Stochastic processes: Assignment sheet 2

(1) Diffusion with constant-concentration boundary condition

Solve the diffusion equation

$$\frac{\partial}{\partial t}C(x,t) = D\frac{\partial^2}{\partial x^2}C(x,t) \tag{1}$$

for the concentration field $C(x,t)$ in a one-dimensional system with the diffusion coefficient D for the following initial and boundary conditions: at $x = 0$ a constant concentration C_0 is applied, at $x = \infty$ assume natural boundary conditions (i.e., C with all its derivatives vanishes); at $t = 0$ the interval $(0, \infty)$ is empty ($C(x, 0) = 0 \ \forall \ x > 0$).
(i) Calculate the "breakthrough curve" $C(x,t)$ for a given, finite x and the "residual breakthrough curve" $1 - C(x,t)/C_0$; show that the latter has a power-law asymptote.
(ii) Calculate the reduced moments $\langle x^n(t)\rangle/C_0$ for $n = 1, 2$ for this problem.

(2) Ornstein-Uhlenbeck process

Solve the Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t) = L_{\mathrm{FP}}(x)P(x,t) \tag{2}$$

with the Fokker-Planck operator

$$L_{\mathrm{FP}} = \frac{\partial}{\partial x}\frac{V'(x)}{m\eta} + K\frac{\partial^2}{\partial x^2} \tag{3}$$

for the PDF $P(x,t)$ with natural boundary conditions at $\pm\infty$, in the harmonic potential

$$V(x) = \frac{1}{2}m\omega^2x^2. \tag{4}$$

(i) Calculate the PDF of this process for a sharp initial condition $P(x, 0) = \delta(x - x_0)$, including the associated stationary solution. Show that this stationary PDF is the Boltzmann distribution for the given $V(x)$. In the above, η is the friction coefficient, m is the mass of the test particle, and ω is a frequency. Check physical dimensions.
(ii) Calculate the first and second moments $\langle x(t)\rangle$ and $\langle x^2(t)\rangle$, as well as the mean-squared displacement $\langle(x(t) - \langle x(t)\rangle)^2\rangle$. Express the MSD as deviation from the thermal value $\langle x^2\rangle_{\mathrm{th}}$ (second moment of the Boltzmann PDF) reached at $t \to \infty$.