier to estimate than the center of its mass. This translation is given by

$$\vec r \;=\; \vec R + \vec r_1 = \vec R - \frac{\mu_*}{1+\mu_*}D\vec{e_x} = \vec R \;-\; r_1^* D\vec{e_x} \quad , \tag{279}$$

[6]For the moduli of the distances of the primaries one may write: $r_{1/2} \;=\; \left|x_{1/2}\right|$ with $D = r_1 + r_2$.

and with it follows:

$$
\begin{aligned}
-\nabla\Phi_G \;&=\; \ddot{\vec{R}} \,-\, r_1^*\ddot{D}\vec{e_x} \,+\, \dot{\vec{\Omega}}\times\left(\vec{R}\,-\,r_1^*D\vec{e_x}\right) \\
&\quad +2\vec{\Omega}\times\left(\dot{\vec{R}}\,-\,r_1^*\dot{D}\vec{e_x}\right)\,+\,\vec{\Omega}\times\left(\vec{\Omega}\times\left(\vec{R}-r_1^*D\vec{e_x}\right)\right) \quad . \qquad (280)
\end{aligned}
$$

At first, we express the inertia terms containing r_1^* and D owing to the shift $\vec{r_1} = -r_1^*\vec{D}$ of the origo of the frame. Therefore, we remind the TBP for $\ddot{\vec{D}}$, with the solution(273) and its time derivative, needed later:

$$
\dot{D} \;=\; e_p \sin f\,\frac{D^2}{L_p^2} \quad . \qquad (281)
$$

According to the conservation of the center of mass we again have:

$$
\begin{aligned}
M\vec{r_1} + m\vec{r_2} \;&=\; Mx_1\vec{e_x} \;+\; mx_2\vec{e_x} \;=\; 0 \\
x_1 \;&=\; -\frac{m}{M}x_2 = -\mu_* x_2
\end{aligned}
$$

and remembering the relations (277) we confirm that

$$
D \;=\; r_1 \,+\, r_2 \quad \Longrightarrow \quad r_1 \;=\; \frac{\mu_* D}{1+\mu_*} \;\;,\;\; r_2 = \frac{D}{1+\mu_*} \quad .
$$

Back to the additional inertia terms in Eq. (280) containing the parameter $r_1^* \;=\; \mu_*/(1+\mu_*) \;=\; \gamma m$. We will show that these inertia forces can be expressed by an additional potential. To this aim an inertia system is used for the eqnation of motion of the third body, index 3 is dropped from now on for simplicity:

$$
\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{\vec{r}-\vec{r_1}}{(1+\mu_*)|\vec{r}-\vec{r_1}|^3} \,-\, \frac{\mu_*}{1+\mu_*}\frac{\vec{r}-\vec{r_2}}{|\vec{r}-\vec{r_2}|^3} \quad , \qquad (282)
$$

and with the transformation $\vec{R} = \vec{r} - \vec{r_1}$ one obtains:

$$
\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\vec{r_1}}{\mathrm{d}t^2} \;=\; -\,\frac{1}{1+\mu_*}\frac{\vec{R}}{R^3} \,-\, \frac{\mu_*}{(1+\mu_*)}\cdot\frac{\vec{R}+\vec{r_1}-\vec{r_2}}{|\vec{R}+\vec{r_1}-\vec{r_2}|^3}
$$

With $\vec{D} \;=\; \vec{r_2}-\vec{r_1}$, $\vec{r_1} \;=\; -r_1\,\vec{e_x}$, as well as with the motion of the primary:

$$
\frac{\mathrm{d}^2\vec{r_1}}{\mathrm{d}t^2} \;=\; -\frac{\mu_*}{1+\mu_*}\frac{\vec{r_1}\,-\,\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} = -r_1^*\,\frac{\vec{D}}{D^3} \;=\; -\,r_1^*\,\nabla\left[\Omega^2\left(\vec{R}\cdot\vec{D}\right)\right] \quad . \qquad (283)
$$

In summary: one obtains the gradient of a potential as a result of the shift of the origo so that the equation of motion in an inertial system finally reads

$$
\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} \;=\; -\,r_2^*\,\frac{\vec{R}}{R^3} - r_1^*\left(\frac{\vec{R}-\vec{D}}{|\vec{R}-\vec{D}|^3} - \frac{\vec{D}}{D^3}\right) \;=\; -\nabla\Phi_G' \qquad (284)
$$

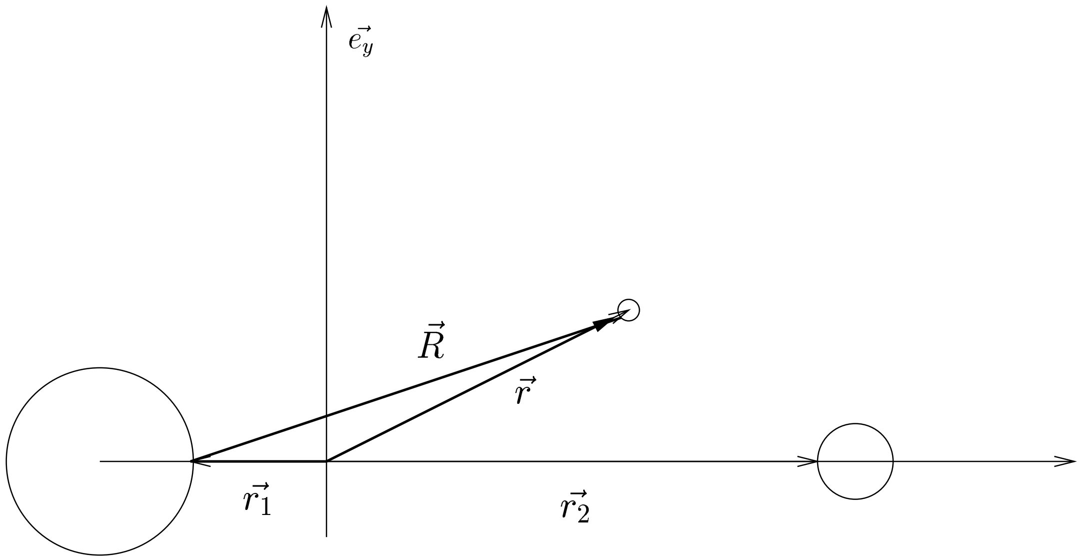


Abbildung 7: Shift of the origo of the coordinate frame into the center of the Primary.

with the full potential

$$\Phi'_G \;=\; -\frac{r_2^*}{R} \;-\; r_1^* \left[\frac{1}{|\vec{R}-\vec{D}|} \;+\; \Omega^2(D)\,\left(\vec{R}\cdot\vec{D}\right)\right] \;=\; \Phi_G \;-\; r_1^*\,\Omega^2(D)\,\left(\vec{R}\cdot\vec{D}\right) \quad . \tag{285}$$

The additional potential term describes a tiny "wobbling" motion of the central body (massive primary) about the center of mass of both primaries. This effect has been successfully used for the detection of *Exo-planets* – i.e. planets around other stars than our Sun.
The description via $\vec{R}$ and $\vec{D}$ has the advantage that the distance between the primaries can be estimated quite accurately whereas often the masses, and thus the center of mass remains vague.

Now we turn back to the co-rotating frame where one has to account also for the inertia pseudo forces, so that the equation of motion now reads

$$\ddot{\vec{R}} + \dot{\vec{\Omega}}\times\vec{R} + 2\vec{\Omega}\times\dot{\vec{R}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{R}) \;=\; -\nabla\Phi'_G \quad . \tag{286}$$

Circular Restricted Three-Body Problem

A further simplification of the RTB-problem is obtained by assuming the primaries to move in circular orbits $\dot{\Omega}=0,\ \vec{\Omega}=\vec{e_z}$ with normalized distances as
$D=1\ ;\ \ddot{D}=\dot{D}=0\ ;\ \Omega=1$, then the equation of motion becomes:

$$\ddot{\vec{R}} + 2\vec{e_z}\times\dot{\vec{R}} + \vec{e_z}\times(\vec{e_z}\times(\vec{R}-r_1^*\vec{e_x})) \;=\; -\nabla\Phi_G \quad , \tag{287}$$

being the *circular restricted three-body problem* (henceforth cRTB-problem). It is a nice **homework** to show that a transition $\Phi_G \to \Phi'_G$ results in a vanishing (appearance) of

the term containing r_1^* on the left hand side [hints: double cross product-expansion, analyzing the term $-\nabla(\Phi_G' - \Phi_G)$].
Expanding the double cross-vector product and write each component separately one obtains the following equation of motion of the cRTB-problem:

$$\ddot{\vec{R}} + 2\vec{e_z} \times \dot{\vec{R}} + z\vec{e_z} - \vec{R} + r_1^*\vec{e_x} \quad = \quad -\nabla\Phi_G \quad , \tag{288}$$

and this written in spatial cartesian coordinates of the vector $\vec{R} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ – where x points in the direction from the primary to the secondary $\vec{e}_x = \vec{D}/D$ and y measures the orbital motion direction if the primaries ($\vec{e}_z \times \vec{e}_x = \vec{e}_y$) – reads

$$\begin{aligned} \ddot{x} - 2\dot{y} - (x - r_1^*) &= -\partial_x\Phi_G \\ \ddot{y} + 2\dot{x} - y &= -\partial_y\Phi_G \\ \ddot{z} &= -\partial_z\Phi_G \end{aligned} \tag{289}$$

Jacobi-Integral

Multiplying Eqs. (289) with the corresponding speeds $\dot{x},\ \dot{y}\ ,\ \dot{z}$ and subsequently adding up the resulting equations one obtains (*note, that Coriolis-terms cancel out*):

$$\begin{aligned} \ddot{x}\dot{x} - 2\dot{y}\dot{x} - (x - r_1^*)\dot{x} &= -\dot{x}\partial_x\Phi_G \\ \ddot{y}\dot{y} + 2\dot{x}\dot{y} - y\dot{y} &= -\dot{y}\partial_y\Phi_G \\ \ddot{z}\dot{z} &= -\dot{z}\partial_z\Phi_G \\ \Longrightarrow \ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} - (x - r_1^*)\dot{x} - y\dot{y} &= -\frac{\mathrm{d}\Phi_G}{\mathrm{d}t} \end{aligned}$$

which can then be re-formulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2} - \frac{(x - r_1^*)^2}{2} - \frac{y^2}{2}\right) + \frac{\mathrm{d}\Phi_G}{\mathrm{d}t} \quad = \quad 0 \quad .$$

This equation can be directly integrated so that one is lead to the *Jacobi-integral*:

$$\boxed{\frac{v^2}{2} - \frac{(x - r_1^*)^2}{2} - \frac{y^2}{2} + \Phi_G = \mathcal{C} \qquad \text{Jacobi-Integral}} \tag{290}$$

Equation (290) comprises the kinetic energy and the gravitational potential (276).

Attention: The Jacobi-Integral is NOT identical with the conservation of the orbital energy.

Example:

With the change of the semi-major axis a if the test particle the specific orbital energy according to $E \sim -\frac{\mu}{2a}$ – which is always possible with the full *Three Body Problem*. In

the field of space sciences this property of the cRTB-problem has been applied to perform *Swing-by* of a space craft at a planet – the so-called gravity assist where elliptical orbits can even be transformed to hyperbolic ones [ellipses $E < 0 \Rightarrow$ hyperbelolas $(E > 0)$] – demonstrated at the *Voyager*-space probes (in principle with all deep-space space vehicles: e.g. *Pioneer, Galileo, Ulysses, Cassini, New Horizon*, just to mention a view). With this maneuver the space ship "steals" energy needed for this transition from the orbital energy of the planet – in sum the mechanical energy of all three bodies is, of course, conserved.
Summarizing: $\mathcal{C}$ is just a pseudoenergy!

$$\frac{v^2}{2} + \Phi_{eff} \;=\; \mathcal{C} \tag{291}$$

with the three-body effective potential

$$\Phi_{eff} \;=\; -\frac{(x-r_1)^2 + y^2}{2} - \frac{r_2}{R} - \frac{r_1}{S} \tag{292}$$

where the distance vector between test particle and secundary (smaller primary) is defined as $\vec{S} = \vec{R} - \vec{D} = \vec{R} - \vec{e_x}$ and the scaling $n = r_1 + r_2 = 1$.

5.1.1 Lagrange (fixed) points

The Lagrange points are *fixed points* of the nonlinear dynamical cRTP-system – also dubbed Lagrange points. They are defined by the following conditions:

$$\boxed{\dot{\vec{v}} = 0 \quad\Longleftrightarrow\quad \dot{x}_i \;=\; 0\;;\;\; \partial_{x_i}\Phi_{eff} \;=\; 0} \tag{293}$$

Further, also all other higher derivatives $\ddot{x}_i = ... = \mathrm{d}^n x_i/\mathrm{d}t^n = 0$ vanish too (which is to be shown in a **homework**). In other words, particles rest forever at these points (whether this is stable is another issue discussed later).
For the cRTB-problem we had the effective potential (292) - which we write here once again

$$\boxed{\Phi_{eff} \;=\; -\frac{(x-r_1)^2 + y^2}{2} - \frac{r_2}{R} - \frac{r_1}{S} \quad .}$$

The geometry gives us the following distances:

$$\begin{aligned} R^2 &= x^2 + y^2 + z^2 \\ S^2 &= (x-1)^2 + y^2 + z^2 \quad , \end{aligned}$$

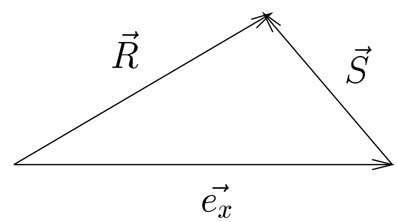


Abbildung 8:

while from the Eqs. of motion (289) we have

$$\ddot{x} - 2\dot{y} = -\partial_x \Phi_{eff} \tag{294}$$

$$\ddot{y} + 2\dot{x} = -\partial_y \Phi_{eff} \tag{295}$$

$$\ddot{z} = -\partial_z \Phi_{eff} \quad . \tag{296}$$

Using the definitions (293) for the fixed points to these relations right above one obtains:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= -\partial_x \Phi_{eff} = 0 \\ \ddot{y} + 2\dot{x} &= -\partial_y \Phi_{eff} = 0 \\ \ddot{z} &= -\partial_z \Phi_{eff} = 0 \\ \Longrightarrow \ddot{x} = \ddot{y} = \ddot{z} &= \dot{x} = \dot{y} = \dot{z} = 0 \end{aligned}$$

It is easy to check that all higher derivates must also vanish

$$\begin{aligned} \frac{\mathrm{d}^3 x}{\mathrm{d}t^3} - 2\ddot{y} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\partial_x \Phi_{eff}\right) &= -\dot{x}\frac{\partial^2 \Phi_{eff}}{\partial x^2} - \dot{y}\frac{\partial^2 \Phi_{eff}}{\partial y \partial x} - \dot{z}\frac{\partial^2 \Phi_{eff}}{\partial z^2} = 0 \\ \Longrightarrow \frac{\mathrm{d}^3 x}{\mathrm{d}t^3} &= \frac{\mathrm{d}^3 y}{\mathrm{d}t^3} = \frac{\mathrm{d}^3 z}{\mathrm{d}t^3} = 0 \quad . \end{aligned}$$

In order to identify the location of the *Lagrange-points* in the configuration space we need to calculate following expressions:

$$\begin{aligned} -\partial_x \Phi_{eff} &= 0 \\ -\partial_y \Phi_{eff} &= 0 \\ -\partial_z \Phi_{eff} &= 0 \quad . \end{aligned} \tag{297}$$

As a result of this exercise we get:

$$\partial_x \Phi_{eff} = -\left(x - r_1\right) + r_2 \frac{x}{R^3} + r_1 \frac{x-1}{S^3} = 0 \tag{298}$$

$$\partial_y \Phi_{eff} = -y + y\frac{r_2}{R^3} + y\frac{r_1}{S^3} =$$

$$= -y \left(1 - \frac{r_2}{R^3} - \frac{r_2}{S^3}\right) = 0 \tag{299}$$

$$\partial_z \Phi_{eff} = z \left(\frac{r_2}{R^3} + \frac{r_1}{S^3}\right) = 0 \ . \tag{300}$$

For the identification of the zero-roots one easily recognizes that $z = 0$ must hold, which means all Lagrange points are located in the orbital plane of the primaries! Determined by the structure of Eqs.(298)-(299) requires that there are 5 roots (**homework**), i.e. there have to be five Lagrange (Libration) points. In the following we will distinguish between two *triangular* points ($y \neq 0$; $(\cdots) \to 0$ in Eq.(299)) and for $y = 0$ the *co-linear* ones with Eq.(298) $\to 0$.

Triangular Lagrange-points: $\mathcal{L}_4$ und $\mathcal{L}_5$

For $y \neq 0$ the bracket term in in Eq.(299) must vanish

$$1 - \frac{r_2}{R^3} - \frac{r_1}{S^3} = 0 \tag{301}$$

where with the normalizations $r_1 + r_2 = D = 1$ one easily concludes that it must hold that:

$$R = S = 1 \ , \tag{302}$$

so that the *two* Lagrange-points, $\mathcal{L}_4$ and $\mathcal{L}_5$, form *two* equilateral triangles! Further, with the result (302), Eq. (298) vanishes also trivially (**homework**).
In order to calculate the exact coordinates $(x_{4,5}, y_{4,5})$ of these points[7], again geometrical relations, deduced from Fig. (5.1.1), symmetries and the Eqs. (301)-(302) are used to give:

$$\left.\begin{array}{rcl} R^2 = x^2 + y^2 & = & 1 \\ S^2 = (x-1)^2 + y^2 & = & 1 \end{array}\right\} \implies x = \frac{1}{2} \ , \ y = \pm\sqrt{\frac{3}{4}}$$

so that we obtain for Lagrange points $\mathcal{L}_4$ and $\mathcal{L}_5$ the coordinates:

$$\boxed{\mathcal{L}_{4/5} = (1/2, \ \pm\sqrt{3}/2, \ 0)} \tag{303}$$

With these loci the (energetic) value of the Jacobi-constant can be evaluated at these Lagrange-points $\mathcal{L}_{4/5}$ by using Eq. (290), while using the necessary conditions $|\vec{v}| = 0$ and $r_1 + r_2 = 1$, then read:

$$\mathcal{C}_{4/5} = \Phi_{eff} = -\frac{(x - r_1)^2}{2} - \frac{y^2}{2} - 1 = -\frac{3}{2} + \frac{r_1}{2} - \frac{r_1^2}{2}$$

$$= -\frac{3}{2} + \frac{r_1}{2}(1 - r_1) = -\frac{3}{2} + \frac{r_1 r_2}{2} \ . \tag{304}$$

[7]Here we use the traditional indices 4 and 5 for the triangular points.

Before we will start investigating the stability of the Lagrange-points, we are going to check which type of points $\mathcal{L}_{4/5}$ are – i.e. are they extrema or saddle-points in the potential landscape $\Phi_{eff}(x,y)$. To mention result right away – the triangular points $\mathcal{L}_{4/5}$ are *maxima*, whereas the co-linear points $\mathcal{L}_{1/2/3}$ are *saddle-points*.
The necessary condition for extrema (maximum) are given by vanishing first derivatives of the effective cRTP-Potential $\partial_{x_i}\Phi_{eff} = 0$ (x_i stands for x, y, z). Furthermore, the *Hessian* determinante has to take positive definite values. In order to calculate this determinant, higher derivatives of Φ_{eff} are needed:

$$
\begin{aligned}
\frac{\partial^2 \Phi_{eff}}{\partial x^2} &= \Phi_{,xx} = -1 + \frac{r_2}{R^3} - \frac{3r_2 x^2}{R^5} + \frac{r_1}{S^3} - \frac{3r_1 (x-1)^2}{S^5} \\
\frac{\partial^2 \Phi_{eff}}{\partial y^2} &= \Phi_{,yy} = -1 + \frac{r_2}{R^3} - \frac{3r_2 y^2}{R^5} + \frac{r_1}{S^3} - \frac{3r_1 y^2}{S^5} \\
\frac{\partial^2 \Phi_{eff}}{\partial z^2} &= \Phi_{,zz} = \frac{r_2}{R^3} - \frac{3r_2 z^2}{R^5} + \frac{r_1}{S^3} - \frac{3r_1 z^2}{S^5} \\
\frac{\partial^2 \Phi_{eff}}{\partial x \partial y} &= \Phi_{,,xy} = -\frac{3r_2 xy}{R^5} - \frac{3r_1 y\,(x-1)}{S^5} = \frac{\partial^2 \Phi_{eff}}{\partial y \partial x} \\
\frac{\partial^2 \Phi_{eff}}{\partial y \partial z} &= \Phi_{,yz} = -3yz\left(\frac{r_2}{R^5} + \frac{r_1}{S^5}\right) = \frac{\partial^2 \Phi_{eff}}{\partial z \partial y} \qquad (305) \\
\frac{\partial^2 \Phi_{eff}}{\partial z \partial x} &= \Phi_{,zx} = -3\frac{xzr_2}{R^5} - 3\frac{z(x-1)r_1}{S^5} = \frac{\partial^2 \Phi_{eff}}{\partial x \partial z} \qquad (306)
\end{aligned}
$$

Inserting the coordinates (303) for Lagrange-points $\mathcal{L}_4$ and $\mathcal{L}_5$, we have :

$$
\Phi_{,xx} = -\frac{3}{4}\,;\ \Phi_{,yy} = -\frac{9}{4}\,;\ \Phi_{,zz} = 1\ ;\ \Phi_{,xy} = \frac{3\sqrt{3}}{4}\,(r_2 - r_1)\ ;\ \Phi_{,xz} = \Phi_{,yz} = 0\,; \qquad (307)
$$

(the index $_{eff}$ has been dropped for brevity, and we denote $\Delta r = r_2 - r_1$) giving for the *Hessian* Determinant:

$$
\begin{aligned}
H_{\mathcal{L}_4/\mathcal{L}_5} = \begin{vmatrix} \Phi_{,xx} & \Phi_{,xy} \\ \Phi_{,xy} & \Phi_{,yy} \end{vmatrix} &= \begin{vmatrix} -\frac{3}{4} & \pm\frac{3\sqrt{3}}{4}\Delta r \\ \pm\frac{3\sqrt{3}}{4}\Delta r & -\frac{9}{4} \end{vmatrix} \\
&= \left(-\frac{3}{4}\right)^2 \begin{vmatrix} 1 & \sqrt{3}\Delta r \\ \sqrt{3}\Delta r & 3 \end{vmatrix} = \frac{27}{16}\left(1 - \Delta r^2\right) > 0 \quad .
\end{aligned}
$$

The positive definite value of the Hessian for $\mathcal{L}_4$ and $\mathcal{L}_5$ points to extrema, and negative 2. derivatives $\Phi_{,xx} = -3/4 < 0$ bzw. $\Phi_{,yy} = -9/4 < 0$ identify that we have maxima at the triangular points.

Colinear Lagrange-points: $\mathcal{L}_1, \mathcal{L}_2$, and$\mathcal{L}_3$

For $y = 0$ one obtains with Eq. (297)-(298) the coordinates of the Lagrange-points $\mathcal{L}_{1/2/3}$:

$$
-(x - r_1) + \frac{r_2}{x^2} + \frac{r_1}{(x-1)^2} \quad = \quad 0
$$

$$\Longrightarrow -(x-r_1)(x-1)^2x^2 \,+\, r_2(x-1)^2 \,+\, r_1x^2 \;=\; 0 \quad .$$

For $x \approx 1$ entwickeln wir linear die Koordinate x um die Lage des Sekundärkörpers, $x \approx 1 \pm h$ with $h \ll 1$ so that $\mathcal{L}_1$ and $\mathcal{L}_2$ can concluded from as

$$\begin{aligned}
-(1+h-r_1) \,+\, \frac{r_2}{(1+h)^2} \,+\, \frac{r_1}{h^2} &= 0 \\
-1-h+r_1+r_2(1-2h+\cdots)+\frac{r_1}{h^2} &= 0 \\
-h^3-2r_2h^3+r_1 &= 0 \quad .
\end{aligned} \tag{308}$$

Assuming now very different masses of the primaries: i.e. $r_2 \to 1$ and $r_1 = \mu_*/(1+\mu_*) \to 0$ from Eq. (308)

$$\begin{aligned}
\Longrightarrow \qquad 3h^3 &= r_1 \\
\Longrightarrow \qquad h_{1/2} &= \pm\sqrt[3]{\frac{r_1}{3}} + O\left(r_1^{2/3}\right) \;\approx\; \sqrt[3]{\frac{\mu_*}{3(1+\mu_*)}} \quad .
\end{aligned} \tag{309}$$

defining the Hill-scale h – a crucial quantity of the RTB-problem. It approximately measures the radius of the sphere of gravitational influence of the secondary (the smaller of the primaries), and it marks the border between the ranges of gravitational influences of either primaries.
As examples we give the Hill-scales of a satellite with the mass m_s and a semi-major axis a_s in orbit about a planet (mass M_p)

$$\begin{aligned}
h &= a_S\sqrt[3]{\frac{m_s}{3(M_p+m_s)}} = a_s\sqrt[3]{\frac{r_1^*}{3}} \\
&= \sqrt[3]{\frac{r_1}{3}} \quad ,
\end{aligned} \tag{310}$$

where the last expression concerns the normalized value ($a = D = 1$).
For the Earth-Sun system we obtain the Hill-scale of the Earth as:

$$h_\oplus \;=\; 1\mathrm{AE}\sqrt[3]{\frac{M_\oplus}{3(M_\odot+M_\oplus)}} \approx 1,5\cdot 10^6\mathrm{km} \quad ,$$

meaning, it is the 1.5 million km large. In other words, our "good old moon" lies with a fifth of that distance quite well inside the gravitational range of the Earth.

The third Lagrange point $\mathcal{L}_3$ is located on the other side of the larger primary, so that we insert $x_3 \,=\, -1 \,+\, \delta$ with $\delta \ll 1$ in Eq. (308) and obtain:

$$\begin{aligned}
-(-1+\delta-r_1) - \frac{r_2}{(\delta-1)^2} - \frac{r_1}{(\delta-3)^2} &= 0 \\
1-\delta+r_1-(1+2\delta\cdots)r_2-\frac{r_1}{4}(1-\delta\cdots) &= 0 \\
\Longrightarrow \delta = \frac{7}{12}r_1, \;\; \delta \ll h = \sqrt[3]{\frac{r_1}{3}}
\end{aligned}$$

Summarizing we have for the Lagrange-points:

$$\boxed{\begin{array}{c}\mathcal{L}_{1/2} = (1 \pm h + O(r_1^{2/3}),\ 0,\ 0) \\ \mathcal{L}_3 = (-1 + \frac{7}{12} r_1,\ 0,\ 0) \\ \mathcal{L}_{4/5} = (\frac{1}{2},\ \pm\frac{\sqrt{3}}{2},\ 0)\end{array}} \tag{311}$$

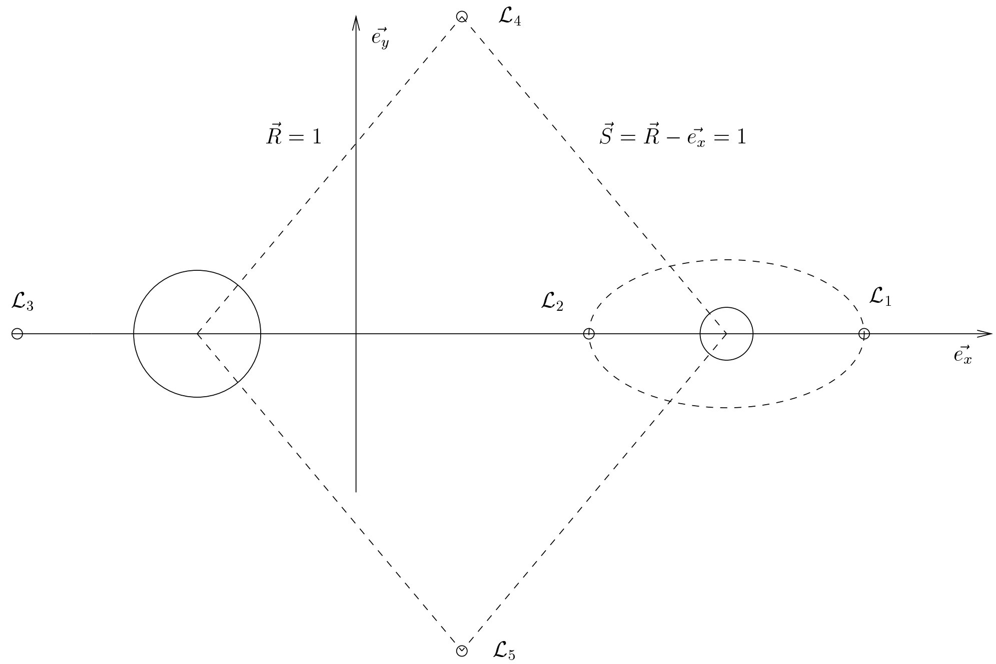


Abbildung 9: The loci of all Lagrange points in the restricted.three body problem (Attention: the y-axis must be shifted into the center of the primary by the distance r_1).

The co-linear Lagrange points are *saddle*-points which suggests instability, as the following stability analysis shows. One direction is stable (y) – the other (x) not. The triangular points are quasi-stable - test particles oscillate around them,

5.1.2 Linear Stability of the Lagrange-points.

First we will present a general linear analysis of the stability of a solution (e.g. fixed point $\vec{Z}_0$) of a system of nonlinear dynamical equations describing the the state-values $\vec{Z}(t)$:

$$\dot{\vec{Z}} \ = \ \vec{\mathcal{F}}(\vec{Z}, \vec{\mathcal{P}}) \ \ , \tag{312}$$

with the (non-linear) vectorfunction $\vec{\mathcal{F}}$ which further – apart from the state $\vec{Z}$– depends on parameters $\vec{\mathcal{P}}$. In our case of the cRTB-problem these are the mass-parameters r_1, and r_2. Linearization of this function around a fixed point $\vec{Z}_0$ gives:

$$\dot{\vec{Z}} \ = \dot{\vec{\xi}} \ = \ \hat{\mathbf{J}} \cdot (\vec{Z} - \vec{Z}_0) \ = \hat{\mathbf{J}} \cdot \vec{\xi} \tag{313}$$

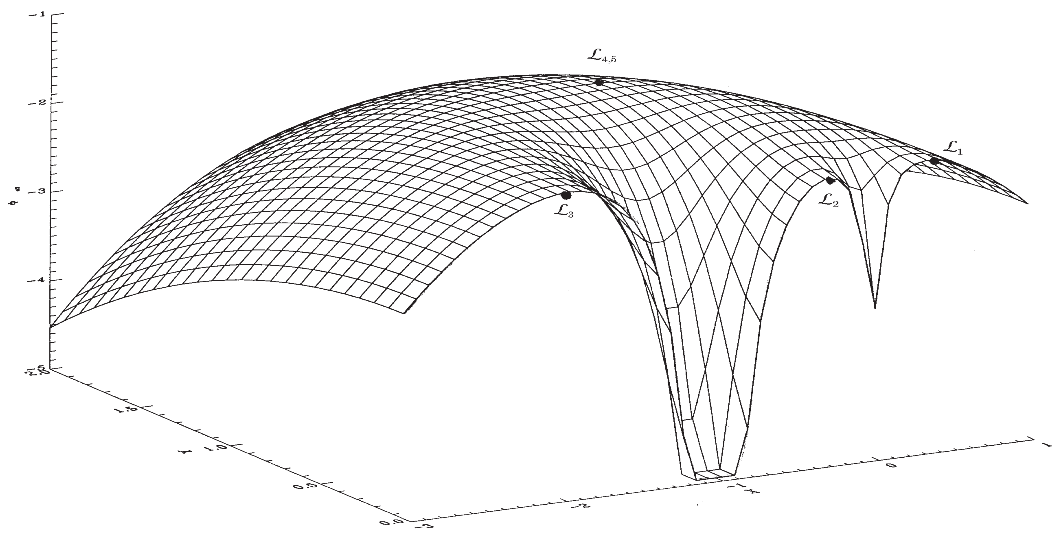


Abbildung 10: Lagrange points in the *landscape* of the effective potential Φ_{eff} of the restricted-three body problem of a planet and its moon

.

with $\vec{\xi} = \vec{Z} - \vec{Z}_0$, and the coefficients of the Jacobi-matrix – *Jacobian*:

$$J_{ij} \;=\; \partial_{Z_j} \mathcal{F}_i \big|_{\vec{Z}_0} \quad . \tag{314}$$

The Eigenvalues λ_k and Eigenfunctions $\vec{\xi_k}$, i.e. the solutions of the equation

$$(\hat{\mathbf{J}} - \lambda_k \hat{\mathbf{I}}) \cdot \vec{\xi_k} = 0 \tag{315}$$

provide the base for the construction of a general solution of the linearized stability problem (see below).

Back to our cRTB-problem – the stability analysis of the Lagrange-points $\mathcal{L}_i(x_0, y_0, z_0 = 0)$, with $i = 1, ..., 5$. Expanding the spatial coordinates around the points $\mathcal{L}_i$ and taking into account only the linear terms, one obtains:

$$\begin{pmatrix} x = x_0 + \xi \\ y = y_0 + \eta \\ z = z_0 + \zeta \overset{z=0}{=} \zeta \end{pmatrix} \quad \Longrightarrow \begin{pmatrix} \dot{x} = \dot{\xi} \\ \dot{y} = \dot{\eta} \\ \dot{z} = \dot{\zeta} \end{pmatrix} \Longrightarrow \begin{pmatrix} \ddot{x} = \ddot{\xi} \\ \ddot{y} = \ddot{\eta} \\ \ddot{z} = \ddot{\zeta} \end{pmatrix} \tag{316}$$

the linear perturbations equations based upon the basic non-linear equations of motion (287) as:

$$\boxed{\begin{array}{l} \ddot{\xi} - 2\dot{\eta} = -\partial_\xi \Phi \\ \ddot{\eta} + 2\dot{\xi} = -\partial_\eta \Phi \\ \ddot{\zeta} \qquad = -\partial_\zeta \Phi \end{array}} \tag{317}$$

The right-hand sides of Eqs. (317) are obtained by (linearly) expanding the effective potential (292) to give

$$\Phi(x, y, z) \;=\; \Phi(x_0, y_0, z_0) + \sum_{i,j,k} \frac{1}{i!j!k!} \frac{\partial^{i+j+k}\Phi}{\partial^i x \partial^j y \partial^k z}\bigg|_0 (x - x_0)^i (y - y_0)^j z^k$$

$$= \quad \Phi(x_0, y_0) + \left.\frac{\partial \Phi}{\partial x}\right|_0 \xi + \cdots + \frac{1}{2}\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_0 \xi^2 + \cdots + \left.\frac{\partial^2 \Phi}{\partial x \partial y}\right|_0 \xi\eta + \cdots \qquad .$$

Because the expansion starts around the Lagrange points, all first spatial derivatives must vanish [see Eqs. (294)-(296)] just leaving higher orders of the expansion

$$\begin{aligned} \Phi(x, y, z) \quad = \quad & \Phi(x_0, y_0, z_0) + + \frac{1}{2}\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_0 \xi^2 + \frac{1}{2}\left.\frac{\partial^2 \Phi}{\partial y^2}\right|_0 \eta^2 + \frac{1}{2}\left.\frac{\partial^2 \Phi}{\partial z^2}\right|_0 \zeta^2 \\ & + \left.\frac{\partial^2 \Phi}{\partial x \partial y}\right|_0 \xi\eta + \left.\frac{\partial^2 \Phi}{\partial y \partial z}\right|_0 \eta\zeta + \left.\frac{\partial^2 \Phi}{\partial x \partial z}\right|_0 \xi\zeta \qquad . \end{aligned}$$

The Eqs. (317) now take the form of the linearized system of equations

$$\ddot{\xi} - 2\dot{\eta} + \Phi^0_{xx}\xi + \Phi^0_{xy}\eta + \Phi^0_{xz}\zeta = 0 \tag{318}$$

$$\ddot{\eta} + 2\dot{\xi} + \Phi^0_{xy}\xi + \Phi^0_{yy}\eta + \Phi^0_{yz}\zeta = 0 \tag{319}$$

$$\ddot{\zeta} \qquad + \Phi^0_{xz}\xi + \Phi^0_{yz}\eta + \Phi^0_{zz}\zeta = 0 \tag{320}$$

Formulating these equations as a system of six 1. order differential equations – by introducing velocities (u, v, w)

$$\dot{\xi} = u \tag{321}$$

$$\dot{\eta} = v \tag{322}$$

$$\dot{\zeta} = w \tag{323}$$

$$\dot{u} = 2v - \Phi^0_{xx}\xi - \Phi^0_{xy}\eta - \Phi^0_{xz}\zeta \tag{324}$$

$$\dot{v} = -2u - \Phi^0_{xy}\xi - \Phi^0_{yy}\eta - \Phi^0_{yz}\zeta \tag{325}$$

$$\dot{w} = - \Phi^0_{xz}\xi - \Phi^0_{yz}\eta - \Phi^0_{zz}\zeta \quad . \tag{326}$$

This is a system of linear differential equations with constant coefficients, whose solutions are constructed with the aid of the solution of the Eigen-values problem with the Jacobi-matrix $\hat{\mathbf{J}}$:

$$\boxed{\begin{array}{c} \dot{\vec{\xi}} = \hat{\mathbf{J}} \cdot \vec{\xi} \quad \text{with} \quad \vec{\xi} = (\xi, \eta, \zeta, u, v, w) \\ \text{characteristic equation:} \quad \mathrm{Det}\left(\hat{\mathbf{J}} - \lambda I\right) = 0 \end{array}} \tag{327}$$

The quantities λ constitute the Eigenvalues of that system – obtained as the solution of the above (algebraic) *characteristic equation* – and with it one obtains the Eigenvectors as

$$\left[\hat{\mathbf{J}} - \lambda_i \hat{I}\right] \vec{\xi_i} = 0 \tag{328}$$

with $i = (1, 2, ..., 6)$, as the fundamental system of the Eqs. (321) - (326) with the base of Eigenvectors $\vec{\xi_i} = (\xi_i, \eta_i, ..., w_i)$. With this one may formulate the general solution

$$\vec{\xi_i} = \exp(\lambda_i t)\,(C_1, ..., C_6)^T \quad ; \quad C_i = \text{const.} \quad , \tag{329}$$

where the constants C_i can be determined by the initial conditions $\vec{\xi_0} = \vec{\xi}(t=0)$. For the characterization of the stability, in the first term, only the solution of the characteristic equation for the complex quantity λ_i is relevant. With their knowledge we can interpret the stability by applying the following criteria for the real-part of λ_i

$$\Re\lambda_i < 0 \quad \Longrightarrow \text{(absolutely) stable} \tag{330}$$

$$\Re\lambda_i > 0 \quad \Longrightarrow \text{unstable} \tag{331}$$

$$\Re\lambda_i = 0 \quad \Longrightarrow \text{stable librations} \quad . \tag{332}$$

Back to the stability of our Lagrange points $\mathcal{L}_1 \cdots \mathcal{L}_5$. All those points are located in the plane $z = 0$, so that for the derivatives (305)-(306) we have $\Phi_{iz} \propto z$, so that we have for the expansion coefficients in the plane $\Phi^0_{xz} = \Phi^0_{yz} = 0$, so that the lateral and vertical components of motion decouple to yield

$$\ddot{\xi} - 2\dot{\eta} + \Phi^0_{xx}\xi + \Phi^0_{xy}\eta = 0 \tag{333}$$

$$\ddot{\eta} + 2\dot{\xi} + \Phi^0_{xy}\xi + \Phi^0_{yy}\eta = 0 \tag{334}$$

$$\ddot{\zeta} + \Phi^0_{zz}\zeta = 0 \quad . \tag{335}$$

With the ansatz $\xi = A\exp(\lambda t)$, and $\eta = B\exp(\lambda t)$ one obtains $\dot{\xi} = \lambda\xi$, and $\dot{\eta} = \lambda\eta$ so that one may write for the lateral component (ξ, η) of the solution[8]:

$$\begin{pmatrix} (\lambda^2 + \Phi^0_{xx}) & (\Phi^0_{xy} - 2\lambda) \\ (\Phi^0_{yx} + 2\lambda) & (\lambda^2 + \Phi^0_{yy}) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \quad . \tag{336}$$

The solubility conditions of this homogeneous system of equations (coefficient determinant $\rightarrow 0$) provides the the characteristic equation for λ (**homework:** derive the same from the system (321)-(326), hint: consider only the reduced lateral equations system of dimension $D = 4$):

$$\Longrightarrow \boxed{\lambda^4 + (\Phi^0_{xx} + \Phi^0_{yy} + 4)\lambda^2 + \Phi^0_{xx}\Phi^0_{yy} - (\Phi^0_{xy})^2 = 0} \tag{337}$$

For the potential-coefficients in case of $\mathcal{L}_{4,5}$ as given in Eqs.(303), (307), and (337) giving $\Phi^0_{xx} = -3/4,\ \Phi^0_{yy} = -9/4,\ \Phi^0_{xy} = 3\sqrt{3}\Delta r/4$, and $\Delta r = r_2 - r_1$:

$$\lambda^4 + \lambda^2 + \frac{27}{4}\frac{\mu_*}{(1+\mu_*)^2} = 0$$

$$\lambda^4 + \lambda^2 + \frac{27}{4}r_1 r_2 = 0$$

Now, substituting $\lambda^2 = \Lambda$, we obtain a quadratic equation with the corresponding solution:

$$\Lambda_{1/2} = -\frac{1}{2}\left(1 \pm \sqrt{1 - 27r_1r_2}\right) \tag{338}$$

$$\Longrightarrow \quad \lambda_{1,2,3,4} = \pm\sqrt{\Lambda} = \pm\sqrt{-\frac{1}{2}\left(1 \pm \sqrt{1 - 27r_1r_2}\right)} \tag{339}$$

[8]Note that the Eigenvalues λ_i are generally complex numbers!

These solutions are pure oscillation frequenciess$_i$ with $\lambda_i = \imath\, s_i$ as long as the expression under the square-root remains negativ! The latter is fulfilled for the following mass parameters $r_1 = \mu_*/(1+\mu_*) \approx \mu_*$, $r_2 = 1 - r_1 \approx 1$:

$$\overset{\text{discriminant}\geq 0}{\Longrightarrow} \quad 1 - 27 r_1 r_2 \approx 1 - 27\mu_* \geq 0 \quad ;$$

$$\Longrightarrow \qquad \mu_*^{(c)} = \left.\frac{m}{m+M}\right|_c \approx 3.8 \cdot 10^{-2}$$

With this the motion close to the Lagrange-points $\mathcal{L}_{4/5}$ is stable if, and only if $0 \leq \mu_* \leq \mu_*^c = 4 \cdot 10^{-2}$. Interestingly, for ratio's $\mu_* > \mu_*^{(c)}$ – for instance, for double-star systems – the real value $\Re\lambda_i \neq 0$, i.e. the oscillatory solution becomes unstable or relaxes to the fixed points $\mathcal{L}_{4/5}$.
Otherwise, for $\mu_* < \mu_*^{(c)}$ the particles oscillate around the Lagrange points so that one may use the ansatz:

$$\xi = \sum_i^4 \{C_i \cos(s_i t) + S_i \sin(s_i t)\}$$

$$\eta = \sum_i^4 \left\{\tilde{C}_i \cos(s_i t) + tildeS_i \sin(s_i t)\right\}$$

Remark: the type of the above solutions can also be obtained by using the above solution sketch (planar case $z = 0$) $\vec{\xi}_i = \exp(\lambda_i t) \times \left[C_1^{(i)}, ..., C_4^{(i)}\right]$ by applying the Eulerian-formula for the imaginary exponential function (**homework**).

Just for simplicity, we will assume $C_i = \tilde{C}_i = S_i = \tilde{S}_i$ for all $i \in (2,4)$, i.e. with the exception of $i = 1$ we separate long-term periodic solution branch — i.e. for the moment we have selected exclusively the mode $s = s_1$ for an analysis:

$$\xi = C\cos(st) + S\sin(st) \tag{340}$$

$$\eta = \tilde{C}\cos(st) + \tilde{S}\sin(st) \tag{341}$$

$$\Longrightarrow \xi\tilde{C} - C\eta = \left(\tilde{C}S - C\tilde{S}\right)\sin(st) \tag{342}$$

$$\tilde{S}\xi - S\eta = \left(S\tilde{C} - \tilde{S}C\right)\cos(st) \quad . \tag{343}$$

Squaring and then adding Eqs. (342) & (343) one obtains an implicit equation of second order in the (linearized) state variables ξ and η

$$\boxed{\xi^2\left(\tilde{S}^2 + \tilde{C}^2\right) + \eta^2\left(S^2 + C^2\right) - \xi\eta\left(S\tilde{S} + C\tilde{C}\right) = \underbrace{\left(S\tilde{C} - C\tilde{S}\right)^2}_{B^2>0}} \tag{344}$$

which characterizes an elliptical form. With a main axis transformation (solving the Eigen-value problem for the matrix $\hat{\mathbf{A}}$; see below) one may simply re-derive the normal form of an *ellipse* (vanishing coefficients of mixed terms $\propto \xi\eta$) and can determine the direction of the main axes. The elliptic quadratic form (344) can be expressed by $\vec{\xi}^T \hat{\mathbf{A}} \vec{\xi} \;=\; B^2$ with the matrix[9]

$$\begin{pmatrix} (\tilde{S}^2 + \tilde{C}^2) & \frac{1}{2}(S\tilde{S} + C\tilde{C}) \\ \frac{1}{2}(S\tilde{S} + C\tilde{C}) & (S^2 + C^2) \end{pmatrix} \;=\; \hat{\mathbf{A}} \;. \tag{345}$$

The coefficients can finally calculated by using the initial conditions $(\xi_0 = \xi(t=0), \eta_0, u_0, v_0)$ and by inserting the ansatzes (340)-(341) in the linearized differential Eqs. (333)-(334) $\Rightarrow$

$$\begin{aligned} \xi &= C\cos(st) + S\sin(st) & \eta &= \tilde{C}\cos(st) + \tilde{S}\sin(st) \\ \dot{\xi} &= -sC\sin(st) + sS\cos(st) & \dot{\eta} &= -s\tilde{C}\sin(st) + s\tilde{S}\cos(st) \\ \ddot{\xi} &= -s^2C\cos(st) - s^2S\sin(st) & \ddot{\eta} &= -s^2\tilde{C}\cos(st) - s^2\tilde{S}\sin(st) \end{aligned}$$

so that one obtains the relations between the coefficients:

$$\boxed{\tilde{C}_i = \frac{2s_iS_i + \Phi^0_{xy}C_i}{S_i^2 - \Phi^0_{yy}} \;;\; \tilde{S}_i = \frac{2s_iC_i - \Phi^0_{xy}S_i}{\Phi^0_{yy} - S_i^2}} \tag{346}$$

For the points $\mathcal{L}_{4/5}$ there are quasi-stable oscillations, provided that the for the mass parameters holds $0 \leq r_1 < r^{(c)}$.
Complementary, for larger mass ratios $r^c \leq r_1 \leq 1/2$, i.e. $d = 1 - 27r_1r_2 \;<\; 0$ it holds

$$\Lambda \;=\; -\frac{1}{2}\left(1 \mp \sqrt{d}\right) \;=\; \frac{1}{2}\left(-1 \pm \sqrt{d}\right)$$

$$\text{with} \quad .\sqrt{d} \;=\; \imath a \quad\Longrightarrow\quad \Lambda \;=\; \frac{1}{2}\left(-1 \pm \imath a\right)$$

For the frequency value $a = \sqrt{-d} \;=\; \sqrt{27r_1r_2 - 1} \;\leq\; \frac{\sqrt{23}}{2} = 2,398$ one finally obtains the real frequency λ:

$$\lambda_k \;=\; \pm\sqrt{\Lambda} = \alpha_k + i\beta_k \qquad \forall k = 1, ..., 4$$

The real part of the value $\Re\lambda_k$ is now of main interest, and because the system is Hamiltonian, so that holds the relation $\sum_k \Re\lambda_k = 0$; i.e. there must necessarily also be positive values $\Re\lambda_k \;>\; 0$. That means, the motion of a test particle around $\mathcal{L}_{4/5}$ for a mass ratio of about $\mu_* > 3.8 \times 10^{-4}$ of the primaries can be expected to be *unstable*.

5.1.3 Curves of Zero-Velocity

As already indicated by the title, here we will investigate the role of curves in the sub-phase space (hyperplane) where the velocity of the test particle is zero: $|\vec{v}| \;=\; v = 0$. Depending on the "landscape" of the effective potential Φ_{eff} in the configuration space

[9]Calculating the Eigen-values and Eigen-vectors one may construct the normal form of an ellipse and model approximately the shape of the trajectories near $\mathcal{L}_4$ and $\mathcal{L}_5$.

(see Fig.) and the quantity of the Jacobi-constant $\mathcal{C}$ different types of limiting curves emerge

$$\begin{aligned} \frac{v^2}{2} + \Phi_{eff}(\vec{r}) &= \mathcal{C}(\vec{r}) \\ \Longrightarrow \quad \frac{v^2}{2} &= \mathcal{C} - \Phi_{eff} \ \geq \ 0 \quad . \end{aligned} \tag{347}$$

The left-hand side of the above equation must be positive which set constraints for the right-hand side. The limiting curves in the phase-space are given by the "equal" sign on the right-hans side relation, i.e. $v = 0$ which means:

$$\begin{aligned} \mathcal{C}(x_0, y_0, z_0, r_1) \ &- \ \Phi_{eff}(x, y, z, r_1) \ = \ 0 \\ \Longrightarrow \quad \mathcal{C} \ &= \ \Phi_{eff} \quad , \end{aligned} \tag{348}$$

where one has to emphasize that $v = 0$ does not mean that the particle do not move – remember that the description of the cRTB-problem takes place in a co-rotating frame! But also it means, that the motion is also restricted to the plane $z = 0$ because $v = 0$ also means $\dot{\zeta} = 0$. For the potential (292) we have:

$$\begin{aligned} \Phi_{eff} = \mathcal{C} \quad &= \quad -\frac{(x - r_1)^2}{2} - \frac{y^2}{2} - \frac{r_2}{R} - \frac{r_1}{S} \\ &\overset{z=0}{=} \quad -\frac{R^2}{2} - \frac{r_2}{R} + x r_1 - \frac{r_1}{S} - \frac{r_1^2}{2} \qquad (349) \\ &= \quad -\frac{R^2}{2} - \frac{r_1}{S}(1 - xS) - \frac{r_2}{R} \qquad (350) \end{aligned}$$

Different Cases:
Here we assume, for simplicity, that the secondary is much less massive than the primary, i.e. $\mu_* = m/M \ll 1$ and thus also $r_1 = \mu_*/(1 + \mu_*) \ll 1$ as well as $r_2 = 1 - r_1 \approx 1$.

Motion near the primary: Here for $-\infty < \mathcal{C} < \mathcal{C}_2$, the distance to the larger primary (mass M) is quite small $R \to 0$ (also meaning $S \to 1$), i.e. for Eq. (350) one may approximate

$$\Phi_{eff} \ \approx \ -\frac{r_2}{R} - r_1 \ = \ \mathcal{C} \quad .$$

This means these curves are circles because we have

$$\frac{r_2}{R} \ = \ -(\mathcal{C} + r_1) \ = \ \textit{constant} \tag{351}$$

which also means that it has to be $R \ \approx \ \textit{constant}$ — which defines a circle (see Fig. 11).

Motion near the secondary: In this case $(-\infty < \mathcal{C} < \mathcal{C}_2)$ it inversely holds, the test particle is always close to the secondary (mass m) $S \to 0$ (also meaning: $R \to 1$ and also $x \to 1$). With this and neglecting terms $O(r_1^2)$ we have from Eq. (349) approximately

$$\Phi_{eff} \ \approx \ -\frac{3}{2} - \frac{r_1}{S} + r_1 \ = \ \mathcal{C}$$

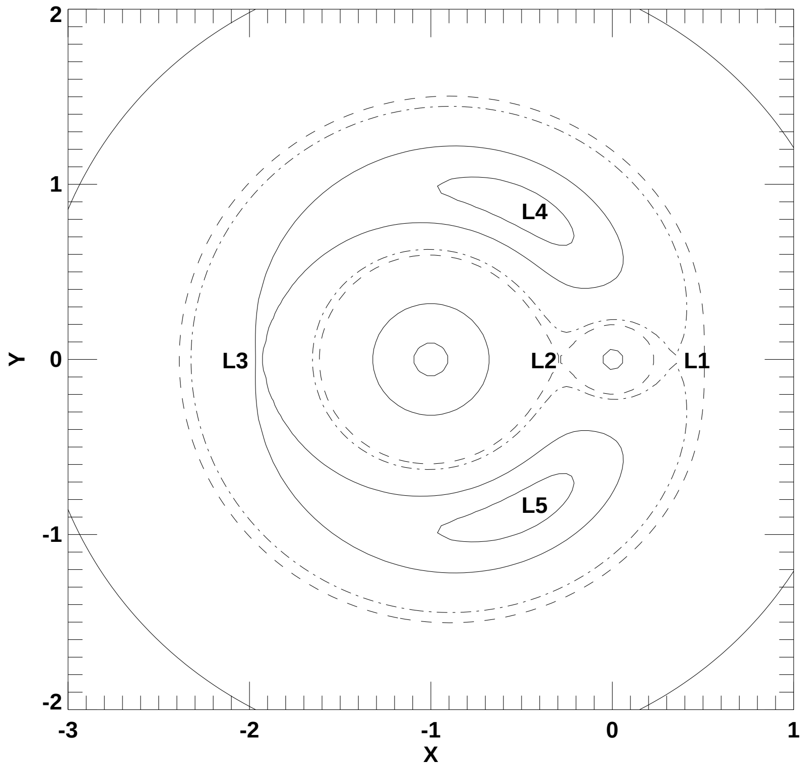


Abbildung 11: The “zoo” of the *zero-velocity curves*. **Attention:** Here a shifted x'- axis is used with $x' = x - 1$!

— please, have in mind, that here we have used $x \approx 1$ and $R \approx 1$. Again we have a form

$$\frac{r_1}{S} \; = \; -(\mathcal{C} + \frac{3}{2} - r_1) \; = \; constant \tag{352}$$

meaning circles (see Fig. 11)!

Motion between both Primaries: For Jacobi-constants $\mathcal{C}_2 < \mathcal{C} < \mathcal{C}_1$, the zero-velocity curves form a *lying eight*.

Summarizing:

1. $\Phi_{eff}(x, y) \ll \Phi_{eff}(\mathcal{L}_2) = \mathcal{C}_2 \implies$ circles around the primaries
2. $\Phi_{eff}(\mathcal{L}_2) \leq \Phi_{eff} \leq \Phi_{eff}(\mathcal{L}_1) \implies$ lying eight
3. $\Phi_{eff}(\mathcal{L}_1) \leq \Phi_{eff} \leq \Phi_{eff}(\mathcal{L}_3) \implies$ “horse-shoe”
4. $\Phi_{eff}(\mathcal{L}_3) \leq \Phi_{eff} \leq \Phi_{eff}(\mathcal{L}_{4/5}) \implies$ “tad-poles”

5.1.4 Non-linear Phenomena of the cRTB-problem

In order to include non-linear effects one has to consider higher orders of the taylor-expansion (317) of the effective potential. As a result the characteristic frequencies s_k do depend also on the state values $\vec{\xi} = (\xi, \eta, ..., v, w)^T$. Also the shape and size in the phase-space of the orbit will also have influence on the frequencies s_k (Eigenvalues).

$$T(\epsilon, \vec{x}) = T_{OL}(\mu) + P(\epsilon x, ..., \mu) \qquad \text{Horns Theorem} \tag{353}$$

In this context the function $P(\mu, \epsilon)$ is analytical (Cauchy-Riemann) which means it can be expanded in a Taylor-series according with respect to the small parameter ϵ.

Summary: cRTB-problem

1. It exists an integral of motion – the *Jacobi*-integral;
2. there exist 5 fixed points, called *Lagrange*-points.
   $\Longrightarrow$ a fix-solution of the cRTB-problem;
   (a) 3 co-linear points $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, located on the axe connecting the primaries, are saddle-points which that solution in their vicinity are certainly unstable!
   (b) 2 triangular points $\mathcal{L}_4, \mathcal{L}_5$ which are stable for mass parameters $r_1 \leq 0,04$. In their vicinity test particle librate around them. There are long-term periodic motions with frequencies $s_1 \approx \frac{1}{2}\sqrt{27 r_1}$ and short periodic ones $s_2 \approx 1 - \frac{27}{8} r_1$!
3. For small mass parameters (large differences between primary masses) $r_1 < r^c$ one obtains Zero-velocity curves, which constitute a good proxi for the real particle trajectories. There are the following shapes of curves:
   - Keplerbahnen $\mathcal{C} \ll \mathcal{C}_2$
   - "horse-shoe" orbits for $\mathcal{C}_1 < \mathcal{C} < \mathcal{C}_3$
   - "tadpole" orbits (libration-ellipses) for $\mathcal{C}_3 < \mathcal{C} < \mathcal{C}_{4/5}$

5.2 The Hill-Problem

Two steps are necessary to derive the *Hill-problem* from the cRTB-problem (here we consider the problem planet-satellite-particle):

1. shift of the origin of the coordinate frame $\Longrightarrow$ into the secondary (satellite)
2. expanding the potential of the planet and cancelling the centrifugal force!

Starting point are again the equations of motion of the cRTB-problem (289)

$$\begin{aligned}
\ddot{x} - 2\dot{y} &= (x - r_1) - \frac{r_2 x}{R^3} - \frac{r_1 (x-1)}{S^3} \\
\ddot{y} + 2\dot{x} &= y - \frac{r_2 y}{R^3} - \frac{r_1 y}{S^3} \\
\ddot{z} &= -z \left(\frac{r_2}{R^3} + \frac{r_1}{S^3} \right)
\end{aligned}$$

where we now shift the frame-origin $\tilde{x} = x - 1$, so that we obtain for the $\tilde{x}$-component (the $\tilde{y}$ component remains unchanged)

$$\begin{aligned}
\ddot{\tilde{x}} - 2\dot{\tilde{y}} &= (1 + \tilde{x} - r_1) - \frac{r_2\,(1+\tilde{x})}{\tilde{R}^3} - \frac{r_1\tilde{x}}{\tilde{S}^3} \\
\tilde{R}^2 &= (1+\tilde{x})^2 + \tilde{y}^2 + \tilde{z}^2 \\
\tilde{S}^2 &= \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 \quad .
\end{aligned}$$

For brevity in the following we will skip the “tilde”. Next we will expand the potential of the central body (planet)

$$\Phi_P = -\frac{r_2}{R} = -r_2 \left\{ \left(1+x^2\right)^2 + y^2 + z^2 \right\}^{-1/2} \quad ,$$

around the new origin $(x, y, z) = 0$, the location of the secondary. **Attention:** we do NOT expand the effective potential, but just a part of it accounting for the central body (planet):

$$\Phi_{eff} = -\frac{(1 + x - r_1)^2 + y^2}{2} - \underbrace{\frac{r_2}{R}}_{\equiv \Phi_P} - \frac{r_1}{S} \quad .$$

With this, expanding Φ_P we obtain:

$$\Phi_P = \Phi(0) + \nabla\Phi_P\big|_0 \cdot \vec{S} + \sum_{ij=1}^{2} \frac{1}{i!j!} \frac{\partial^2\Phi_P}{\partial x_i \partial x_j}\bigg|_0 x_i x_j + ... + \quad . \tag{354}$$

It has to be noted, at the starting point of the expansion $(0, 0, 0)$ — the position of the secundary with the mass m — we also have for the distances $R = 1; S = 0$. With this we obtain

$$\Longrightarrow \quad \frac{\partial\Phi_P}{\partial x} = \frac{r_2\,(1+x)}{R^3} \quad \longrightarrow \quad \frac{\partial\Phi_P}{\partial x}\bigg|_0 = r_2$$

$$\frac{\partial\Phi_P}{\partial y}\bigg|_0 = \frac{r_2 y}{R^3}\bigg|_0 = \frac{\partial\Phi_P}{\partial z}\bigg|_0 = \frac{r_2 z}{R^3}\bigg|_0 = 0$$

$$\frac{\partial^2\Phi_P}{\partial x^2} = -2r_2 \;;\; \frac{\partial^2\Phi_P}{\partial y^2} = r_2 \;;\; \frac{\partial^2\Phi_P}{\partial z^2} = r_2 \;;\; \frac{\partial^2\Phi_P}{\partial x \partial y} = 0 \quad .$$

With the relation $r_2 = 1 - r_1$ one obtains for the central potential:

$$\begin{aligned}
-\frac{\partial\Phi_P}{\partial x} &= -r_2 + 2r_2 x = -(1-r_1)(1-2x) = -1 + 2x + r_1 \\
-\frac{\partial\Phi_P}{\partial y}\bigg|_0 &= -y + r_1 y
\end{aligned} \tag{355}$$

where one must have in mind that in Eq. (354) the first derivatives of the central potential not necessarily vanish – no, moreover the they cancel the centrifugal forces.

$$\Phi_p = r_2(x-1) - r_2 x^2 + \frac{r_2}{2} y^2 + \frac{r_2}{2} z^2 \; . \tag{356}$$

Herewith, the components of the negative gradient, neglecting quadratic small terms as $r_1 x \;\; , \; r_1^2 \to 0$ and having in mind that $r_1 + r_2 = 1$:

$$-\frac{\partial \Phi_P}{\partial x} \;=\; 2r_2 x - 1 + r_1 \;\to\; 2x + r_1 - 1 \tag{357}$$

$$-\frac{\partial \Phi_P}{\partial y} \;=\; -y(1 - r_1) \;\to\; -y \tag{358}$$

$$-\frac{\partial \Phi_P}{\partial z} \;=\; -z(1 - r_1) \;\to\; -z \quad . \tag{359}$$

With this, the differential equations of the Hill-problem read:

$$\ddot{x} \,-\, 2\dot{y} \;=\; 3x \,-\, \frac{r_1 x}{S^3} \,+\, O(x r_1) \tag{360}$$

$$\ddot{y} \,+\, 2\dot{x} \;=\; -\frac{r_1 y}{S^3} \,+\, O(y r_1) \tag{361}$$

$$\ddot{z} \;=\; -z \,-\, \frac{r_1 z}{S^3} \,+\, O(z r_1) \; . \tag{362}$$

Please note, that the expansion around the secondary means that the coordinates are small quantities $(x, y, z) \ll 1$, as $r_1 \ll 1$ is too. As in the cRTB-case, now we multiply Eqs. (360)-(362) with the corresponding velocities $(\dot{x}, \dot{y}, \dot{z})$, respectively, and adding them up, we can derive an integral of motion

$$\frac{v^2}{2} \,+\, \Phi_H \;=\; C \tag{363}$$

analogous to the Jacobi-integral, with a new effective potential:

$$\Phi_H \;=\; -\frac{3}{2}x^2 + \frac{z^2}{2} - \frac{r_1}{S} \quad . \tag{364}$$

5.2.1 Lagrange-Points & Zero-Velocity Curves:

Also here all time derivatives of x, y, z vanish, where with Eqs. (360)-(362), exactly as in case of the cRTB, following conditions characterize the fixed points:

$$\nabla \Phi_H \;=\; 0 \quad . \tag{365}$$

From the y and z derivatives of that potential one can directly deduce that $y = z = 0$ must hold at the Lagrange-points – exactly the same situation as for the co-linear points $\mathcal{L}_{1/2}$ of the cRTB-problem. Remains the vanishing x-component of the Hills-force:

$$\partial_x \Phi_H \;=\; -3x \,+\, r_1 \frac{x}{S^3} \;=\; x\left[-3 \,+\, \frac{r_1}{x^3}\right] = 0 \quad , \tag{366}$$

i.e. the square-bracket must vanish to hold:

$$-3 \,+\, \frac{r_1}{x^3} \;=\; 0 \quad , \tag{367}$$

which directly gives the Hill-scale, already calculated in the case of the cRTB-problem:

$$x_{1/2} \;=\; \pm h \;=\; \pm\left\{\frac{r_1}{3}\right\}^{1/3} \quad . \tag{368}$$

5.2.2 Scaling of the Hill-equations!

The dynamical equations (360)-(362) can be transformed by expressing all length-values x_i and their time-derivatives by the above introduced Hill-scale (368). In other words, we set $x_i = h\hat{x}_i$ so that the Eqs. of motion read

$$\ddot{\hat{x}} - 2\dot{\hat{y}} - 3\hat{x} = -\frac{3\hat{x}}{\hat{S}^3} \tag{369}$$

$$\ddot{\hat{y}} + 2\dot{\hat{x}} = -\frac{\hat{y}}{\hat{S}^3} \tag{370}$$

$$\ddot{\hat{z}} + \hat{z} = -\frac{3\hat{z}}{\hat{S}^3} \quad , \tag{371}$$

where the crucial fact is — these equations, scaled in Hill-units, do NOT contain the mass parameter r_1 anymore. In other words, once a solution is found it holds for different masses (ratios) of the primaries as long as $r_1 \ll 1$ is fulfilled.

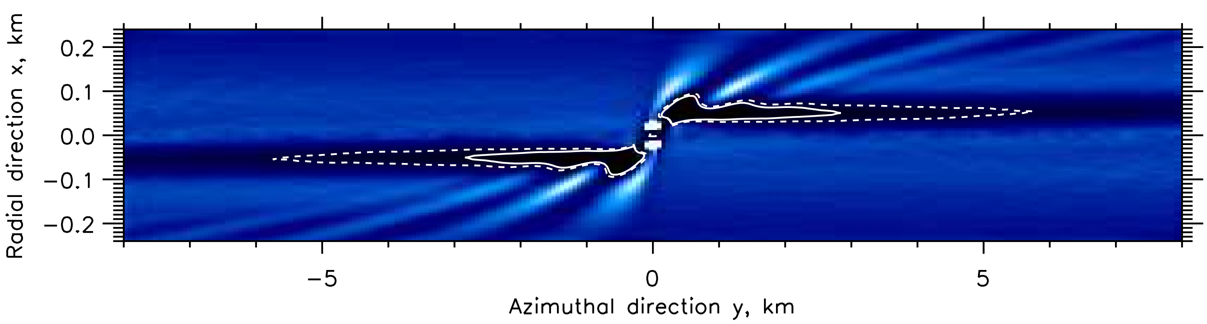


Abbildung 12: A simulation of a propeller structure where the gravitational scattering by the moonlet has been carried out by using numerical solutions of the *Hill-problem.*

5.3 Elliptisches DKP

Hierbei gelten die Annahmen $r_1 = 1 - r_2 \ll r_2 \approx 1$. Nimmt man nun an, daß die beiden Primärkörper eine pulsierende Bewegung vollführen, so ist eine Normierung $D = 1$ nicht mehr möglich. Der Abstand zwischen beiden Primärkörpern D ist natürlich wieder durch das Zweikörperproblem (ZKP) beschrieben:

$$\begin{aligned}
\ddot{D} - \Omega^2 D &= -\frac{\mu}{D^2} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(D^2\Omega\right) &= 0 \\
\left(D^2\Omega\right)^2 &= \mu\left(1 - e^2\right) \quad ,
\end{aligned}$$

wobei hier die Halbachse des ZKP wie gehabt $a_D = 1$ normiert wurde. Dann gilt die folgende Bewegungsgleichung:

$$\ddot{\vec{R}} + \dot{\vec{\Omega}} \times \vec{R} + 2\vec{\Omega} \times \dot{\vec{R}} + \vec{\Omega} \times \left(\vec{\Omega} \times \left(\vec{R} - \tilde{r_1}\vec{e_x}\right)\right) = -\nabla\Phi_G \tag{372}$$

Mit der Normierung aller Längen gemäß $\vec{R} = D\vec{s}$ – wobei $\vec{s}$ normierter Ortsvektor des zu beschreibenden Testeilchens ist – folgt dann:

$$\begin{aligned}
\ddot{D}\vec{s} + 2\dot{D}\dot{\vec{s}} + D\ddot{\vec{s}} + D\dot{\vec{\Omega}} \times \vec{s} + 2\vec{\Omega} \times \left(\dot{D}\vec{s} + D\dot{\vec{s}}\right)& \\
-D\left(\vec{\Omega} \times \left(\vec{\Omega} \times \left(\vec{s} - r_1\vec{e_x}\right)\right)\right) &= -\nabla\Phi_G
\end{aligned}$$

Weiterhin gilt $r_1 + r_2 = 1$ und somit $r_1 D(t) + r_2 D(t) = D$ wobei wie immer gelten muss:

$$r_1 = \frac{\mu_*}{1 + \mu_*} \qquad r_2 = \frac{1}{1 + \mu_*}$$

Entwickelt man nun die Vektorprodukterme der Bewegungsgleichung, setzt das Keplerproblem ein

$$\ddot{D} = \frac{L^2}{D^3} - \frac{\mu}{D^2} \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(D^2\Omega\right) = 0 \quad , \quad D = \frac{1 - e^2}{1 + e\cos f} \quad , \tag{373}$$

so ergibt sich mit

$$\vec{z} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

das Gleichungssystem für das elliptische DKP:

$$D\ddot{\xi} - D\frac{\dot{\Omega}}{\Omega}\dot{\xi} - 2D\Omega\dot{\eta} - \Omega^2\left(\frac{D^2}{1 - e^2}\xi - r_1 D\right) = -\partial_x\Phi_G \tag{374}$$

$$D\ddot{\eta} - D\frac{\dot{\Omega}}{\Omega}\dot{\eta} + 2D\Omega\dot{\xi} - \Omega^2\frac{D^2}{1 - e^2}\eta = -\partial_y\Phi_G \tag{375}$$

$$D\ddot{\zeta} = -\partial_z\Phi_G \tag{376}$$

Leitet man nun nicht nach der Zeit sondern nach der wahren Anomalie ab, so ergibt sich:

$$\frac{\mathrm{d}}{\mathrm{d}t} \longrightarrow \frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}f} = \Omega\frac{\mathrm{d}}{\mathrm{d}f} \Longrightarrow \qquad \dot{\xi} = \Omega\xi'$$

$$\ddot{\xi} = \Omega\Omega'\xi' + \Omega^2\xi''$$

Setzt man dies nun in die obigen Formeln ein, kürzen sich einige Terme raus und es verbleiben:

$$\xi'' - 2\eta' - \left(\frac{D}{1-e^2}\xi - r_1\right) = F_x \tag{377}$$

$$\eta'' + 2\xi' - \left(\frac{D}{1-e^2}\eta\right) = F_y \tag{378}$$

$$\zeta'' + \left(1 - \frac{D}{1-e^2}\right)\zeta = F_z \tag{379}$$

Für die Terme der Gravitationskraft erhalten wir (Division von D etc. in Gln. (381)-(383))

$$\begin{aligned}
\vec{F}_{x/y/z} &= -\frac{1}{D^3\Omega^2}\left(r_2\frac{\vec{z}}{z^3} + r_1\frac{(\vec{z}-\vec{e_x})}{\tilde{S}^3}\right) = -\frac{D}{\mu\left(1-e^2\right)}\left(r_2\frac{\vec{z}}{z^3} + r_1\frac{\vec{S}}{S^3}\right) \\
\text{mit} \qquad & n^2a^3 = \mu = \gamma\left(M+m\right) = 1 \\
\Longrightarrow \qquad & \vec{F} = -\frac{D}{1-e^2}\left(r_2\frac{\vec{z}}{z^3} + r_1\frac{\vec{S}}{S^3}\right)
\end{aligned}$$

Bringt man all Terme auf der linken Seite, die nur Koordinaten (ξ, η, ζ) enthalten auch auf die rechte Seite, kann ein Quasi-Potenzial formuliert werden:

$$\Phi_e = -\frac{D(f)}{1-e^2}\left(\frac{\left(\xi - r_1\left(1 + e\cos f\right)\right)^2 + r_2 + r_1 + \eta}{2}\right) \tag{380}$$

mit dem die Differenzialgleichungen letztliche lauten:

$$\xi'' - 2\eta' - = -\frac{\partial\Phi_e}{\partial\xi} \tag{381}$$

$$\eta'' + 2\xi' = -\frac{\partial\Phi_e}{\partial\eta} \tag{382}$$

$$\zeta'' = -\frac{\partial\Phi_e}{\partial\zeta} \quad . \tag{383}$$

Φ_e ist somit indirekt zeitabhängig, da $f = f(t)$. Eine Ähnlichkeit mit dem Jakobi-Integral ist schon zu erkennen. Benutzt man nun:

$$\begin{aligned}
\dot{\Phi}_e(\vec{R}) &\Longleftrightarrow \dot{\vec{R}}\cdot\nabla\Phi_e \\
\mathrm{d}\Phi_e(f,\vec{z}) &= \frac{\partial\Phi_e}{\partial f}\,\mathrm{d}f \;+\; \mathrm{d}\vec{z}\cdot\nabla\Phi_e \\
\vec{z}'\cdot\nabla\Phi_e &= \frac{\mathrm{d}\Phi_e}{\mathrm{d}f} \;-\; \Phi_e'
\end{aligned}$$

und geht komponentenweise Ähnlich wie beim eingeschränkten DKP vor, so erhält man für das Potential:

$$\Phi_e \quad = \quad -\left(1+e\cos f\right)^{-1}\left(\frac{1}{2}\left(\xi - r_1\left(1+e\cos f\right)\right)^2 + \frac{\eta^2}{2} + \frac{r_1}{\tilde{S}} + \frac{r_2}{z}\right)$$

$$\overset{e,r_1\ll 1}{\Longrightarrow} \qquad \Phi_e = -\frac{1}{1+e\cos f}\Phi_{eff}^{DKP}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}f}\left(\frac{\xi'^2}{2}+\frac{\eta'^2}{2}+\frac{\zeta'^2}{2}+\Phi_e\right) &= -\Phi_e' \\ \Longrightarrow \frac{v^2}{2}+\Phi_e &= -\int \mathrm{d}f\,\Phi_e'\end{aligned}$$

Damit ergeben sich auch hier analoge Darstellungen:

$$\begin{gathered}\boxed{\frac{v^2}{2}+\Phi_{eff}^{DKP}\left(1-2e\cos f\right)=\mathcal{C}\left(\vec{z_0},\dot{\vec{z_0}}\right)} \\ \boxed{\Phi_{eff}^{DKP}=\left(1+2e\cos f\right)\mathcal{C}=\mathcal{C}_*}\end{gathered} \tag{384}$$

Für das elliptische DKP bleiben die Lagrange-Punkte und ihre Stabilitätseigenschaften erhalten. Auch hier erweisen sich Störungen als kleine Schwingungen. $\mathcal{C}_*$ oszilliert um $\mathcal{C}$ und die Nullgeschwindigkeitskurven pulsieren.

Inhaltsverzeichnis