Stochastic processes: Assignment sheet 3

(1) Derive the backward Fokker-Planck equation

Consider the forward Fokker-Planck equation for a probability density $P(x, t|x_0, t_0)$,

$$\frac{\partial}{\partial t}P(x,t|x_0,t_0) = -\frac{\partial}{\partial x}\Big[\mu(x,t)P(x,t|x_0,t_0)\Big] + \frac{\partial^2}{\partial x^2}\Big[D(x,t)P(x,t|x_0,t_0)\Big], \qquad (1)$$

which corresponds to the stochastic differential equation (see later in the course)

$$dX_t = \mu(X_t,t)dt + \sqrt{2D(X_t,t)}dW_t. \qquad (2)$$

Show that the backward Fokker-Planck equation then has the form

$$\frac{\partial}{\partial t_0}P(x,t|x_0,t_0) = -\left[\mu(x_0,t_0)\frac{\partial}{\partial x_0} + D(x_0,t_0)\frac{\partial^2}{\partial x_0^2}\right]P(x,t|x_0,t_0). \qquad (3)$$

For the derivation, employ the two following approaches.

(i) Adjoint-operator method

First define the forward Fokker-Planck operator and its adjoint,

$$L_{\mathrm{FP}}f(x) = -\frac{\partial}{\partial x}\Big[\mu(x,t)f(x)\Big] + \frac{\partial^2}{\partial x^2}\Big[D(x,t)f(x)\Big], \qquad (4a)$$

$$L^{+}_{\mathrm{FP}}f(x) = \mu(x,t)\frac{\partial}{\partial x}f(x) + D(x,t)\frac{\partial^2}{\partial x^2}f(x). \qquad (4b)$$

Then prove the adjointness relation under the L^2 inner product

$$\int_{-\infty}^{\infty}(L_{\mathrm{FP}}\varphi)(x)\psi(x)\,dx = \int_{-\infty}^{\infty}\varphi(x)(L^{+}_{\mathrm{FP}}\psi)(x)dx, \qquad (5)$$

for smooth and rapidly decaying test functions $\varphi(x)$ and $\psi(x)$.

Conclude that if $P(x, t|x_0, t_0)$ satisfies the forward equation (1) in x, then as a function of the initial position x_0 it satisfies

$$\frac{\partial}{\partial t_0}P(x,t|x_0,t_0) = -L^{+}_{\mathrm{FP}}P(x,t|x_0,t_0), \qquad (6)$$

where derivatives are taken with respect to x_0.

(ii) Direct test-function method

Start from the forward Fokker-Planck equation, multiply both sides by a test function $\psi(x)$, and integrate over x, holding $\tau = t - t_0$ fixed.

Then use integration by parts and change of variables to show that the same backward equation is satisfied, i.e. equation (3).

(2) Analytical solution of the backward Fokker-Planck equation

Consider the backward Fokker-Planck equation with constant coefficients,

$$\frac{\partial}{\partial t_0} P(x, t|x_0, t_0) = -\left[\mu \frac{\partial}{\partial x_0} + D \frac{\partial^2}{\partial x_0^2}\right] P(x, t|x_0, t_0), \tag{7}$$

with the final condition

$$P(x_0, T) = \sqrt{\frac{\lambda}{\pi}} \exp\Big(-\lambda x_0^2\Big), \quad \lambda > 0, \tag{8}$$

given at $t_0 = T$.

First change the time variables to $\tau = T - t_0$ and rewrite the equation in terms of τ such that $P(x_0, \tau)$ evolves forward in $\tau \in [0, T]$.

Then solve the resulting equation for $P(x_0, \tau)$ using the Fourier transform method.

(3) Mean first-passage time

Show that the mean-first passage time $\langle \tau \rangle = \int_0^\infty t \wp(t) dt$ for a Brownian particle in the interval $[-L, L]$ with absorbing boundaries and starting point $x_0 \in [-L, L]$ satisfies the relation

$$L^+_{\text{FP}} \langle \tau \rangle (x_0) = -1, \tag{9}$$

where

$$L^+_{\text{FP}} = \mu \frac{\partial}{\partial x} + D \frac{\partial^2}{\partial x^2}. \tag{10}$$

Hint: Use the relation between the survival probability $\mathscr{S}(t) = \int_\Omega P(x, t) dx$ and the first-passage time density $\wp(t) = -d\mathscr{S}(t)/dt$.

Then show that the mth moment of the first-passage time satisfies

$$\left(L^+_{\text{FP}}\right)^m \langle \tau^m \rangle (x_0) = (-1)^m \Gamma(1 + m). \tag{11}$$

Compute $\langle \tau \rangle$ for the case $\mu = 0$ and $D > 0$.

Finally solve $\langle \tau \rangle$ for the general case $\mu \neq 0$ and $D > 0$.