Stochastic processes: Assignment sheet 4

(1) Fractional Fokker-Planck equation

Consider the fractional Fokker-Planck equation for the probability density $P(x,t)$,

$$\frac{\partial}{\partial t}P(x,t) = {}_0D_t^{1-\alpha}\left(\frac{\partial}{\partial x}\frac{V'(x)}{m\eta_\alpha} + K_\alpha\frac{\partial^2}{\partial x^2}\right)P(x,t), \tag{1}$$

where the fractional Riemann-Liouville operator is defined as $(0 < \alpha \leq 1)$

$${}_0D_t^{1-\alpha}P(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{P(x,t')}{(t-t')^{1-\alpha}}dt'. \tag{2}$$

Here, $V(x)$ is an external potential, m is the mass of the test particle, and η_α and K_α are the generalised friction and diffusion coefficients. Finally, $\Gamma(\cdot)$ denotes the complete Gamma function. For the initial condition, use $P(x,0) = \delta(x-x_0)$.

(i) First consider the linear potential $V(x) = ax$. Calculate the transport moments $\langle x(t)\rangle$, $\langle x(t)^2\rangle$, and $\left\langle\left(x(t) - \langle x(t)\rangle\right)^2\right\rangle$. Discuss the case $0 < \alpha < 1$ in comparison to the case $\alpha = 1$.

Next, plot the probability density $P(x,t)$ for different times by help of the subordination relation

$$P_\alpha(x,t) = \int_0^\infty \mathscr{E}_\alpha(s,t)P_1(x,s)ds, \tag{3}$$

where $P_1(x,t)$ is the solution of the fractional Fokker-Planck equation for $\alpha = 1$, and $P_\alpha(x,t)$ corresponds to the subdiffusive case. Use the explicit form

$$\mathscr{E}_{1/2}(s,t) = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{s^2}{4t\tau}\right). \tag{4}$$

What do you observe in comparison to the normal-diffusive case $\alpha = 1$?

(ii) Now consider the case of the fractional Ornstein-Uhlenbeck process in the harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$ with the frequency ω. Calculate the moments and their relaxation behaviour at long times in terms of the Mittag-Leffler function $E_\alpha(\cdot)$ (see the handout sent around). Use the thermal value $\langle x^2\rangle_{\text{th}}$ for the second moment.

In the next step use the known Brownian solution for this process and plot the probability density function for different times for the cases $\alpha = 1$ and $\alpha = 1/2$, including the stationary limit at long times.

Finally, use the method of separation of variables and split the fractional Fokker-Planck equation into the eigenequations for the spatial and temporal parts (write $P(x,t) = X(x)T(t)$ and determine the eigenvalues. Use the resulting series expression for $P(x,t)$ to plot and compare the results to the subordination procedure.