WiSe 2024-25. StochPro I.

Lectures 1-2.

Plan. StochPro - Part I.

I. Random Variables

1. Basic definitions and properties
2. Central Limit Theorem
3. Alpa-stable probability laws
4. Generalized Central Limit Theorem

II. Stochastic Processes

1.

- van Kampen
- Gardiner
- Capasso, Bakstein Stoch. Processes...
- Baschnagel Stoch. Processes from Phys to Finance

- Paulos Nimtz
- Maximilian Steuer
- Alek Schiffer

Lectures 1-2 WiSe 24-25

Stochastic Processes and Statistical Methods

I. STOCHASTIC VARIABLES.

CONTENT

1. Basic definitions.
2. Averages.
3. Characteristic function.
4. Canonical probability densities.
5. Multivariate distributions.
6. Addition of stochastic variables.
7. Transformation of variables.
8. Central Limit Theorem. A pedestrian approach.
9. Stable (α - stable) probability laws. Generalized Central Limit Theorem.

ABBREVIATIONS
PDF = probability density function
CF = characteristic function
CDF = cumulative distribution function
r.v. = random variable
i.r.v. = independent random variables
iid r.v. = independent identically distributed random variables
CLT = Central Limit Theorem
GCLT = Generalized Central Limit Theorem
lhs = left hand side (of an equation)
rhs = right hand side (of an equation)

NOTATIONS
$\div$ Fourier (Laplace) transform pair
$X \overset{d}{=} Y$ = equality in distribution

NOTATIONS

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CLT = Central Limit Theorem

1. Basic definitions

1.1. Stochastic variable (random variable, random number)

Definition. A "*random number*" or "*stochastic variable*" X is an object defined by

- set of possible values Ω (called "*range*", "*set of states*", "*sample space*" or "*phase space*");
- probability distribution over this set;

1.2. Set Ω can be

- discrete (number of particles in a reacting mixture);
- continuous (velocity component of Brownian particle, $-\infty < v_x < \infty$; kinetic energy of Brownian particle, $0 < E < \infty$);
- continuous and discrete (energy of electrons in presence of binding states);
- multidimensional; Notation: $\vec{X}$.

1.3. Probability density function (PDF) (probability distribution)

Notation. $p_X(x)$ or simply $p(x)$.

Meaning. $p(x)dx = \Pr\{x \le X < x + dx\}$

Properties.

(i) $p(x) \ge 0$

(ii) $\int_\Omega p(x)dx = 1$

1.4. Cumulative distribution function (CDF).

$$P(x) := \Pr\{X \le x\} = \int_{-\infty}^{x^+} p(x')dx' .$$

Remark: "$-\infty$" at a lower bound with no loss of generality. Suppose $a \le X \le b$. Then

$$p(x) \to \bar{p}(x) = \begin{cases} p(x) & , \quad a \le x \le b \\ 0 & , \quad otherwise \end{cases} .$$

2. Averages

2.1. Average (expectation value)

r.v. X, function $g(X)$ of the random variable X

$$\langle g(X) \rangle := \int_\Omega g(x)p(x)dx \qquad \text{Math notation: } \mathbf{E}[g(X)] \text{ (Expectation value)}$$

2.2. Moments

$g(X) = X^n$, $n = 0.1,2,$

$$\langle X^n \rangle := \int_{\Omega} x^n p(x)dx \equiv \mu_n \quad .$$

$\to$ first moment (mean, average) $\langle X \rangle$, $\mathbf{E}[X]$

$\to$ second moment $\langle X^2 \rangle$, $\mathbf{E}\left[X^2\right]$

$\to$ variance (dispersion) $\sigma^2 := \left\langle \left(X - \langle X \rangle\right)^2 \right\rangle = Var[X] = ... = \mu_2 - \mu_1^2$,

$\to$ standard deviation σ .

The second moment characterizes intensity of stochastic process, variance characterizes intensity of fluctuations. $\mapsto$ next page: Digression

2.3. Chebyshev inequality (*important in data analysis*)

$\boxed{\Pr\left(|X - \mu_1| \geq r\sigma\right) = ?}$, r positive constant .

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu_1)^2 p(x)dx = \left(\int_{-\infty}^{\mu_1 - r\sigma} + \int_{\mu_1 - r\sigma}^{\mu_1 + r\sigma} + \int_{\mu_1 + r\sigma}^{\infty} \right)(x - \mu_1)^2 \, p(x)dx \geq$$

$$\geq \left(\int_{-\infty}^{\mu_1 - r\sigma} + \int_{\mu_1 + r\sigma}^{\infty} \right)(x - \mu_1)^2 \, p(x)dx \;.\sim$$

In the last two integrals $(x - \mu_1)^2 \geq r^2\sigma^2$, thus if we replace $(x - \mu_1)^2 \to r^2\sigma^2$, then

$$\sigma^2 \geq r^2\sigma^2 \left(\int_{-\infty}^{\mu_1 - r\sigma} + \int_{\mu_1 + r\sigma}^{\infty} \right) p(x)dx = r^2\sigma^2 \Pr\left\{|x - \mu_1| \geq r\sigma\right\} \quad ,$$

therefore

$$\boxed{\Pr\left\{|X - \mu_1| \geq r\sigma\right\} \leq \frac{1}{r^2}} \qquad \text{(Chebyshev)}$$

Lecture 1

Consequences.
- $r < 1$ trivial;
- $r = \sqrt{2}$ max 50% of the total probability is beyond $\sqrt{2}$ times standard deviation;
- $r = 2$ max 25% beyond two standard deviations;
- $r = 3$ max 11.(1)% beyond 3 standard deviations;

Equivalent form. $r\sigma \equiv \varepsilon$

$$\boxed{\Pr\left\{|X - \mu_1| \geq \varepsilon\right\} \leq \frac{\sigma^2}{\varepsilon^2}} \qquad (1.2)$$

any distribution with finite 2^{nd} mom.

Remark.

(i) *Three-sigma rule of thumb for Gaussian distribution*: $r = 3 \Rightarrow \sim 0.9973$.

Digression. R.v. *X* discrete set $\Omega = \{x_1, x_2, ..., x_N\}$

$p_X(x) = ?$

$$p_X(x) = p_1\delta(x - x_1) + p_2\delta(x - x_2) + ... + p_N\delta(x - x_N) = \sum_{i=1}^{N} p_i\delta(x - x_i)$$

- Normalization

$$\int_{-\infty}^{\infty} p_X(x)dx = 1 \quad \Rightarrow \quad \sum_{i=1}^{N} p_i = 1$$

- Expectation value

$$\langle X \rangle = \int_{-\infty}^{\infty} x\, p_X(x)dx = \sum_{i=1}^{N} x_i p_i$$

- M trials (probabilistic experiment), M >> 1.

$X = x_1$ in M_1 trials,
$X = x_2$ in M_2 trials
.... M1 + M2 + … + MN = M
$X = x_N$ in M_N

•• experimental probabilities

$$p_{1\exp} \approx \frac{M_1}{M}, \quad p_{2\exp} \approx \frac{M_2}{M}, \quad ..., \quad p_{N\exp} \approx \frac{M_N}{M}$$

•• experimental mean

$$\langle X \rangle_{\exp} = \frac{1}{M}\{x_1 M_1 + ... + x_N M_N\} = \frac{1}{M}\sum_{i=1}^{N} x_i M_i$$

•• at $M \to \infty \quad \Rightarrow \quad p_{i\exp} \to p_i, \quad \langle X \rangle_{\exp} \to \langle X \rangle$

Back to Chebyshev inequality

3. Characteristic function (CF)

Definition.

$$\hat{p}(k) := \int_{-\infty}^{\infty} e^{ikx} p(x)dx \equiv \left\langle e^{ikX} \right\rangle \quad x, k \in \mathbf{R}$$

Fourier transform pair (symbol $\div$):

$$\hat{p}(k) \div p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{p}(k)dk \quad .$$

Note prefactor

Properties.

1) $\hat{p}(0) = 1$.

2) $|\hat{p}(k)| \le \int_{-\infty}^{\infty} \left|e^{ikx}\right| p(x)dx = 1$.

3) $\hat{p}(-k) = \hat{p}^*(k)$.

4) CF as the moment generating function

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)dx = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(ikx)^n}{n!} p(x)dx = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \int_{-\infty}^{\infty} x^n p(x)dx = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n =$$

$$= 1 + ik\mu_1 - \frac{k^2}{2}\mu_2 + \ldots \quad ,$$

thus

$$\mu_1 = \frac{1}{i} \frac{d\hat{p}(k)}{dk}\bigg|_{k=0} \quad ,$$

$$\mu_2 = \frac{1}{i^2} \frac{d^2 \hat{p}(k)}{dk^2}\bigg|_{k=0} = -\frac{d^2 \hat{p}(k)}{dk^2}\bigg|_{k=0} \quad ,$$

Lecture 1

and so on. The general formula reads

$$\boxed{\mu_n = \frac{1}{i^n} \frac{d^n \hat{p}(k)}{dk^n}\bigg|_{k=0}} \quad , \qquad (1.3)$$

5) CF as generating function for cumulants

Use

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad , \quad x^2 < 1, x = -1 \quad .$$

Let $1 - x = \hat{p}(k)$ (note property 2 of the CF). Then

$$\hat{p}(k) = \exp\left[-\sum_{n=1}^{\infty} \frac{(1-\hat{p}(k))^n}{n} \right] .$$

On the other hand (see property 4 of the CF)

$$\hat{p}(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n ,$$

or

$$\hat{p}(k) - 1 = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \mu_n ,$$

Therefore

$$\hat{p}(k) = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{n=1}^{\infty} \frac{(ik)^m}{m!} \mu_m \right)^n \right] ,$$

or

$$\boxed{\ln \hat{p}(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \kappa_n} , \qquad (1.4)$$

where κ_n are called *cumulants* (compare with (1.3). From (1.4) one has

$$\boxed{\kappa_n = \frac{1}{i^n} \frac{d^n}{dk^n} (\ln \hat{p}(k)) \Big|_{k=0}} . \qquad (1.5)$$

<u>Consequences</u>.

$$\kappa_1 = \mu_1 ,$$

$$\kappa_2 = \mu_2 - \mu_1^2 = \sigma^2$$

$$\kappa_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$$

$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4$$

Lecture 1

4. Canonical probability densities

4.1. Gaussian PDF (normal PDF) as arising from the first two non-zero cumulants

Assume κ_1 , $\kappa_2 \neq 0$, $\kappa_3 = \kappa_4 = ... = 0$. Then, the CF takes the form

$$\ln \hat{p}(k) = ik\kappa_1 - \frac{k^2}{2}\kappa_2 \quad , \quad \hat{p}(k) = \exp\left(ik\mu_1 - \frac{k^2}{2}\sigma^2 \right) .$$

Making an inverse Fourier transform, one gets the *Gaussian PDF*

$$\boxed{g(x;\mu_1,\sigma)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma^2}\right) \quad \div \quad g(k;\mu_1,\sigma)=\exp\left(ik\mu_1-\frac{k^2}{2}\sigma^2\right)} \qquad (1.6)$$

Indeed,

$$\hat{p}(k;\mu_1,\sigma)=\int_{-\infty}^{\infty} e^{ikx}\frac{e^{-\frac{(x-\mu_1)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}dx=\text{<introduce } y=\frac{x-\mu}{\sigma}\text{>}=$$

$$=\frac{e^{ik\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(y^2-2ik\sigma y-(k\sigma)^2+(k\sigma)^2\right)\right]dy=$$

$$=\frac{e^{i\mu k-\sigma^2k^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-ik\sigma)^2}=e^{i\mu k-\sigma^2k^2/2}\quad.$$

Important in data analysis ! The higher cumulants κ_n (n = 3,4,) give the quantitative measures how an arbitrary distribution deviates from the *symmetric* Gaussian distribution, $\kappa_1 = 0$.

- *skewness*

$$K_3=\frac{\mu_3}{\sigma^3}\quad;$$

- *kurtosis*

$$K_4=\frac{\kappa_4}{\sigma^4}=\frac{\mu_4-3\mu_2^2}{\sigma^4}=\frac{\mu_4}{\sigma^4}-3\quad.$$

Moments:

$$\langle x\rangle=\mu \qquad \left\langle x^2\right\rangle=-\left.\frac{d^2\hat{p}(k)}{dk^2}\right|_{k=0}=\sigma^2+\mu^2$$

$$\left\langle\left(x-\langle x\rangle\right)^2\right\rangle=\left\langle\left(x^2-2x\langle x\rangle+\langle x\rangle^2\right)\right\rangle=\left\langle x^2\right\rangle-\langle x\rangle^2=\sigma^2$$

4.2. Laplace PDF (double exponential PDF)

$$\boxed{p(x;\mu,b)=\frac{1}{2b}e^{-\frac{|x-\mu|}{b}} \quad \div \quad \hat{p}(k;\mu,b)=\frac{e^{i\mu k}}{1+b^2k^2}} \qquad (1.7)$$

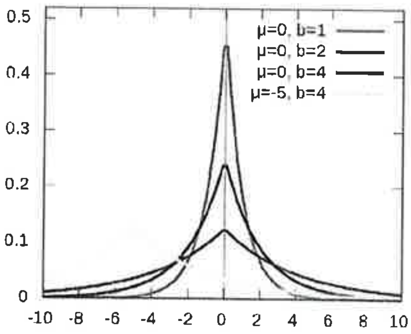
Remarks. 1) Kurtosis (sometimes defined as) $K=\frac{\mu_4}{\sigma^4}$ (no "-3")

greater likelihood of extreme events → • Leptokurtic distribution $K>3$

• Mesokurtic $K=3$

• Platykurtic $K<3$

2) Risk-seeking investors: focus on investments whose returns follow leptokurtic distr.



5.

$$\hat{p}(k;\mu,b)=\frac{1}{2b}\int_{-\infty}^{\infty} e^{ikx-\frac{|x-\mu|}{b}}dx = < y=\frac{x-\mu}{b} > =$$

$$=\frac{e^{ik\mu}}{2}\left\{\int_0^{\infty} e^{(ikb-1)y}dy+\int_0^{\infty} e^{(-ikb-1)y}dy\right\}=\frac{e^{ik\mu}}{2}\left\{\frac{1}{1-ikb}+\frac{1}{1+ikb}\right\}=\frac{e^{i\mu k}}{1+b^2k^2} .$$

- *First moment*

$$\langle X\rangle=\frac{1}{i}\frac{d\hat{p}(k)}{dk}\bigg|_{k=0}=\ldots=\mu \ ,$$

- *Second moment*

$$\langle X^2\rangle=-\frac{d^2\hat{p}(k)}{dk^2}\bigg|_{k=0}=\mu^2+2b^2 \ ,$$

- *Variance*

$$\sigma^2=\left\langle\left(X-\langle X\rangle\right)^2\right\rangle=\langle X^2\rangle-\langle X\rangle^2=2b^2 .$$

Exercise. Skewness and kurtosis.

Q. Leptokurtic or Platykurtic ?...

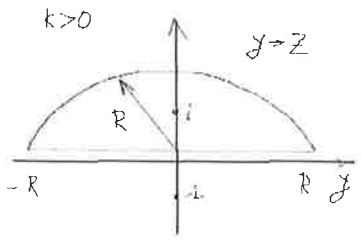
4.3. Cauchy PDF (Cauchy-Lorentz PDF)

$$\boxed{p(x;\mu,b)=\frac{1}{\pi}\frac{b}{(x-\mu)^2+b^2} \quad \div \quad \hat{p}(k;\mu,b)=e^{i\mu k-b|k|}} \qquad (1.8)$$

a) Characteristic function.

$$\hat{p}(k;\mu,b)=\frac{b}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{(x-\mu)^2+b^2}dx = < y=\frac{x-\mu}{b} > = \frac{e^{i\mu k}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikby}}{y^2+1}=\frac{e^{i\mu k}}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikby}}{(y+i)(y-i)} .$$

Reminder. *Residue Theorem.*
For $k>0$ make a closed contour in the upper half-plane:



$$\int_C \frac{e^{ikbz}}{(z+i)(z-i)}dz = 2\pi i\,\mathrm{Res}\{z=i\} = 2\pi i\frac{e^{-bk}}{2i} = \pi e^{-bk}, \quad k>0,$$

$$\int_C ...dz = \int_{-\infty}^{\infty} ...dy + \int_{arc} ...dz \quad .$$

Due to Jordan's lemma

$$\int_{arc} \frac{e^{ikbz}}{1+z^2}dz \le \pi R \sup_{arc}\left|\frac{e^{ikbz}}{1+z^2}\right| \le \pi R \sup_{arc}\left|\frac{1}{1+z^2}\right| \sim \frac{1}{R} \to 0, \text{ since } R\to\infty ,$$

thus

$$\int_{-\infty}^{\infty} \frac{e^{ibky}}{y^2+1} = \pi e^{-bk} \quad , \quad k>0 \quad ,$$

and

$$\hat{p}(k;\mu,b) = e^{i\mu k - bk} , \quad k>0 \quad .$$

Similarly, for $k<0$ we make a closed contour in the lower half-plane to get

$$\hat{p}(k;\mu,b) = e^{i\mu k + bk} , \quad k<0 \ ,$$

and finally

$$\hat{p}(k;\mu,b) = e^{i\mu k - b|k|} \ .$$

$$\langle X\rangle = \frac{b}{\pi}\left\{ \int_0^{\infty} ... + \int_{-\infty}^{0} ... \right\} = \infty - \infty$$

~~undefined~~ not defined (does not exist)

b) moments

Put $\mu = 0$ for simplicity

$$\langle X^2\rangle = \int_{-\infty}^{0} ... + \int_{0}^{\infty} ... = \infty + \infty = \infty$$

infinite

The mean

$$\langle X\rangle = \int_{-\infty}^{\infty} xp(x)dx = \frac{b}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+b^2}dx = \frac{b}{\pi}\lim_{\substack{L_1\to-\infty\\ L_2\to\infty}}\int_{L_1}^{L_2}\frac{x}{x^2+b^2}dx =$$

$$= \lim_{\substack{L_1\to-\infty\\ L_2\to\infty}} \ln\left(x^2+b^2\right)\Big|_{L_1}^{L_2} = \frac{b}{\pi}\lim_{\substack{L_1\to-\infty\\ L_2\to\infty}} \ln\left|\frac{L_2}{L_1}\right| \quad \text{undefined} \quad .$$

~~*All integer moments* $\langle X^n\rangle$, $n=1,2,...,$ *are undefined.*~~

Remark. *The Cauchy principal value.*

Exercise. Fractional moments for the Cauchy distribution.

4.4. Exponential PDF

Positive r.v. T, PDF $p_T(t)$

a) Laplace transform pair (symbol $\div$ for the Laplace transform pair, same as for the Forier transform pair, tilde ~ to denote Laplace transform)

$$p_T(t) \div \tilde{p}_T(s) \equiv \left\langle e^{-sT} \right\rangle \equiv \mathbf{E}e^{-sT} = \int_0^\infty e^{-st} p_T(t)dt \quad \div \quad p_T(t) = \frac{1}{2\pi i} \int_{Br} e^{st} \tilde{p}_T(s)ds$$

Laplace transform is more convenient than Forier transform for positive r.v.

b) Definition. *Exponential PDF*

$$\boxed{p(t;\lambda) = \begin{cases} \lambda e^{-\lambda t} & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}} \quad . \tag{1.9}$$

c) *Laplace tranform*

$$\tilde{p}(s;\lambda) = \frac{\lambda}{\lambda + s} \quad \div \quad p(t;\lambda)$$

d) *moments*

$$\left\langle T^n \right\rangle = \int_0^\infty t^n p(t;\lambda)dt = (-1)^n \left. \frac{d^n \tilde{p}(s;\lambda)}{dt^n} \right|_{s=0} = \frac{n!}{\lambda^n} \quad .$$

5. Multivariate distributions

5.1. Joint probability density function (PDF, probability distribution)

r.v. $\vec{X} = \{X_1, X_2, ..., X_n\}$ on n-dimensional set Ωn.

a) meaning

$$p_n(x_1, x_2, ..., x_n)\, dx_1 dx_2 ... dx_n =$$
$$= \Pr\{x_1 \le X_1 < x_1 + dx_1; x_2 \le X_2 < x_2 + dx_2; ...; x_n \le X_n < x_n + dx_n\} \ .$$

b) non-negativity

$$p_n(x_1, x_2, ..., x_n) \ge 0 \ .$$

c) normalization

$$\int_{\Omega_n} p_n(x_1, x_2, ..., x_n)\, dx_1 dx_2 ... dx_n = 1 \ .$$

ADD Example. Bivariate Gaussian distribution

5.2. Cumulative distribution function

$$P_n(x_1, x_2, ..., x_n) = \Pr\{-\infty < X_1 \le x_1; -\infty < X_2 \le x_2; ...; -\infty < X_n \le x_n\} \ .$$

Remark. "-∞" with no loss of generality.

5.3. Marginal probability density (probability distribution)

Subset $X_1, X_2, \ldots, X_s$, $s < n$

$$p_s(x_1, x_2, ..., x_s) = \int p_n(x_1, ..., x_s, x_{s+1}, ..., x_n)\, dx_{s+1} ... dx_n$$

5.4. Conditional probability density

Fix $X_{s+1}, ..., X_n$.

Conditional PDF

$$p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n)$$

Normalization:

$$\int p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n)\, dx_1 dx_2 ... dx_s = 1 \ .$$

5.5. Bayes' rule

$$p_n(x_1, x_2, ..., x_n) = p_{n-s}(x_{s+1}, ..., x_n)\, p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n) \ .$$

$$\boxed{p_{s|n-s}(x_1, ..., x_s \mid x_{s+1}, ..., x_n) = \frac{p_n(x_1, x_2, ..., x_n)}{p_{n-s}(x_{s+1}, ..., x_n)}} \ . \qquad (5.1)$$

5.6. Moments

$$\left\langle X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n} \right\rangle = \int x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \, p_n(x_1, x_2, \ldots, x_n) \quad .$$

5.7. Characteristic function

$$\hat{p}_n(k_1, k_2, \ldots, k_n) \equiv \left\langle \exp(ik_1 X_1 + ik_2 X_2 + \ldots + ik_n X_n) \right\rangle =$$

$$= \int e^{ik_1 x_1 + ik_2 x_2 + \ldots + ik_n x_n} \, p_n(x_1, x_2, \ldots, x_n) =$$

$$= \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(ik_1)^{m_1} (ik_2)^{m_2} \ldots (ik_n)^{m_n}}{m_1! \, m_2! \ldots m_n!} \langle X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n} \rangle$$

5.8. Covariance matrix and correlation coefficients

$$C_{ij} = \left\langle (X_i - \langle X_i \rangle)(X_j - \langle X_j \rangle) \right\rangle = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle \quad .$$

Diagonal elements : variances, non-diagonal : covariances.

$$\rho_{ij} = \frac{\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle}{\sqrt{\left(\langle X_i^2 \rangle - \langle X_i \rangle^2\right)\left(\langle X_j^2 \rangle - \langle X_j \rangle^2\right)}} \quad .$$

Exercise. Prove 1) $-1 \le \rho_{ij} \le 1$; 2) if $\rho_{12} = \pm 1 \;\Rightarrow\; X_2 = aX_1 + b$, $a, b = const$.

5.9. Statistical independence

Definition. Take $n = 2$. X_1 and X_2 are *independent random variables* (i.r.v.) if

– joint PDF factorizes,

$$p_2(x_1, x_2) = p(x_1) p(x_2) \quad ,$$

or equivalently,

– all moments factorize,

$$\left\langle X_1^{m_1} X_2^{m_2} \right\rangle = \left\langle X_1^{m_1} \right\rangle \left\langle X_2^{m_2} \right\rangle \quad ,$$

or equivalently

– characteristic function factorizes,

$$\hat{p}_2(k_1, k_2) = \left\langle \exp(ik_1 X_1 + ik_2 X_2) \right\rangle = \left\langle e^{ik_1 X_1} \right\rangle \left\langle e^{ik_2 X_2} \right\rangle = \hat{p}(k_1) \hat{p}(k_2) \quad .$$

Definition. X_1 and X_2 are *uncorrelated* if $C_{12} = 0$. Weaker than statistical independence.

irv $X_1, X_2 \Rightarrow$ uncorrelated

uncorr. $X_1, X_2 \not\Rightarrow$ always irv