Add Example. X1 and X2 are uncorrelated, but not statistically independent

Exercise. Find elements of the covariance matrix of the bivariate Gaussian distribution

$$p(x,y) = const \bullet \exp\left[-\frac{1}{2}\left(ax^2 + 2bxy + cy^2\right)\right] \quad ,$$

where $ac - b^2 > 0$, $a, c > 0$.

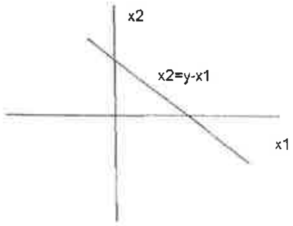
Show that for such distribution the notions of "uncorrelated" and "independent" are equivalent.

↦ 5.10. Multivariate Gaussian (next page)

6. Addition of stochastic variables

Consider r.v. X_1 and X_2, $-\infty < X_1, X_2 < \infty$, joint PDF $p_2(x_1, x_2)$.
$Y = X_1 + X_2$, $p_Y(y) = ?$

6.1. General case: X_1, X_2 are not i.r.v.



Cumulative distribution function:

$$P(y) = \Pr\{Y \le y\} = \int_{-\infty}^{y} p_Y(z)dz = \Pr\{X_1 + X_2 \le y\} = \iint_{x_1 + x_2 \le y} p_2(x_1, x_2) dx_1 dx_2 =$$

$$= \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{y - x_1} dx_2 \; p_2(x_1, x_2) \; ,$$

therefore, the PDF

$$p_Y(y) = \frac{dP(y)}{dy} = \int_{-\infty}^{\infty} dx_1 \frac{d}{dy} \int_{-\infty}^{y - x_1} dx_2 \; p_2(x_1, x_2) \quad ,$$

$$\boxed{p_Y(y) = \int_{-\infty}^{\infty} p_2(x_1, y - x_1) \, dx_1} \quad . \qquad (6.1)$$

<u>*Reminder.*</u> $\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,z)dz = f(x, b(x))\frac{db}{dx} - f(x, a(x))\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,z)dz$.

Remark. Multivariate Gaussian distribution.

$$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}, \quad \vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$\mathbb{C} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}, \quad |\mathbb{C}| = \det \mathbb{C}$$

$$p_n(\vec{x}) = \frac{1}{(2\pi)^{n/2} |\mathbb{C}|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})^T \mathbb{C}^{-1} (\vec{x}-\vec{\mu})\right]$$

$\mathbb{C}^{-1}$ inverse of $\mathbb{C}$, "T" transposed

Isserlis/Wick's Theorem $\vec{\mu} = 0$

$$\langle X_p \ldots X_q \rangle \equiv C_{p\ldots q}$$

$$\underbrace{C_{p\ldots q}}_{\text{odd}} = 0, \quad \underbrace{C_{p\ldots q}}_{\text{even}} = \sum C_{pq} \ldots C_{rs}$$

sum over all possible pairs

Example. $\langle X_1 X_2 X_3 X_4 \rangle = \langle X_1 X_2 \rangle \langle X_3 X_4 \rangle + \langle X_1 X_3 \rangle \langle X_2 X_4 \rangle + \langle X_1 X_4 \rangle \langle X_2 X_3 \rangle$

Back to previous page #6.ua Addition of r.v.

6.2. X_1 and X_2 are i.r.v.

$$p_2(x_1, x_2) = p_{X_1}(x_1) p_{X_2}(x_2)$$

$$\boxed{p_Y(y) = \int_{-\infty}^{\infty} p_{X_1}(x_1) p_{X_2}(y - x_1)\, dx_1} \quad \text{(convolution integral)} \ . \tag{6.2}$$

The same via the CF:

$$\hat{p}_Y(k) = \left\langle e^{ikY} \right\rangle = \left\langle e^{ik(X_1 + X_2)} \right\rangle = \left\langle e^{ikX_1} \right\rangle \left\langle e^{ikX_2} \right\rangle \ ,$$

$$\boxed{\hat{p}_Y(k) = \hat{p}_{X_1}(k) \hat{p}_{X_2}(k)} \ . \tag{6.3}$$

Is (6.3) equivalent to (6.2) ? Let us check.

$$\hat{p}_Y(k) = \int_{-\infty}^{\infty} e^{iky} p_Y(y) = \int_{-\infty}^{\infty} e^{iky} \int_{-\infty}^{\infty} p_{X_1}(x_1) p_{X_2}(y - x_1) dx_1 dy = \left\{ \text{change the order of integration} \right\} =$$

$$= \int_{-\infty}^{\infty} dx_1\, e^{ikx_1} p_{X_1}(x_1) \int_{-\infty}^{\infty} dy\, e^{ik(y - x_1)} p_{X_2}(y - x_1) = \left\{ y - x_1 = x_2 \right\} =$$

$$= \int_{-\infty}^{\infty} dx_1\, e^{ikx_1} p_{X_1}(x_1) \int_{-\infty}^{\infty} dx_2\, e^{ikx_2} p_{X_2}(x_2) = \hat{p}_{X_1}(k) \hat{p}_{X_2}(k) \quad . \text{OK}$$

6.3. Three rules concerning the moments of i.r.v. X_1, X_2, $Y = X_1 + X_2$

(i) the means

$$\boxed{\langle Y \rangle = \langle X_1 \rangle + \langle X_2 \rangle} \ . \tag{6.4}$$

(ii) the variances

$$\boxed{\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2} \ . \tag{6.5}$$

(iii) the CFs

$$\boxed{\hat{p}_Y(k) = \hat{p}_{X_1}(k) \hat{p}_{X_2}(k)} \ . \tag{6.6}$$

Remark. (6.4) – (6.6) *are also valid for sums with* $n > 2$.

6.4. Instructive example: discrete-time random walk

7. Tranformation of variables

We have r.v. X, PDF $p_X(x)$, $Y = f(X)$, $p_Y(y) = ?$

Examples :

(i) plotting on a log-scale $\quad Y = \ln X$

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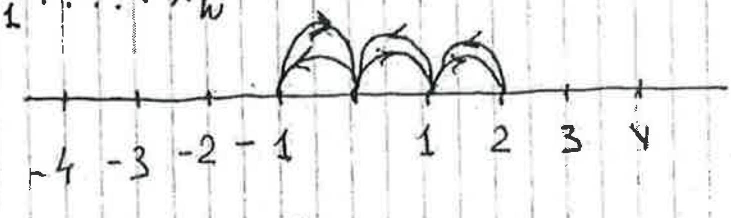
6.4. Instructive example: discrete-time random walk

each set $X_1, X_2, \ldots, X_n$ i.r.v.   time $n \equiv$ time

$$p(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$$

$Y = X_1 + \ldots + X_n$



$$\langle X \rangle = \int_{-\infty}^{\infty} x\, p(x)\, dx = 0 \Rightarrow \langle Y \rangle = 0$$

$$\text{MSD } \langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\, dx = 1 \Rightarrow \langle Y^2 \rangle = n\langle X^2 \rangle = n$$

$\boxed{\text{MSD} \sim \text{number of steps (time)}}$ hallmark of normal diffusion

$$\hat{p}_X(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\, dx = \frac{1}{2}e^{ik} + \frac{1}{2}e^{-ik} = \cos k$$

$$\hat{p}_Y(k) = \hat{p}_{X_1}(k)\,\hat{p}_{X_2}(k)\cdots\hat{p}_{X_n}(k) = \left[\hat{p}_X(k)\right]^n =$$

$$= \frac{1}{2^n}\left(e^{ik} + e^{-ik}\right)^n = \cos^n(k)$$

$$p_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos^n(k)\, e^{-iky}\, dk =$$

$$= \frac{1}{2^n}\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{ik} + e^{-ik}\right)^n e^{-iky}\, dk \Rightarrow$$

Digression. Binomial theorem

$$(x+y)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} y^l ,$$

where

$$\binom{n}{l} = \frac{n!}{(n-l)!\,l!} , \quad \sum_{l=0}^{n} \binom{n}{l} = 2^n$$

$$\Rightarrow \quad \frac{1}{2^n} \frac{1}{2\pi} \sum_{l=0}^{n} \frac{n!}{(n-l)!\,l!} \int_{-\infty}^{\infty} e^{i(n-l)k} e^{-ilk} e^{-iky} dk$$

$$= \frac{1}{2^n} \sum_{l=0}^{n} \frac{n!}{(n-l)!\,l!} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(n-2l-y)k} dk \Rightarrow$$

Digression. $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iky} dk = \delta(y)$

$$\Rightarrow \quad P_Y(y) = \frac{1}{2^n} \sum_{l=0}^{n} \frac{n!}{(n-l)!\,l!} \delta(n-2l-y)$$

Normalization.

$$\int_{-\infty}^{\infty} P_Y(y)\,dy = \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} = 1 \qquad \text{OK}$$

Q. What happens if $n \to \infty$ $(t \to \infty)$?

Very preliminary estimate

$$\hat{p}_Y(k) = \cos^n k$$

$$p_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos^n k \, e^{-iky}\, dy \approx$$

$$\approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[1 - \frac{k^2}{2} + O(k^4)\Big]^n e^{-iky}\, dy =$$

$\uparrow$ neglect (?!) naively: k small?

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{n \ln\left(1-\frac{k^2}{2}\right)} e^{-iky} dk \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{nk^2}{2}} e^{-iky}\, dk$$

$$= \frac{1}{\sqrt{2\pi n}}\, e^{-\frac{y^2}{2n}}$$

Gaussian with

$\langle Y \rangle = 0$

$\langle Y^2 \rangle = n \; (= t^1)$,

see above

Do the same, but differently:

X, Y only integer numbers

$P_Y(y) \rightarrow P_n(j)$ Probability of arriving at site j after n steps

$$P_n(j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos^n k \, e^{-ikj}\, dk =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos^n k \left(\cos kj - i\underbrace{\sin kj}_{=0}\right) dk =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos^n k \cos kj \, dk = \left\{ \begin{matrix} \text{tabulated integral:} \\ \text{MATHEMATICA} \end{matrix} \right\}$$

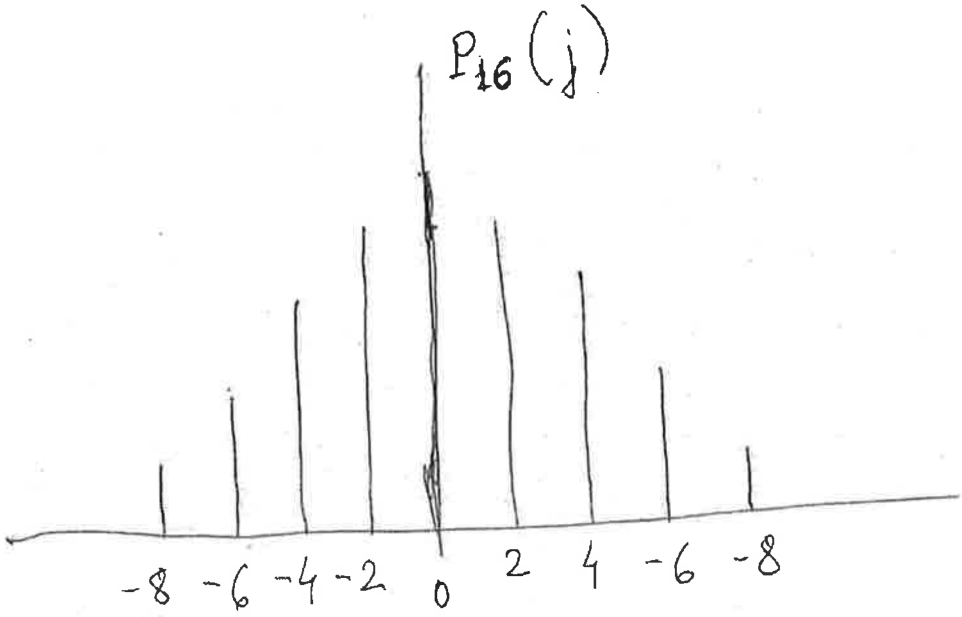
$$= \frac{1}{2^{n+1}}\left[1 + (-1)^{n+j}\right] \frac{n!}{\left(\frac{n+j}{2}\right)!\left(\frac{n-j}{2}\right)!} \qquad (*)$$

Consequences of (*)

1. $P_n(j) = 0$ if n and j have different parities:
   After even number of steps n the walker can reside only on a site with even number j (even $\leftrightarrow$ odd)
2. $P_n(j) = 0$ if $j > n$ due to $(n-j)! = \infty$, $j > n$:
   The displacement of walker ~~never exceeds~~ after n steps never exceeds n

Inset 4

Schematic view



Q. What happens (with $P_n(j)$) if $n \to \infty$?

Reminder. Stirling formula (effective approximation for factorial)

$$n! \simeq \sqrt{2\pi n}\, n^n e^{-n} \qquad (\text{for } n = 2 \text{ relative error} \sim 2\%).$$

$$P_n(j) \simeq \frac{2}{\sqrt{2\pi n}} \exp\left(-\frac{j^2}{2n}\right)$$ Probability to get j after n steps

To interpret as the formula for PDF of j: divide by 2: the sites with either only even or only odd numbers can be reached

$$p_n(j) = \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{j^2}{2n}\right) \leftrightarrow p_Y(y) = \frac{1}{\sqrt{2\pi n}} e^{-y^2/2n}$$

Exercise. Asymmetric random walk

$$p(x) = p\delta(x-1) + q(x+1), \quad p+q = 1$$

7. Transformation of variables

r.v. X, PDF $p_X(x)$, $Y=f(X)$, $p_Y(x)=?$

Examples (i) $Y=\frac{1}{X}$; $Y=\ln X$.

7.1. General case

$$P_Y(y)=\int_{-\infty}^{y} p_Y(z)dz=\Pr\{Y\le y\}=\int_{f(x)\le y} p_X(x)dx=\int_{-\infty}^{\infty} p_X(x)\Theta(y-f(x))dx \quad ,$$

Where $\Theta(x)$ is the Heaviside step function, $\Theta(x)=\begin{cases}1 & , \quad x\ge 0\\ 0 & , \quad x<0\end{cases}$, and $\dfrac{d\Theta(x)}{dx}=\delta(x)$.

Then,

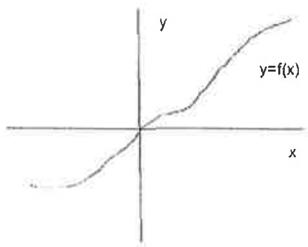
$$\boxed{\frac{dP_Y(y)}{dy}=p_Y(y)=\int_{-\infty}^{\infty} p_X(x)\delta(y-f(x))\,dx} \quad . \qquad (7.1)$$

7.2. CF $\hat{p}_Y(k)$

$$\hat{p}_Y(k)=\int_{-\infty}^{\infty} e^{iky}p_Y(y)dy=\int_{-\infty}^{\infty} e^{iky}\int_{-\infty}^{\infty} p_X(x)\delta(y-f(x))\,dxdy=\int_{-\infty}^{\infty} p_X(x)e^{ikf(X)}dx \ .$$

$$\boxed{\hat{p}_{Y=f(X)}(k)=\left\langle e^{ikf(X)}\right\rangle} \quad . \qquad (7.2)$$

7.3. Particular case: single valued *f*(*X*)



$Y=f(X) \quad \Rightarrow \quad X=f^{-1}(Y)$

Recall. $p_X dx=\Pr\{x\le X<x+dx\}$. Then

$$\Pr\{x_1\le X<x_2\}=\int_{x_1}^{x_2} p_X(x)dx=$$

$$=\{\text{change of variable of integration: } y_{1,2}=f(x_{1,2}), \quad dx=\left|\frac{df^{-1}(y)}{dy}\right|dy\ldots\}=$$

$$= \int_{y_1}^{y_2} p_X\left(x = f^{-1}(Y)\right)\left|\frac{df^{-1}(y)}{dy}\right| dy = \int_{y_1}^{y_2} p_Y(y)dy = \Pr\{y_1 \le Y < y_2\} ,$$

$$\boxed{p_Y(y) = p_X\left(x = f^{-1}(y)\right)\left|\frac{df^{-1}(y)}{dy}\right|} . \tag{7.3}$$

Remark. Keep this in mind as $p_Y(y)dy = p_X(x)dx$ *, but then* $p_Y(y) = p_X(x)\left|\frac{dx}{dy}\right|$.

7.4. Multivalued function $f(X)$

$Y = f(X)$, $X = \varphi_1(Y),\ldots, \varphi_n(Y)$

$$\boxed{p_Y(y) = \sum_{j=1}^{n} p_X\left(x = \varphi_j(y)\right)\left|\frac{d\varphi_j(y)}{dy}\right|} . \tag{7.4}$$

Remark. Eqs. (7.3) and (7.4) can be obtained from (7.1) by using the properties of δ*-function.*

7.5. Linear transformation of variables

$Y = a_1 + a_2 X$, $X = \frac{1}{a_2}Y - \frac{a_1}{a_2}$

$f(X) = a_1 + a_2 X$, $f^{-1}(Y) = \frac{1}{a_2}Y - \frac{a_1}{a_2}$

Use (7.3).

(*a*) PDF

$$\boxed{p_Y(y) = \frac{1}{a_2} p_X\left(x = \frac{y - a_1}{a_2}\right)} . \tag{7.5}$$

(*b*) CF

$$\hat{p}_Y(k) = \left\langle e^{ikY}\right\rangle = \left\langle e^{ik(a_1 + a_2 X)}\right\rangle = e^{ika_1}\left\langle e^{ika_2 X}\right\rangle .$$

$$\boxed{\hat{p}_Y(k) = e^{ika_1}\hat{p}_X(a_2 k)} . \tag{7.6}$$

(*c*) Example

X is a Gaussian r.v. with

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2}{2}\right] \div \exp\left(-\frac{k^2}{2}\right) ,$$

where $\langle X\rangle = 0$, $Var[X] = \left\langle X^2\right\rangle - \langle X\rangle^2 = 1$. Standard normal Gaussian distribution.

Take $a_1 = \mu$, $a_2 = \sigma$, $Y = \mu + \sigma X$, then

$$p_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] \div \exp\left(i\mu k - \frac{\sigma^2 k^2}{2}\right) ,$$

where $\langle Y\rangle = \mu$, $Var[Y] = \langle Y^2\rangle - \langle Y\rangle^2 = \sigma^2$.

7.6. Multidimensional generalization

$\vec{X} = \{X_1, X_2\}$

$Y_1 = f(X_1, X_2)$, $X_1 = \varphi_1(Y_1, Y_2)$, $\varphi_1 = f_1^{-1}$

$Y_2 = f_2(X_1, X_2)$, $X_2 = \varphi_2(Y_1, Y_2)$, $\varphi_2 = f_2^{-1}$

$$\boxed{p_{Y_1,Y_2}(y_1, y_2) = p_{X_1,X_2}(x_1 = \varphi_1(y_1, y_2), x_2 = \varphi_2(y_1, y_2)) \left|\frac{\partial(\varphi_1, \varphi_2)}{\partial(y_1, y_2)}\right|} , \qquad (7.7)$$

where Jacobian $\equiv \left|\frac{\partial(\varphi_1, \varphi_2)}{\partial(y_1, y_2)}\right| = \begin{vmatrix} \frac{\partial\varphi_1}{\partial y_1} & \frac{\partial\varphi_1}{\partial y_2} \\ \frac{\partial\varphi_2}{\partial y_1} & \frac{\partial\varphi_2}{\partial y_2} \end{vmatrix}$.

7.7. Example: Maxwell -> energy distribution

7. <u>Example</u>. Maxwell → Energy distr. (2.10)

3d → 1d

$$p_M(\vec{v}) = \frac{m^{3/2}}{(2\pi kT)^{3/2}} e^{-\frac{mv^2}{2kT}},$$

$$\vec{v} = \{v_x, v_y, v_z\}$$

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right)$$

$$\underline{p_E(E)} = \int \delta\left(E - \frac{1}{2}mv^2\right) p_M(\vec{v})\, d\vec{v} =$$

$$= \int_0^\infty dv\, v^2 \underbrace{\int_0^\pi d\theta \sin\theta}_{} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\, p_M(\vec{v})\, \delta\left(\frac{1}{2}mv^2 - E\right) \Rightarrow$$

$$\hookrightarrow -\int_0^\pi d(\cos\theta) = -\cos\theta \Big|_0^\pi = 2$$

Reminder $\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x-a) + \delta(x+a)\right]$

$$\delta\left(\frac{mv^2}{2} - E\right) = \frac{2}{m}\delta\left(v^2 - \frac{2E}{m}\right) =$$

$$= \frac{2}{2m\sqrt{\frac{2E}{m}}}\left[\delta\left(v + \sqrt{\frac{2E}{m}}\right) + \delta\left(v - \sqrt{\frac{2E}{m}}\right)\right]$$

$$\Rightarrow \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{4\pi}{m\sqrt{\frac{2E}{m}}} \int_0^\infty \left[\delta\left(v + \sqrt{\frac{2E}{m}}\right) + \delta\left(v - \sqrt{\frac{2E}{m}}\right)\right] \times$$

$$\times v^2 e^{-\frac{mv^2}{2kT}}\, dv =$$

$$= \underline{2\pi^{-1/2}(kT)^{-3/2} E^{1/2} e^{-E/kT}}$$

Check

$$\int_0^\infty p_E(E)\, dE = 1$$