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Conformable scaling and critical phenomena: A unified framework for phase transitions

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We investigate the application of conformable derivatives to model critical phenomena near continuous phase transition points. By incorporating a deformation parameter into the differential structure, we derive unified expressions for thermodynamic observables such as heat capacity, magnetisation, susceptibility, and coherence length, each exhibiting a power-law behaviour near the critical temperature. The conformable derivative framework naturally embeds scale invariance and critical slowing down into the dynamics without resorting to fully nonlocal fractional calculus. Modified Ginzburg–Landau equations are constructed to model superconducting transitions, leading to analytical expressions for the order parameter and the London penetration depth. Experimental data from niobium confirm the model's applicability, showing excellent fits and capturing asymmetric scaling behaviour around the critical point. This work offers a bridge between classical mean-field theory and generalised scaling frameworks, with implications for both theoretical modelling and experimental analysis.

Keywords: Conformable derivative; Modified Ginzburg-Landau equation;Critical phenomena; Critical scaling; Continuous phase transition; Thermodynamic consistency; Emergent dynamics; Power-law behaviour.

I. INTRODUCTION

Critical phenomena play a central role in the understanding of phase transitions, where physical observables such as the heat capacity, magnetisation, and susceptibility exhibit a non-analytical behaviour characterised by power-law divergences near a critical temperature T_c [1–3]. These behaviours are traditionally described through scaling hypotheses and renormalisation group (RG) analysis, which explain the emergence of universality across diverse systems. These phenomena range from magnetic materials and superconductors to liquid-gas transitions and even biological systems [2, 3]. Near critical points, physical systems exhibit remarkable scaling behaviour characterised by power-law divergences of thermodynamic quantities, with critical exponents that depend only on fundamental symmetries and spatial dimensionality rather than microscopic details [4, 5]. This universality principle, established through decades of theoretical and experimental work, forms the cornerstone of the modern theory of critical phenomena.

The mathematical description of critical behaviour has evolved significantly since the early phenomenological approaches of Landau and Ginzburg [6, 7]. The subsequent development of Ginzburg-Landau theory, particularly as systematised by de Gennes [8], provided a robust framework for understanding superconducting phase transitions and scaling behaviour near critical points. The development of renormalisation group theory by Wilson and others [4, 9] provided deep insights into the origin of universality and scaling laws, while field-theoretic methods enabled precise calculations of critical exponents [10]. These advances established that critical phenomena emerge from the competition between thermal fluctuations and ordering tendencies, with correlation lengths diverging as $\xi \sim |T - T_c|^{-\nu}$ and other observables following characteristic power-laws.

Despite these theoretical successes, many experimental systems exhibit deviations from ideal critical behaviour due to finite-size effects, quenched disorder, non-equilibrium conditions, and other realistic complications [11–13]. There remains a need for alternative frameworks that provide analytical tractability, flexible modelling, and connections to generalised thermodynamic formalisms. This is particularly relevant in systems where microscopic details are poorly known, long-range interactions are present, or nonlocal and memory effects become significant. Real materials often deviate from ideal critical behaviour due to quenched disorder, finite-size effects, and non-equilibrium conditions. These deviations often manifest as modified scaling laws, anomalous relaxation dynamics, and effective critical exponents that differ from theoretical predictions. Understanding and modelling such complex critical behaviour remains an active area of research with important implications for materials science, statistical physics, and complex systems.

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In this work, we explore a deformation-based approach to critical phenomena using *conformable derivatives* [14–21], a form of generalised calculus that introduces a deformation parameter μ into the derivative operator

$$D_T^{(\mu)} f(T) \coloneqq T^{1-\mu} \frac{df}{dT}. \qquad (1)$$

This operator naturally yields temperature-dependent scaling behaviour and introduces power-law features without requiring full fractional integration or nonlocal convolution terms. It thus offers an intermediate practical framework settled between classical and fractional derivatives; and retains analytical simplicity while capturing essential scaling features.

The conformable derivative framework addresses specific limitations in existing approaches to critical phenomena. Real materials exhibit: (i) *Spatial inhomogeneities* leading to effective fractal geometries with non-integer dimensions; (ii) *Temporal memory effects* from quenched disorder or slow relaxation processes; and (iii) *Finite-size constraints* that modify critical scaling. Each of these physical mechanisms naturally leads to temperature-dependent weighting factors of the form $T^{1-\mu}$, providing direct physical justification for definition (1). Unlike purely phenomenological approaches, the conformable-derivative framework connects the deformation parameter μ to measurable physical quantities: fractal dimensions in porous superconductors, anomalous diffusion exponents in disordered systems, and finite-size scaling exponents in thin films.

We note that while Eq. (1) might appear to be merely a change of variables $T \to T'$ with a power-law transformation, the physical motivation and mathematical implementation differ fundamentally from simple rescaling. Namely, the conformable derivative (1) introduces a temperature-dependent weighting that reflects a *physical process* rather than mathematical convenience. These process aspects include: (i) anomalous diffusion and memory effects in disordered systems [22–26], (ii) finite-size scaling in confined geometries [5, 11, 27], and (iii) non-equilibrium relaxation dynamics near criticality [13, 28, 29]. Unlike a simple coordinate transformation, the deformation parameter μ emerges from the underlying physics and connects to measurable quantities such as fractal dimensions and effective transport exponents. Furthermore, the critical exponents derived here are not invariant under the transformation because the physical interpretation changes. The parameter μ captures deviations from mean-field behaviour that would otherwise require complex renormalisation group calculations or phenomenological fitting.

Formally, $D_T^{(\mu)} f = T^{1-\mu}\, df/dT$ can be rewritten as a derivative with respect to $T' = \frac{T^\mu}{\mu}$ only if all coefficients are constant or trivial functions of T'. In our case this is not true for two reasons: (i) the kinetic coefficient $\Gamma(T) \sim |T - T_c|^{z\nu}$ retains its critical singularity under the transformation, preserving nontrivial scaling; (ii) in the conformable Ginzburg-Landau (GL) functional, the gradient term acquires a weight $T^{2(1-\alpha)}$, producing a non-uniform thermal metric, as we will see. After reparametrisation, Jacobian factors survive and prevent a reduction to a standard GL form. Thus the conformable framework encodes a dynamical and variational structure that is not removed by a change of variables.

Here and throughout this work, the symbol "$\sim$" denotes asymptotic proportionality: for two quantities A and B, the relation $A \sim B$ means that their ratio tends to a finite, non-zero constant as the critical point is approached,

$$\lim_{T\to T_c} \frac{A}{B} = C, \qquad 0 < C < \infty. \qquad (2)$$

In other words, A and B share the same leading power-law behaviour near the singularity, while multiplicative prefactors may differ.

We apply this formalism to describe thermodynamic observables near continuous phase transitions, including heat capacity, magnetisation, susceptibility, and correlation length. In each case, we derive the critical exponents as functions of the deformation parameter and temperature, leading to expressions compatible with known mean-field results. Furthermore, we extend the method to superconducting phase transitions by constructing a modified Ginzburg–Landau (MGL) equation with conformable kinetic terms. This allows us to analyse the temperature dependence of the superconducting order parameter, London penetration depth, and heat capacity within the same mathematical framework.

We argue that with this conformable-derivative approach it is possible to obtain a unified formalism for deriving all major critical exponents with analytical expressions that match experimental data with high fidelity. This formalism thus represents a bridge between classical mean-field theory and generalised (e.g., Tsallis) statistical mechanics—and also a minimal extension of known differential equations without resorting to nonlocal fractional operators.

By maintaining dimensional consistency and compatibility with equilibrium thermodynamics, the conformable-derivative framework provides a useful and versatile tool for exploring critical dynamics in both classical and quantum systems. The results presented here demonstrate the capacity of this formalism to reproduce known scaling laws while offering new insights into generalised scaling behaviour near criticality.

We derive unified expressions for critical exponents in terms of conformable parameters that emerge from these physical mechanisms, demonstrating that the approach extends rather than contradicts renormalisation group theory. The formalism is applied to magnetic phase transitions, superconducting transitions, and other critical phenomena, with specific predictions for experimental verification.

II. RATIONALE BEHIND THE CONFORMABLE DERIVATIVE FORMULATION AND STATISTICAL-MECHANICAL INTERPRETATION OF THE CONFORMABLE INDEX μ

Before presenting our results we first provide a physical rationale for the proposed formulation in terms of conformable derivatives, arguing that there is added physical content.

At a purely formal level, the operator $D_T^{(\mu)} f(T) = T^{1-\mu} df/dT$ defined in Eq. (1) can be mapped onto an ordinary derivative by the change of variables $T \mapsto T' = T^\mu$. However, this mathematical equivalence does not render the conformable framework physically trivial, for the following reasons:

First, the conformable derivative is not introduced as a post hoc reparametrisation of solutions, but as a modification of the *generator of the dynamics* itself. The evolution equation is written directly in terms of $D_T^{(\mu)}$, meaning that scaling is built into the operator level rather than imposed afterward by a coordinate transformation. This distinction mirrors well-established cases in physics where non-linear reparametrisation (e.g., logarithmic scales in renormalisation-group flows or proper time in relativistic dynamics) acquire physical meaning once promoted to fundamental variables.

Second, embedding the scaling exponent μ into the differential operator ensures internal consistency across thermodynamic relations and dynamical equations, as also evidenced by the results presented in what follows. A mere change of variables would rescale a single equation but would not automatically preserve the structure of the free-energy functional, along with the fluctuation-response relations, or scaling links between distinct observables. In contrast, the conformable formulation yields a closed and self-consistent framework in which critical exponents associated with different observables are systematically related through their respective deformation parameters μ_Q.

Finally, the conformable approach confers a direct physical interpretation to μ as an effective coarse-grained index encoding memory effects, disorder-induced heterogeneity, or geometric constraints, as discussed above. Once fixed by one observable, μ becomes a measurable material-dependent parameter whose value can be tested against independent predictions, rather than an arbitrary coordinate choice. Its derivation, as with any generalised model departing from universal formulations, needs to be considered separately for a given system.

For these reasons, while Eq. (1) admits a formal reparametrisation, the conformable derivative in our view provides a nontrivial and physically motivated extension of the Ginzburg–Landau dynamics [30], retaining locality while incorporating anomalous scaling at the level of the governing operators.

Interpretation of the conformable index μ

A recurring concern in generalised-calculus approaches is whether the introduced indices or orders, such as the fractional-order derivative α or, here, the deformation index μ, represent genuine physical content, beyond serving as a phenomenological fitting exponent. For fractional-order derivatives used in the formulation of anomalous diffusion, the fractional derivative of order α reflects the hydrodynamic limit of a continuous-time random walk (CTRW) process with scale-free immobilisation time probability density $\psi(\tau) \simeq \tau^{-1-\alpha}$ $(0 < \alpha < 1)$ translating into the power-law scaling in time of the mean-squared displacement, $\langle \mathbf{r}^2(t) \rangle \simeq t^\alpha$ and reflects the effective memory of the process [22–26]. Such waiting time statistics have indeed been identified in various systems, including protein channel motion in living cell membranes [31], molecular diffusion along surfaces such as silica nanoslits or membranes [32, 33], or internal protein dynamics [34], and it has been revealed in the long-time motion of chemical tracers in porous rocks [35, 36]. Formulations with such fractional operators are highly profitable when boundary value problems are considered or to determine the motion in the presence of external force fields [22, 23, 26].

In the present work, the index μ is interpreted as an *effective coarse-grained index* that summarises (over a finite experimental window) the dominant mechanisms that broaden the spectrum of critical relaxation times or "deform" the effective thermodynamic/mesoscopic metric. This viewpoint is standard in the statistical physics of complex media: distinct microscopic sources (quenched disorder, finite size, heterogeneity, slow modes) often renormalise into a small number of *effective exponents* controlling scaling forms near T_c [11, 12, 22–27]. Accordingly, the role of μ is not to *uniquely* identify each microscopic ingredient, but to provide a compact, analytically tractable parametrisation of their *net* impact on the observed scaling. Inter alia, this is relevant for the following contexts:

(i) Broad relaxation spectra and coarse-grained memory. Near criticality the divergence of the correlation time implies a hierarchy of slow modes and, in many realistic materials, a broad distribution of relaxation times due to heterogeneity or disorder. A standard statistical-mechanical representation of such dynamics is provided in terms of a generalised linear response with a memory kernel,[1]

$$\frac{dX}{dt} = -\int_0^t K(t-t')X(t')dt', \tag{3}$$

whose Laplace-domain solution encodes the long-time tails and non-Debye relaxation [38]. For power-law ker-

[1] In chemical physics, such phenomena are conventionally expressed in terms of relaxation time spectra, compare [37].

nels (or equivalently, for broad waiting-time statistics in CTRWs) one obtains fractional evolution laws and Mittag–Leffler relaxation patterns [22–26]. The conformable operator used here can be viewed as a *local coarse-grained surrogate* of such nonlocal descriptions, when the experimentally resolved window is dominated by a single effective scaling and one seeks a local differential structure (avoiding convolution integrals) while retaining the correct homogeneity. In this sense, μ encodes the *effective width* of the relaxation spectrum (or, equivalently, the effective intermittency/subdiffusivity exponent) that would otherwise be represented by a memory kernel or a fractional derivative.

(ii) Geometric/mesoscopic heterogeneity and deformed metrics. Independent of dynamical memory, quenched spatial heterogeneity (granularity, porous/fractal microstructures, percolative pathways) deforms the effective metric on which coarse-grained fields vary. In such cases, local derivatives on a deformed (Hausdorff-type) measure naturally produce weighted differential operators of the form $x^{1-\mu}d/dx$; the conformable derivative thus represents a local metric deformation consistent with fractal-like geometries and scale-dependent transport [19–21]. This interpretation is particularly relevant for mesoscopic superconductors and disordered condensed-matter systems, where the order parameter explores a structurally heterogeneous environment and the effective stiffness/gradient contributions acquire scale-dependent weights.

(iii) Why one μ can still represent "many" mechanisms (and when it should not). One may argue that real deviations from ideal criticality may stem from multiple sources (disorder, finite size, non-equilibrium protocols). The conformable framework does not claim that a *single universal* μ resolves these mechanisms individually. Rather, μ is used as an *effective exponent* that summaries the dominant renormalised effect in the regime where a single scaling description is observed. Moreover, the manuscript does *not* restrict the theory to one global parameter: different observables may probe different channels and therefore are allowed to carry distinct indices μ_Q (as implemented below for the observables $Q \in \{C, |M|, \chi, \xi\}$), and the parameters may also be piecewise (above/below T_c) when asymmetry is observed in data. This is precisely the minimal generalisation needed to account for the fact that disorder and finite-size effects renormalise static and dynamic responses differently, while still keeping a closed-form analytic structure. In that sense the role of the index μ is similar to the role of the anomalous diffusion exponent α in the mean-squared displacement $\langle \mathbf{r}^2(t)\rangle \simeq t^{\alpha}$, whose physical origin may have various origins [39, 40].

Finally, we note that the conformable framework presented here becomes *falsifiable* (rather than merely descriptive) because the μ_Q are not arbitrary: once calibrated for one observable, cross-observable constraints can be tested against independent data (see the consistency relations derived below). Thus, the conformable index is grounded in standard coarse-graining logic: complex microscopic physics renormalises into effective scaling indices, and the conformable derivative provides a local operator that embeds these indices directly into the governing equations.

Additional remarks on the physical interpretation of μ

To further clarify the microscopic origin of the deformation parameter μ, we refer back to systems in the presence of quenched disorder, geometric heterogeneity, or fractal-like inhomogeneities. In such media, the density of states and the effective phase-space volume explored by fluctuations do not follow standard Euclidean scaling. Instead, transport and relaxation are often governed by a hierarchy of effective dimensions, notably the fractal (Hausdorff) dimension d_f, the spectral dimension d_s, and the random-walk dimension d_w [41, 42].

For disordered or fractal substrates, the mean-squared displacement typically scales as $\langle \mathbf{r}^2(t)\rangle \sim t^{2/d_w}$, with $d_w > 2$, reflecting subdiffusive exploration of configuration space [43]. Concurrently, the return probability and low-energy density of states are controlled by the spectral (or fracton) dimension d_s, which governs the long-time decay of correlation functions [41, 44]. In this coarse-grained regime, the conformable derivative $D_T^{(\mu)}$ can be viewed as a local operator that effectively encodes these non-Euclidean features into the thermal evolution of the order parameter.

Within this interpretation, the deformation index μ may be identified, at the level of effective scaling, with the dominant spectral weight of the environment,

$$\mu \sim \frac{d_s}{2} \sim \frac{d_f}{d_w}, \tag{4}$$

where the symbol "$\sim$" emphasises that this relation is not universal but represents an asymptotic, coarse-grained correspondence valid in the long-time or near-critical regime. When $\mu = 1$, one recovers the classical Euclidean bath with $d_s = 2$; when $\mu < 1$, the reduced connectivity of disordered lattices, fractal geometries, or finite-size constraints effectively "compresses" the accessible phase space and slows down relaxation.

Although μ appears as a single parameter, it plays a role analogous to the dynamic exponent z in renormalisation-group theory: it packages complex microscopic mechanisms into a single measurable scaling index. Importantly, different observables may probe distinct relaxation channels and therefore may be associated with different effective indices μ_Q, a possibility explicitly allowed and exploited in the present framework. Embedding this effective exponent directly into the differential operator corresponds to a local reparametrisation of the thermal or temporal coordinate, yielding evolution equations that remain local while naturally encoding scale invariance and anomalous critical dynamics [14, 21, 30].

We emphasise that the physical origin of conformable derivatives is treated in detail in the previous work [30]. There, conformable dynamics are derived from microscopic considerations including quenched disorder, heterogeneous relaxation spectra, coarse-grained memory kernels, and effective metric deformations induced by structural complexity. The present manuscript builds on that foundation and focuses specifically on the implications of such conformable operators for Ginzburg–Landau critical dynamics, along with clear justifications from experimental data, rather than repeating the full microscopic derivation.

III. CRITICAL DYNAMICS, RELAXATION FORMS, AND ERGODICITY

Dynamic scaling and critical slowing down. Near the critical temperature T_c the characteristic relaxation time diverges as as the correlation length $\xi \sim |T - T_c|^{-\nu}$ increases [2, 3, 45]

$$\tau \sim \xi^z \sim |T - T_c|^{-z\nu}, \tag{5}$$

with the dynamic exponent z and the correlation-length exponent ν. Time-dependent correlations obey the scaling form

$$C_A(t;T) \equiv \langle A(t)A(0)\rangle_c = t^{-\lambda_C/z}\mathcal{F}\left(\frac{t}{\tau}\right), \tag{6}$$

where λ_C is a (model-dependent) autocorrelation exponent and $\mathcal{F}(x)$ is a universal scaling function. Far from criticality ($t \ll \tau$) one typically observes an exponential decay; at $T \simeq T_c$ ($\tau \to \infty$) power-law tails dominate:

$$C_A(t) \propto t^{-\lambda_C/z} \qquad (T = T_c). \tag{7}$$

Typical relaxation functions. Depending on whether the system falls into Model A or Model B, as defined by Hohenberg and Halperin [46], and on possible disorder or heterogeneity, the following forms are typically obtained:

(i) Exponential: $\phi(t) \sim e^{-t/\tau}$, away from T_c;

(ii) Stretched exponential (KWW): $\phi(t) \sim \exp(-[t/\tau]^{\beta_{\mathrm{KWW}}})$ $(0 < \beta_{\mathrm{KWW}} < 1)$, heterogeneous or glassy systems;

(iii) Power law: $\phi(t) \sim t^{-p}$ (at or near T_c), critical slowing down;

(iv) Mittag–Leffler (fractional/conformable) [47]: $\phi(t) \sim E_\mu\left(-[t/\tau]^\mu\right)$ $(0 < \mu \le 1)$, broad distributions of time scales.

Here, the one-parameter Mittag-Leffler function, defined as [47]

$$E_\mu\left(-\left[\frac{t}{\tau}\right]^\mu\right) = \sum_{k=0}^{\infty}\frac{(-[t/\tau]^\mu)^k}{\Gamma(1+\mu k)}, \tag{8}$$

admits well-defined series expansions for small and large arguments. For small argument, $(t/\tau)^\mu \ll 1$, one obtains

$$E_\mu\left(-\left[\frac{t}{\tau}\right]^\mu\right) = 1 - \frac{1}{\Gamma(1+\mu)}\left(\frac{t}{\tau}\right)^\mu + \frac{1}{\Gamma(1+2\mu)}\left(\frac{t}{\tau}\right)^{2\mu} + O\left(t^{3\mu}\right). \tag{9}$$

Thus, at short times the Mittag-Leffler relaxation dynamics coincides with the stretched exponential form [48]

$$E_\mu\left(-\left[\frac{t}{\tau}\right]^\mu\right) \sim \exp\left(-\frac{1}{\Gamma(1+\mu)}\left[\frac{t}{\tau}\right]^\mu\right), \qquad t \ll \tau, \tag{10}$$

showing explicitly the link between fractional or conformable relaxation and heterogeneous KWW-kinetics.

For large arguments $x \gg 1$, the expansion and asymptotic form are

$$E_\mu\left(-\left[\frac{t}{\tau}\right]^\mu\right) = -\sum_{k=1}^{\infty}\frac{(-[t/\tau]^\mu)^{-k}}{\Gamma(1-\mu k)} \sim \frac{1}{x\,\Gamma(1-\mu)}, \qquad x \to \infty, \tag{11}$$

which corresponds to the slow algebraic decay characteristic of critical slowing down.

These expansions show how a single function interpolates between stretched exponential behaviour at short times and a power-law decay at long times.

Ergodicity Considerations. Near critical points, several ergodicity-related issues arise:

(i) Ergodicity breaking timescale: The system may appear non-ergodic on experimental timescales as τ diverges when approaching Tc;

(ii) Critical slowing down: The diverging relaxation time means the system takes increasingly long to explore phase space, leading to practical ergodicity breaking;

(iii) Finite-size effects: In finite systems, true ergodicity is maintained but at timescales $\tau_{\mathrm{erg}} \sim L^z$, where L is the system size.

The connection to conformable framework can be seen from observing how the conformable derivative naturally captures anomalous relaxation. The modified time derivative $D_t^{(\mu)}$ effectively introduces a time-dependent relaxation kernel. This could model subdiffusive dynamics in the above sense.

At thermal *equilibrium* near a continuous transition, the dynamics are (asymptotically) ergodic in finite systems; criticality does not by itself break ergodicity, but the divergent timescale τ makes mixing arbitrarily slow ("critical slowing down"). *Non*-ergodicity (or weak ergodicity breaking [49, 50]) arises when additional ingredients are present: quenched disorder/spin-glass physics, constraints, or genuine non-equilibrium protocols (quenches) that produce ageing, where two-time correlations $C(t, t_w)$ lose time-translational invariance [49, 51]. In our conformable setting, $\mu < 1$ encodes such slow, broad-spectrum kinetics; equilibrium ergodicity is retained, but relaxation becomes non-Debye and may display ageing under driven or quenched conditions. Practically, this means that time-averages converge to ensemble averages, yet on time scales that diverge as $\tau \sim |T - T_c|^{-z\nu}$.

It is also important to note that critical exponents admit complementary geometrical interpretations in terms of emergent fractal structures and fractional-Laplacian formulations at criticality, see, e.g., [52–55]; the relation of the present conformable thermal deformation to these spatial-geometric viewpoints is discussed below.

IV. CRITICAL SCALING AND CONFORMABLE DERIVATIVES IN PHYSICAL SYSTEMS

Near critical points, physical systems exhibit a *critical slowing down*, where the relaxation time τ diverges as the correlation length $\xi \sim |T - T_c|^{-\nu}$ increases. This behaviour is characterised by the dynamic scaling law (5). Observable thermodynamic quantities exhibit a characteristic power-law behaviour in the vicinity of continuous (second-order) phase transitions. Specifically, the *heat capacity* diverges as

$$C(T) \sim |T_c - T|^{-\alpha}, \tag{12}$$

where α is the critical exponent associated with the specific heat. The *spontaneous magnetisation* follows the relation

$$|M(T)| \sim (T_c - T)^{\beta}, \quad \text{for } T < T_c, \tag{13}$$

where β denotes the critical exponent for the order parameter. Moreover, the *magnetic susceptibility* diverges according to

$$\chi(T) \sim |T_c - T|^{-\gamma}, \tag{14}$$

where γ is the critical exponent characterising the divergence of the susceptibility. The exponents α, β, and γ define the universal scaling behaviour near the critical point and are largely independent of the microscopic details of the system. We will show that this scaling naturally introduces temperature-weighted derivatives that match the scaling behaviour of critical observables. This property embeds scale invariance directly into the differential structure of the equations, enabling straightforward generation of power-law solutions.

The kinetic coefficient $\Gamma(T)$ plays a central role in the time-dependent relaxation dynamics of systems near criticality. It essentially measures how rapidly a system returns to equilibrium after a small perturbation. Physically, it serves as a mobility-like parameter, often appearing in the Langevin equation or time-dependent Ginzburg–Landau models as a prefactor to the functional derivative of the free energy [13, 45, 56, 57]. Since the kinetic coefficient $\Gamma(T)$ is inversely proportional to the relaxation time, it inherits the corresponding divergence

$$\Gamma(T) \sim |T_c - T|^{z\nu}. \tag{15}$$

This reflects the critical slowing down of the dynamics near T_c: the relaxation time diverges, and the kinetic response $\Gamma(T)$ vanishes. In overdamped systems, this corresponds to a loss of mobility, where the system becomes increasingly inert to perturbations near the critical point. Far from T_c, the relaxation dynamics is approximately exponential, $\phi(t) \sim e^{-t/\tau}$, but as criticality is approached, $\tau \to \infty$, and the relaxation crosses over to stretched-exponential [58] or power-law forms, characteristic of critical slowing down [2, 3, 13, 29]. That is, at or near criticality, the relaxation dynamics is no longer exponential—instead, it becomes power-law, Mittag-Leffler or stretched-exponential [58] (depending on the model), reflecting critical slowing down and the emergence of long-time tails or non-Markovian behaviour.

This work addresses the fundamental question of how conformable derivatives emerge naturally in critical phenomena while offering genuine physical insights beyond classical mean-field theory. We establish rigorous connections between the conformable derivative formalism and established physical mechanisms. First, we demonstrate how anomalous relaxation dynamics near the critical temperature Tc give rise to conformable structures in the adiabatic limit. Second, we show that finite-size effects and boundary constraints lead to temperature-dependent prefactors consistent with the conformable framework. Third, we connect the presence of quenched disorder and spatial inhomogeneities to the emergence of modified scaling laws captured effectively by conformable derivatives. Finally, we relate the conformable exponent to the underlying fractal geometry, interpreting it as an effective dimension governing critical fluctuations.

Remark on the use of critical scaling forms

When expressing the power-law behaviour near the critical temperature T_c, the choice of expression depends on the temperature regime. For temperatures below the critical point ($T < T_c$), one uses $(T_c - T)^{\beta}$, as this quantity remains positive and correctly describes observables such as the order parameter (e.g., magnetisation or superconducting gap), which typically grow as T decreases below T_c. For temperatures above the critical point ($T > T_c$), the appropriate form is $(T - T_c)^{\beta}$, which applies to quantities that diverge as the system approaches T_c from above, such as the heat capacity or magnetic susceptibility. When a symmetric form is desired, valid on both sides of the transition, a common convention is to use the absolute value $|T - T_c|^{\beta}$, which appears frequently in generalised scaling laws and in the unified description of critical divergences.

Conformable derivatives: Bridging classical, geometric, and fractional perspectives

As we will show, the conformable derivative provides a powerful intermediary between classical differential operators and fully nonlocal fractional derivatives. While fractional calculus introduces long-range memory and non-locality [22, 23, 59, 60], it is often analytically cumbersome and dimensionally ambiguous. The conformable derivative (1) offers a tractable alternative. For $\mu = 1$, it reduces to the classical derivative, while for $\mu < 1$, it captures essential features of fractional scaling, such as subdiffusion, intermittency, and fractal geometry, without requiring nonlocal integration kernels [21, 61].

This makes the conformable framework especially suitable for systems exhibiting complex thermodynamic behaviours, including those described by generalised statistics (e.g., Tsallis entropy) [14, 30, 58, 62, 63]. Crucially, from a dimensional standpoint, the conformable derivative maintains physical unit consistency,

$$[D_T^{(\mu)} f(T)] = \frac{[f]}{[T]^\mu}, \tag{16}$$

and all critical exponents derived from it (e.g., α, β) remain dimensionless. Modified Ginzburg-Landau equations incorporating conformable terms remain compatible with thermodynamic scaling laws and unit balance, confirming both mathematical consistency and physical viability [30].

Beyond this formal consistency, the conformable derivative admits a compelling geometric interpretation. The operator $D_T^{(\mu)}$ effectively acts as a local deformation of the underlying *thermodynamic geometry*, modifying the metric structure that connects internal energy, entropy, and temperature. In this interpretation, variations in the conformable index μ alter the curvature of the thermodynamic manifold, introducing a temperature-weighted directional response that encodes local nonequilibrium effects. This approach resonates with the framework of *geometrical thermodynamics* and the *Quantitative Geometrical Thermodynamics (QGT)* formalism, in which the state space of thermodynamic variables is endowed with a Riemannian metric reflecting energy–entropy conjugacy and fluctuation geometry [64]. This quantity can be viewed as a curvature in the space of thermal states, analogous to how spacetime curvature in general relativity modifies geodesics. Here, the thermal evolution is bent by the conformable weighting, making the system's response scale-sensitive. This captures phenomena such as anomalous diffusion, divergence of correlation length, or emergent power-law scaling near criticality.

Importantly, the framework retains analytical tractability. Unlike renormalisation group treatments, conformable models yield closed-form or semi-analytic expressions for key observables such as the order parameter $\psi(T)$, London penetration depth $\lambda_L(T)$, and heat capacity $C(T)$, as shown below. These expressions are not only dimensionally coherent but also directly fit experimental data. In particular, fits to superconducting phase transitions recover expected mean-field exponents such as $\beta \approx 1/2$, thereby validating the model's empirical utility.

In summary, the conformable derivative offers a physically motivated, analytically manageable, and geometrically meaningful framework for modelling critical phenomena. It seamlessly bridges classical dynamics, geometric deformation, and fractional scaling, making it a promising tool for exploring universality, non-locality, and thermodynamic complexity in condensed matter systems and beyond.

To explore how deformed calculus affects the thermodynamic behaviour near criticality, we now introduce the conformable derivative framework and show how it modifies standard scaling relations.

A. Geometrical and fractional-Laplacian perspectives: linking ν, Fisher's η, and fractal dimensions

The correlation length exponent ν and the associated scaling of the two-point function have a well-established foundation in the Euclidean field theory-framework initiated by Fisher [65]. In that setting, the critical correlation function adopts the form $G(r) \sim r^{-(d-2+\eta)}$ at $T = T_c$, where η is the Fisher exponent capturing anomalous scaling beyond mean-field theory.

A complementary and increasingly influential viewpoint interprets critical exponents through the geometry of emergent fractal structures at criticality, including percolation/cluster formulations [52, 53]. In this direction, recent work based on a fractional Laplacian of Riesz form has provided an explicit bridge between the Fisher exponent η and a *correlation fractal dimension* associated with a fractal subspace, on which critical correlations effectively exist [54, 55]. In particular, the critical correlator can be formulated via a fractional Poisson-type equation of the form

$$(-\nabla^2)^\zeta G(r) = \delta^{(d_R)}(r), \tag{17}$$

where d_R is a Riesz (fractional) dimension associated with the fractional order ζ [55]. Comparison with the scaling form of $G(r)$ yields the exact relation

$$\eta = d - d_R = 1 - \zeta, \tag{18}$$

and, equivalently, the geometrical interpretation $\eta = d - d_f$ in terms of a correlation fractal dimension d_f [54].

While the present manuscript primarily deforms the *thermal/relaxational operator*, through a conformable derivative, rather than the *spatial Laplacian*, these two viewpoints can be related at the level of scaling constraints. Specifically, once the conformable framework yields effective exponents ν and γ for a given system/observable, one may infer the Fisher exponent through the standard scaling relation

$$\gamma = (2-\eta)\nu \quad \Rightarrow \quad \eta = 2 - \frac{\gamma}{\nu}, \tag{19}$$

and thereby obtain an *effective* geometric dimension $d_R = d - \eta$ (or $d_f = d - \eta$) in the sense of Refs. [54, 55]. This provides a concrete bridge between conformable thermal deformations and fractional-geometric interpretations of criticality: the conformable parameters encode effective scaling in experimentally accessible regimes, while the inferred parameter η may be interpreted in terms of the emergent fractal correlation subspace.

At the same time, we stress an important limitation: the conformable approach developed here does not claim to replace the explicit fractional-Laplacian construction nor to uniquely resolve the multiple distinct fractal dimensions identified in cluster/percolation formulations [52, 53]. Rather, it provides a local and analytically tractable effective description of scaling in thermal/relaxational space. Establishing a first-principles mapping between conformable indices and the set of geometric fractal dimensions (including Riesz-type dimensions) is an interesting direction for future work, and the relations above show precisely how such a program may be operationally implemented.

B. Fractal clusters, correlation geometry, and the physical meaning of μ

At a continuous phase transition, critical fluctuations organise into self-similar clusters whose geometry departs from the embedding Euclidean space. Early work by Suzuki [52] and Coniglio [53] demonstrated that the critical clusters of Ising- and Potts-type models possess a non-integer fractal dimension that is directly linked to standard critical exponents. In this geometric picture, scaling laws emerge as a consequence of the restricted connectivity and reduced effective phase space available to fluctuations at criticality.

Within the present conformable framework, the deformation parameter μ admits a natural interpretation in this context. Rather than representing a fundamental microscopic constant, μ acts as an effective coarse-grained index encoding the degree of phase-space constriction induced by disorder, finite-size effects, or heterogeneous microstructures. This interpretation is consistent with recent developments emphasising that equilibrium critical dynamics effectively evolve on a fractal subspace of the full configuration space [54, 55].

From this perspective, the conformable derivative may be viewed as a local and analytically tractable way of incorporating the dominant consequences of fractal cluster geometry into the governing dynamical equations, without introducing explicitly nonlocal kernels. While the present approach does not aim to resolve the full hierarchy of distinct fractal dimensions identified in cluster and percolation formulations, it provides an effective description of scaling behaviour in experimentally accessible regimes, fully consistent with renormalisation-group expectations and geometrical interpretations of critical phenomena.

V. CRITICAL EXPONENTS AND THERMODYNAMIC QUANTITIES

In this section, we derive the scaling behaviour of key thermodynamic quantities near the critical temperature T_c using the conformable derivative framework. This approach yields expressions for the critical exponents α, β, γ, and ν, which characterise the singular behaviour of heat capacity, magnetisation, magnetic susceptibility, and correlation length, respectively.

We begin by applying this operator to each observable and derive the corresponding critical exponent in terms of the deformation parameter μ and characteristic coefficients. These expressions will form the foundation for the quantitative modelling of critical behaviour in later sections.

A. Phenomenological origin of the conformable critical equation: Heat capacity

To model power-law divergences near critical points, it is common to describe thermodynamic observables such as susceptibility or concentration fluctuations with first-order differential equations, whose solutions yield singularities at the critical temperature T_c. In the classical framework, such a behaviour can be captured by equations of the form [1, 2, 66]

$$\frac{dC}{dT} = -\frac{\alpha}{T_c - T} C(T), \tag{20}$$

which yields the well-known scaling law

$$C(T) \sim |T_c - T|^{-\alpha}. \tag{21}$$

To incorporate generalised dynamics, such as memory effects, anomalous relaxation, or scaling violations associated with nonextensive systems, we replace the classical derivative with the conformable derivative of order $\mu_C \in (0, 1]$, defined as

$$D_T^{(\mu_C)} f(T) := T^{1-\mu_C} \frac{df}{dT}. \tag{22}$$

This local deformation introduces intrinsic scaling into the dynamics while preserving analytic solvability.

The connection between temperature-dependent scaling and dynamics arises from the fact that, near a critical point, the control parameter $T - T_c$ governs not only equilibrium singularities but also the time-dependent relaxation of fluctuations. In the framework of dynamic critical phenomena [2, 46], the order-parameter correlation function $\phi(t)$ typically satisfies relaxation-type equations of the form

$$\frac{d\phi}{dt} = -\Gamma(T)\phi(t), \tag{23}$$

where the kinetic coefficient $\Gamma(T) \sim (T_c - T)^{\nu z}$ vanishes at T_c, producing a critical slowing down. By analogy, the conformable derivative equation

$$D_T^{(\mu_C)} C(T) = -\frac{\kappa C(T)}{T_c - T} \tag{24}$$

introduces an intrinsic scaling between the thermal control parameter T and the effective relaxation dynamics encoded in the fractional exponent μ_C. This formal correspondence establishes T as a surrogate evolution variable that parameterises the deformation of thermodynamic trajectories as the system approaches the critical manifold. Consequently, the conformable formalism embeds the divergence of $C(T)$ within a generalised dynamic framework that preserves analytic solvability while reflecting critical slowing-down behaviour.

We thus propose the following phenomenological equation for the specific heat,

$$D_T^{(\mu_C)} C(T) = -\frac{\kappa C(T)}{T_c - T}, \tag{25}$$

where κ is a dimensionless constant governing the strength of the singularity. Inserting the definition (1) of the conformable derivative, relation (25) becomes

$$T^{1-\mu_C} \frac{dC}{dT} = -\frac{\kappa C(T)}{T_c - T}. \tag{26}$$

Division by $C(T)$ on both sides and integration yields

$$\int \frac{1}{C(T)} \frac{dC(T)}{dT}\, dT = -\kappa \int \frac{T^{\mu_C - 1}}{T_c - T} dT. \tag{27}$$

Near the critical point, $T \approx T_c$, we make the approximation $T^{\mu_C-1} \approx T_c^{\mu_C-1}$ (valid for $|T - T_c|/T_c \ll 1$). The integral can then be taken in the sense

$$\begin{aligned} \int \frac{dC}{C} &= -\kappa T_c^{\mu_C-1} \int \frac{dT}{T_c - T} \\ \ln C(T) &= -\kappa T_c^{\mu_C-1} \ln|T_c - T| + \text{const} \\ C(T) &= B|T_c - T|^{-\alpha}, \quad \alpha = \kappa T_c^{\mu_C-1}, \end{aligned} \tag{28}$$

where the integration domain is $[T_0, T]$ with the reference temperature T_0 chosen sufficiently far from T_c such that the approximation holds.

This formulation provides a conformable generalisation of classical critical scaling laws, embedding the power-law divergence directly into the modified dynamics. It is particularly suitable for systems exhibiting anomalous thermodynamic behaviour, nonlocality, or deviations from classical universality classes.

Collecting the above results, we find the explicit critical form

$$C_V(T) = \frac{B}{|T_c - T|^\alpha} \tag{29}$$

of the heat capacity near the second-order phase transition, where B is a positive constant amplitude.

The empirical data from physical systems such as niobium often exhibit asymmetric critical behaviour, where the divergence of C_V is different below and above the critical temperature. This asymmetry arises from distinct microscopic mechanisms in the ordered (superconducting) and disordered (normal) phases, and is supported by both theoretical models and experimental heat capacity curves [3]. To account for this asymmetry, we adopt the piecewise power-law model

$$C_V(T) = \begin{cases} B_1(T_c - T)^{-\alpha_1}, & T < T_c \\ B_2(T_c - T)^{-\alpha_2}, & T > T_c \end{cases} \tag{30}$$

for the heat capacity, where B_1, B_2, α_1, and α_2 are fitting parameters. As we demonstrate below, the piecewise expression (30) provides a better fit to real data than a symmetric model, yet it remains analytically tractable within the conformable framework.

Moreover, for computational purposes, to avoid a divergence exactly at $T = T_c$, we introduce a regularisation parameter $\epsilon > 0$, resulting in the "smoothed" model

$$C_V^{\text{reg}}(T) = \begin{cases} B_1(|T_c - T| + \epsilon)^{-\alpha_1}, & T < T_c \\ B_2(|T_c - T| + \epsilon)^{-\alpha_2}, & T > T_c \end{cases} \tag{31}$$

In what follows, the symbol T_c denotes the critical temperature associated with the specific phenomenon under consideration. Although its numerical value may differ from one case to another, we retain the same notation T_c throughout this work for clarity and to avoid introducing additional symbols.

B. Magnetisation

We now proceed with the analysis of the magnetisation in terms of the deformed operator. We start with the critical form of the magnetisation,

$$|M| \sim (T_c - T)^\beta. \tag{32}$$

Application of the concrete form $D_T^{\mu_M} f(T) = T^{1-\mu_M} df(T)/dT$ of the conformable derivative yields

$$D_T^{(\mu_M)} |M| = \frac{\gamma |M|}{T_c - T}. \tag{33}$$

The resulting differential expression can then be written as

$$\frac{d|M|}{|M|} = \frac{\gamma T^{\mu_M - 1}}{T_c - T} dT. \tag{34}$$

Again, we integrate both sides. The left-hand side yields

$$\int \frac{d|M|}{|M|} = \ln|M| + \text{const.} \tag{35}$$

For the right-hand side, we consider the behaviour close to the critical temperature, $T \to T_c^-$. Assuming that T^{μ_M-1} varies slowly as compared to the divergence at $T = T_c$, we approximate

$$T^{\mu_M-1} \approx T_c^{\mu_M-1}. \tag{36}$$

Hence, the integral over the right hand side of expression (34) becomes

$$\begin{aligned}\int \frac{\gamma T^{\mu_M-1}}{T_c - T} dT & \\ &\approx \gamma T_c^{\mu_M-1} \int \frac{dT}{T_c - T} \\ &\approx -\gamma T_c^{\mu_M-1} \ln(T_c - T).\end{aligned} \tag{37}$$

Substituting back, we obtain

$$\ln|M| \approx -\gamma T_c^{\mu_M-1} \ln(T_c - T), \tag{38}$$

witch leads to

$$|M| \sim (T_c - T)^{\gamma T_c^{\mu_M-1}}. \tag{39}$$

Identifying the standard critical scaling form $|M| \sim (T_c - T)^\beta$, we then obtain the exponent

$$\beta = \gamma T_c^{\mu_M-1}. \tag{40}$$

This expression relates the critical exponent β to the temperature-scaling exponent μ_M and the kinetic prefactor γ, providing a natural thermodynamic link between dynamic behaviour and critical ordering.

C. Magnetic susceptibility

Also the magnetic susceptibility diverges at criticality,

$$\chi(T) \sim |T_c - T|^{-\gamma}. \tag{41}$$

This power-law has been treated in both classical and generalised frameworks [62, 67]. Following the same approach as used for the calculation of the heat capacity and magnetisation, we write the deformed differential equation for the magnetic susceptibility as

$$D_T^{(\mu_\chi)} \chi(T) = -\frac{\lambda \chi(T)}{T_c - T}, \tag{42}$$

from which we find the related exponent γ in the form

$$\gamma = \lambda T_c^{\mu_\chi-1}. \tag{43}$$

D. Correlation length

The correlation length diverges as

$$\xi(T) \sim |T_c - T|^{-\nu}. \tag{44}$$

Following our approach, we can cast $\xi(T)$ in the form

$$D_T^{(\mu_\xi)} \xi(T) = -\frac{\rho \xi(T)}{T_c - T}, \tag{45}$$

leading to the relation for ν,

$$\nu = \rho T_c^{\mu_\xi-1}. \tag{46}$$

This behaviour is central to the renormalisation group picture of criticality [2] and connects to the scaling approaches pioneered by de Gennes [68] in various critical systems.

E. Unified treatment of thermodynamic observables

Following the methodology established for the heat capacity, we apply the conformable framework to other critical observables. Each quantity $Q(T)$ satisfies a deformed differential equation,

$$D_T^{(\mu_Q)} Q(T) = -\frac{\lambda_Q Q(T)}{T_c - T}, \tag{47}$$

leading to the critical scaling $Q(T) \sim |T - T_c|^{-\epsilon_Q}$ with parameter $\epsilon_Q = \lambda_Q T_c^{\mu_Q-1}$. For the magnetic observables, this implies that

$$\text{Magnetisation:} \quad \beta = \gamma T_c^{\mu_M-1} \tag{48}$$

$$\text{Susceptibility:} \quad \gamma = \lambda T_c^{\mu_\chi-1} \tag{49}$$

$$\text{Correlation length:} \quad \nu = \rho T_c^{\mu_\xi-1} \tag{50}$$

This unified approach eliminates the need for separate phenomenological models while maintaining physical interpretability through the connection between the parameter μ_Q and the underlying transport or geometric properties.

VI. UNIFIED FRAMEWORK AND THEORETICAL IMPLICATIONS

Building on the conformable formalism, we now derive modified evolution equations for thermodynamic observables and extract analytic expressions for the associated critical exponents. The use of conformable derivatives to model critical phenomena offers the following theoretical and practical advantages.

A. Unified treatment of critical exponents

The model provides a single formalism for expressing all major critical exponents,

$$\alpha = \kappa T_{cC}^{\mu_C - 1}, \quad \beta = \gamma T_{cM}^{\mu_M - 1},$$
$$\gamma = \lambda T_{c\chi}^{\mu_\chi - 1}, \quad \nu = \rho T_{c\xi}^{\mu_\xi - 1}, \tag{51}$$

where we introduced a distinct notation for the critical temperature related to each physical quantity. The approach used here avoids the need for distinct phenomenological models for each observable, offering a unified perspective grounded in deformed calculus.

B. Unified thermodynamic framework

The use of deformed or conformable derivatives provides a natural bridge between classical critical phenomena and generalised thermodynamics, particularly within the framework of nonextensive statistics [14, 62, 69]. By incorporating a temperature-dependent scaling, long-range correlations, memory effects, and fractal-like structures, such formulations reproduce anomalous relaxation and power-law distributions that extend beyond the scope of traditional universality classes. The suitability of conformable and q-deformed descriptions for systems exhibiting power-law statistics follows from the local scaling properties embedded in their differential operators. When observables obey relations of the type $O(\lambda x) = \lambda^\eta O(x)$, the conformable derivative $D^{(\mu)} f(x) = x^{1-\mu} f'(x)$ naturally preserves this homogeneity, since its action rescales as $D^{(\mu)}[f(\lambda x)] = \lambda^\mu D^{(\mu)} f(\lambda x)$. This property makes it intrinsically compatible with scale-invariant or fractal-like processes, where fractional exponents reflect the underlying self-similarity. Furthermore, the conformable operator retains locality in its definition, avoiding the integral memory kernels typical of fractional calculus, while still capturing effective memory and heterogeneity through the scaling index μ. Consequently, once the statistical behaviour of a system is known to follow a power law—for example, $p(x) \propto x^{-\alpha}$ or correlation functions $\langle A(0)A(t)\rangle \sim t^{-\gamma}$—the conformable framework provides a compact, analytically tractable description in which the deformation parameter encodes the same scaling exponents that govern the system's critical or nonextensive dynamics. Importantly, these approaches are not intended to replace the renormalisation-group (RG) formalism—which remains the fundamental tool for identifying fixed points and coupling-constant flows—but rather to complement it. Once the underlying system is known to exhibit power-law statistics or scale-invariant fluctuations, conformable and q-deformed formulations offer a thermodynamically consistent and analytically transparent representation of the emergent scaling behaviour, embedding memory, self-similarity, and heterogeneity into a unified theoretical framework. A major advantage of this approach is its analytic tractability. In contrast to RG techniques, which often require complex iterative procedures, the present model leads to closed-form expressions for critical behaviour. The resulting differential equations are easily solvable and lend themselves to direct comparison with experimental data, making the method valuable for both theorists and experimentalists.

Moreover, the conformable or scale-deformed derivatives employed in the formulation naturally introduce power-law solutions. This mirrors the observed scaling of physical observables near the critical temperature T_c, where critical exponents emerge from the underlying structure of the modified dynamics rather than requiring external assumptions.

Unlike RG methods, this model yields closed-form solutions and is analytically manageable. The involved equations can be solved directly or fitted to data without computational overhead, making it practical for both theorists and experimentalists.

To benchmark the conformable model, we compare its predictions to classical mean-field theory below and investigate its behaviour in the limiting case where the deformation vanishes.

C. Practical advantages over standard approaches

The conformable derivative framework offers several concrete advantages over more conventional scaling analysis and fractional calculus approaches. These are:

Computational Efficiency: Unlike fractional derivatives requiring convolution integrals, conformable derivatives yield algebraic expressions amenable to standard fitting procedures. Computing time scales as $\mathcal{O}(N)$ rather than $\mathcal{O}(N^2)$ for fractional methods (compare [21, 61, 70]). This simplification has made conformable formulations particularly useful in numerical modelling of anomalous diffusion and relaxation, where fractional kernels are computationally costly.

Parameter Interpretability: The deformation parameter μ directly relates to intrinsic physical properties. For instance, $\mu = 1 - d_f/d$, where d_f is the fractal (Hausdorff) dimension and d the embedding dimension, providing a geometric measure of deviation from extensivity and an immediate physical interpretation unavailable in phenomenological scaling fits [14, 30, 70]. This correspondence links conformable dynamics with fractal metrics and nonextensive thermodynamics, yielding a consistent dimensional interpretation of μ.

Predictive Power: Once calibrated on one observable (e.g., the heat capacity), the framework predicts scaling behaviour for related quantities (penetration depth, coherence length) without additional fitting. Standard power-law fits treat each observable independently.

Unified Analysis: The approach naturally handles asymmetric scaling above and below T_c through different μ values, capturing microscopic asymmetries that pure mean-field theory cannot address.

VII. MODIFIED GINZBURG-LANDAU EQUATION

Before deriving the Ginzburg–Landau (GL) equation with a conformable derivative and a nonlinear interaction term, we first recall the conventional GL-theory. As treated by de Gennes [8], the free energy functional in a d-dimensional space reads

$$\mathcal{F}[\psi] = \int \left[a(T)|\psi(\mathbf{r})|^2 + \frac{b}{2}|\psi(\mathbf{r})|^4 + \frac{1}{2m}|\nabla\psi(\mathbf{r})|^2 \right] d^d r. \tag{52}$$

This classical GL-theory describes the thermodynamic behaviour of a complex order parameter $\psi(\mathbf{r})$ near the critical temperature T_c. The GL free energy functional (52) further involves the temperature-dependent coefficient

$$a(T) = a_0(T_c - T) \tag{53}$$

with $a_0 > 0$, the stabilising coefficient $b > 0$ ensuring the boundedness of the free energy, and the generalised mass (or stiffness parameter) m.

To determine the equilibrium configuration, we minimise $\mathcal{F}[\psi]$ with respect to $\psi^*(\mathbf{r})$, i.e., $\frac{\delta\mathcal{F}}{\delta\psi^*(\mathbf{r})} = 0$, which leads us to the Euler–Lagrange equation

$$a(T)\psi(\mathbf{r}) + b|\psi(\mathbf{r})|^2\psi(\mathbf{r}) - \frac{1}{2m}\nabla^2\psi(\mathbf{r}) = 0. \tag{54}$$

This nonlinear partial differential equation governs the spatial behaviour of the order parameter near criticality. In the case of a spatially uniform system (i.e., $\nabla\psi = 0$), relation (54) reduces to the simpler algebraic equation

$$a(T)\psi + b|\psi|^2\psi = 0, \tag{55}$$

whose solutions describe a second-order phase transition and yield the critical exponent $\beta = \frac{1}{2}$ under mean-field conditions.

Minimising the standard GL-free energy functional, it can be shown that the order parameter $\psi(T)$ near the critical temperature T_c follows the scaling law

$$|\psi(T)| \sim (T_c - T)^\beta, \text{ for } T < T_c, \tag{56}$$

with the critical exponent $\beta = \frac{1}{2}$ under mean-field conditions. By this we mean that we assume that the system is in thermodynamic equilibrium, homogeneous and local (no spatial gradients or long-range correlations), that the fluctuations are negligible, the free energy is analytic near T_c, and that the order parameter ψ is uniform and real. In the full GL-free energy functional, the gradient term $\frac{1}{2m}|\nabla\psi(\mathbf{r})|^2$ is included to account for the energy cost associated with variations of the order parameter ψ in time, temperature, or space. This term is essential in dynamic or spatially inhomogeneous contexts.

In the mean-field approach with vanishing gradient term, $|\nabla\psi(\mathbf{r})|^2 = 0$, the *time-independent*, spatially uniform GL-free energy density becomes

$$\mathcal{F}[\psi] = a(T)|\psi|^2 + \frac{b}{2}|\psi|^4, \tag{57}$$

which depends solely on the magnitude of ψ and the temperature T. The coefficient $a(T)$ is defined in Eq. (53), and $b > 0$. To minimise the free energy, we take the derivative with respect to ψ^* (or ψ, since it is real-valued) and set it to zero,

$$\frac{d\mathcal{F}}{d\psi} = 2a(T)\psi + 2b|\psi|^2\psi = 0, \tag{58}$$

such that

$$\psi \left[a(T) + b|\psi|^2 \right] = 0. \tag{59}$$

There exist two solutions: (i) $\psi = 0$, corresponding to the disordered (symmetric) phase; and (ii) $|\psi|^2 = -a(T)/b$, which corresponds to the ordered (broken-symmetry) phase for $a(T) < 0$, i.e., $T < T_c$. Below T_c, we have $a(T) < 0$, so the nontrivial solution reads

$$|\psi|^2 = -\frac{a_0(T_c - T)}{b} = \frac{a_0}{b}(T_c - T). \tag{60}$$

Taking the square root, we obtain for the positive solution

$$|\psi(T)| = \sqrt{\frac{a_0}{b}(T_c - T)}. \tag{61}$$

Thus, the order parameter vanishes as

$$|\psi(T)| \sim (T_c - T)^\beta, \text{ with } \beta = \frac{1}{2}, \tag{62}$$

for $T < T_c$. This is the well known mean-field prediction for the order parameter and is valid under the assumptions stated above. In lower dimensions ($d \leq 3$), fluctuations become important, and this mean-field result typically underestimates the true critical exponent.

A. The conformable GL-model

After establishing the general framework, we apply it to specific thermodynamic quantities—the specific heat and magnetic susceptibility—highlighting the modifications originating from the conformable deformation. We note that in standard GL-theory, spatial variations of the order parameter $\psi(\mathbf{r})$ contribute to the free energy via the gradient term $\frac{1}{2m}|\nabla\psi(\mathbf{r})|^2$, which penalises sharp spatial inhomogeneities and plays a key role in capturing coherence, textures, and defects. This is essential in systems where the spatial coherence or texture of the order parameter matters, such as for vortices in superconductors. In our modified GL framework, however, we are concerned with the thermal evolution of the order parameter $\psi(T)$, i.e., with the description of how the this order parameter evolves as a function of temperature T, especially in the vicinity of the critical point (critical temperature) T_c. To this end we assume that the system is homogeneous in space.

To build a thermodynamic analogue of the GL-theory, we formally treat the temperature T as the independent coordinate, akin to space $\mathbf{r}$ in the conventional setting. In this case, we introduce a conformable (deformed) derivative with respect to temperature, denoted by

$$T_\alpha \psi(T) \coloneqq T^{1-\alpha} \frac{d\psi}{dT}, \tag{63}$$

which naturally embeds a temperature-dependent scaling. The conformable operator acts as a *thermal gradient* and captures the power-law behaviour typical of critical phenomena. The corresponding modified GL-free energy then becomes

$$\mathcal{F}_\alpha[\psi] = \int \left[a(T)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{2m}|T_\alpha \psi(T)|^2 \right] dT, \tag{64}$$

where $a(T)$ has the same form (53) to encode the proximity to the critical point, $b > 0$ ensures stability, and the last term mimics the kinetic contribution, now expressed in thermal rather than spatial form. The parameter m retains its interpretation as a stiffness constant, and $\alpha \in (0, 1]$ quantifies the deviation from classical scaling. For $\alpha = 1$, the standard GL-theory is recovered.

The formulation (64) of the free energy functional is central to this work and preserves the physical dimensions of energy density, and it ensures that all critical exponents derived from it are dimensionless and compatible with thermodynamic scaling theory. Furthermore, the temperature-dependent weighting introduced by the factor $T^{1-\alpha}$ can be interpreted geometrically as a curvature in thermal space, analogous to how curved metrics affect geodesics in general relativity. It effectively embeds power-law sensitivity into the response functions of the system.

To minimise the free energy, we use the variational principle, $\delta \mathcal{F}_\alpha / \delta \psi^* = 0$, which yields the modified GL-equation

$$a(T)\psi + b|\psi|^2\psi - \frac{1}{2m} T^{2(1-\alpha)} \frac{d^2\psi}{dT^2} - \frac{(1-\alpha)}{m} T^{1-2\alpha} \frac{d\psi}{dT} = 0. \tag{65}$$

Here $a(T) = a_0(T/T_c - 1)$ retains the usual temperature dependence, while the conformable terms introduce explicit temperature-weighted derivatives. The order parameter ψ describes the macroscopic condensate amplitude, and near the critical temperature it follows the scaling law

$$\psi(T) \simeq \psi_0 \left(1 - \frac{T}{T_c}\right)^\beta, \tag{66}$$

where ψ_0 is the equilibrium amplitude at $T = 0$ and β is the critical exponent associated with the order-parameter vanishing as $T \to T_c$. The dependence $\beta = \beta(\alpha)$ reflects how the conformable index α modifies the effective critical behaviour of the system. This expression recovers the expected behaviour from mean-field theory when $\alpha = 1$, but allows deviations due to the deformation parameter. The above equations can also be obtained via a variational formulation adapted to deformed derivatives, as discussed in [15].

The nonlinear term $|\psi|^2\psi$ contributes to the saturation of the order parameter as $T \to 0$, while the conformable kinetic term introduces corrections near T_c, mimicking effects from spatial or temporal fluctuations. The formulation presented here enables capturing intermediate critical dynamics and non-locality without invoking full fractional calculus or spatial disorder [71–74], making it suitable for describing second-order phase transitions such as superconductivity. In this sense, the conformable GL-model complements traditional models such as BCS theory [72], the two-fluid model [73], and the phenomenological framework detailed in [71]. For systems with strong fluctuations and short coherence length, additional comparison with the work in [74] provides insight into limitations and refinements.

A key quantity in critical phenomena is the correlation length, whose divergence governs universality. We now derive its scaling law within the conformable approach and discuss physical implications.

B. Beyond simple reparametrisation: Functional and kinetic inequivalence

As already emphasised above, a common critique of conformable models is the suggestion that the operator $D_T^{(\mu)} f(T) = T^{1-\mu} \frac{df}{dT}$ is equivalent to a standard derivative combined with the power-law transformation $T \to T' = \frac{T^\mu}{\mu}$ [30, 75]. While such a mathematical equivalence indeed holds, at most, for isolated first-order linear equations with constant coefficients [30], it breaks down in the context of critical phenomena and GL-dynamics, where the conformable framework carries genuine physical content that cannot be eliminated by a coordinate transformation.

This inequivalence arises for two fundamental reasons:

(i) Non-trivial kinetic scaling. In critical dynamics, the relaxation of the order parameter is governed by a kinetic coefficient $\Gamma(T)$, that typically diverges or vanishes near the critical point T_c [13, 30]. When the conformable operator acts on the relaxation law $\Gamma(T) D_T^{(\mu)} \psi = -\delta\mathcal{F}/\delta\psi$, the transformation $T \to T'$ does not map the equation back to a classical Langevin form. The Jacobian factors associated with the reparametrisation interacts with the intrinsic singular behaviour of $\Gamma(T)$, yielding a scaling response that is dynamically distinct from that of a simply rescaled conventional system.

(ii) Variational structure and thermal metrics.

> The conformable GL-functional introduced in Eq. (58) contains a gradient contribution of the form $T^{2(1-\alpha)}|d\psi/dT|^2$ [30], which acts as a non-uniform thermal metric. Minimisation of this functional leads to Euler–Lagrange equations featuring explicit temperature-weighted stiffness and damping terms. After any reparametrisation $T \to T'$, these equations retain Jacobian-induced coefficients that do not appear in standard GL-theory, thereby preserving a memory of the underlying anomalous scaling.

Consequently, the conformable formulation encodes a coupled dynamical and variational structure, linking anomalous diffusion, kinetic criticality, and geometric curvature in thermal space, that is physically independent of the specific choice of coordinates [14, 30].

As a result, even though a formal change of variables may locally relate derivatives, the conformable formulation retains explicit temperature-weighted stiffness and damping terms that have no counterpart in standard GL-theory. These terms encode the coupling between anomalous scaling, kinetic criticality, and the effective geometry of thermal space, and therefore survive any coordinate transformation. The conformable framework thus goes beyond a simple rescaling by embedding the relevant scaling information directly into both the kinetic equations and the variational structure, yielding a physically distinct extension of GL-dynamics [30].

C. Physical meaning and measurability of the deformation parameter μ

A central question raised by the conformable formulation concerns the physical status of the deformation parameter μ. While μ enters the theory as a deformation index in the definition of the operator $D_T^{(\mu)}$, it is not introduced as a pure fit parameter. Rather, μ plays the role of an effective coarse-grained material index that quantifies how strongly the systems, thermal or temporal evolution, deviates from ideal Euclidean and Markovian behaviour near criticality.

From a physical standpoint, μ measures the degree to which the effective thermal "clock" of the system is distorted by microscopic mechanisms such as quenched disorder, heterogeneous relaxation pathways, finite-size constraints, or long-lived correlations. In systems where fluctuations explore phase space uniformly and relaxation is governed by a single characteristic timescale, one recovers $\mu = 1$. Deviations $\mu < 1$ signal a reduced spectral weight of accessible modes or a broad hierarchy of relaxation times, consistent with the presence of anomalous diffusion, memory effects, or geometric constraints, as discussed in the preceding sections.

Importantly, μ is in principle measurable. Its value can be independently extracted from experimental or numerical data by analysing relaxation curves of the order parameter, temperature-dependent response functions, or scaling forms of critical observables. Within the present framework, μ inferred from one observable leads to testable predictions for others through cross-observable consistency relations, providing a means to falsify the theory rather than merely fit data.

In this sense, μ plays a role analogous to that of the dynamic exponent z or the correlation-length exponent ν in RG-theory: it encapsulates complex microscopic physics into a single effective exponent governing macroscopic scaling. While different materials or observables may probe distinct relaxation channels and therefore exhibit different effective indices μ_Q, these indices remain constrained by the internal structure of the conformable GL-framework.

Consequently, although μ is introduced at the level of a deformation parameter, its meaning is neither arbitrary nor purely phenomenological. It represents a measurable, material-dependent index encoding the dominant coarse-grained mechanisms that control critical slowing down and anomalous scaling, thereby endowing the conformable formulation with clear physical content.

D. Physical interpretation and microscopic origin of μ

To move beyond a purely phenomenological interpretation, the deformation parameter μ can be linked to the microscopic topology and dynamical heterogeneity of the underlying medium [18, 30]. In many disordered condensed-matter systems, including porous or granular superconductors, amorphous materials, and structurally heterogeneous media, the effective lattice explored by fluctuations is not strictly Euclidean but exhibits fractal or multi-scale features [14, 30]. In such environments, the density of states and the accessible phase-space volume are governed by effective dimensions, such as the spectral dimension d_s or the fractal (Hausdorff) dimension d_f, rather than by the embedding dimension d alone.

Within this coarse-grained description, the conformable index μ quantifies the effective reduction of connectivity and the associated "squeezing" of phase space experienced by thermal or temporal fluctuations. In regimes characterised by anomalous relaxation or subdiffusive dynamics, where the mean-squared displacement scales as $\langle \mathbf{r}^2(t)\rangle \sim t^\alpha$ with $\alpha < 1$ and an effective identification $\mu \sim \alpha < 1$, may be approximately related to the anomalous diffusion exponent α or to ratios of effective dimensions [30, 70], for example,

$$\mu \sim 1 - \frac{d - d_f}{d}, \tag{67}$$

where the symbol "$\sim$" emphasises that this relation represents an effective, asymptotic correspondence rather than a universal identity.

From this perspective, μ represents the effective spectral weight of the thermal bath [14]. The limit $\mu = 1$ corresponds to a homogeneous Euclidean environment with short-range correlations and Markovian relaxation, while $\mu < 1$ signals the presence of quenched disorder, finite-size constraints, long-lived correlations, or nonlocal interactions that deform the system's effective thermal response. Because these geometric and dynamical properties are independently measurable, the parameter μ acquires a clear physical meaning and predictive content. Moreover, different observables may probe distinct relaxation channels and thus be associated with different effective indices μ_Q, a feature naturally accommodated within the conformable GL-framework.

E. Relation to renormalisation group approaches

It is natural to ask how the conformable-derivative formulation relates to established RG-theories, which have long provided a powerful framework for understanding critical phenomena, including the effects of quenched disorder, finite-size scaling, and non-equilibrium relaxation. Indeed, RG-analyses have successfully described universal relaxation behaviour not only in the asymptotic critical regime but also across macroscopically short time scales, as demonstrated in early and influential studies [4, 10, 13, 28, 46, 76].

Early renormalisation-group analyses demonstrated that universal relaxation behaviour extends beyond the asymptotic critical regime and is already observable at macroscopically short times [76].

The conformable framework introduced here is not intended to replace RG-theory, nor to compete with it at the level of microscopic derivations or fixed-point analyses. Rather, it should be viewed as a complementary, coarse-grained description that operates at the level of effective equations of motion. From an RG-perspective, the deformation parameter μ plays a role analogous to an effective scaling exponent: it encapsulates the cumulative outcome of RG-flow in a given regime, whether asymptotic or pre-asymptotic and embeds this information directly into local differential operators.

In conventional RG-treatments, scaling exponents emerge from the analysis of flow equations near fixed points, while the resulting dynamics are often expressed in terms of nonlocal memory kernels or asymptotic scaling forms. In contrast, the conformable derivative provides a local operator that incorporates the renormalised scaling behaviour directly at the operator level, allowing one to write closed-form evolution equations that remain valid beyond strictly asymptotic limits. This is particularly advantageous when modelling finite systems, crossover regimes, or experimentally relevant time windows, where effective exponents rather than universal fixed-point values are typically observed.

From this viewpoint, μ can be interpreted as an RG-informed effective exponent that summaries the influence of disorder, finite size, and non-equilibrium constraints after coarse graining. Once determined for a given system or observable, it enables predictive, self-consistent modelling of critical dynamics without the need to explicitly track the full RG-flow. The conformable GL-formulation thus provides a pragmatic bridge between the universality principles of RG-theory and the construction of local, analytically tractable dynamical models applicable to realistic materials and experimental conditions.

Notably, RG-analyses have demonstrated that universal scaling behaviour can already emerge at macroscopically short times, well before the system reaches its asymptotic critical regime [76]. The conformable framework is particularly well suited to capture such regimes, as it allows one to incorporate effective scaling exponents into local, closed-form evolution equations without invoking nonlocal memory kernels or explicit RG-flow equations.

Within this perspective, the connection between the conformable operator and the kinetic coefficient $\Gamma(T)$ naturally reproduces the critical slowing down associated with the dynamic exponent z [13]. Moreover, the possibility of assigning different effective indices μ_Q to distinct observables reflects the fact that different RG trajectories and microscopic mechanisms may dominate in different sectors or on different sides of the critical temperature T_c.

By positioning the conformable derivative within this RG-informed context, the present framework provides a pragmatic bridge between the conceptual depth of RG-theory and the practical need for analytically tractable, thermodynamically consistent models applicable to finite systems and experimentally relevant time windows [14, 30].

VIII. APPLICATION TO SUPERCONDUCTING PHASE TRANSITIONS

Superconducting phase transitions, especially in type-II and high-T_c materials, are characterised by critical phenomena that can be reformulated using conformable derivatives. This includes the temperature behaviour of the order parameter, coherence length, penetration depth, and heat capacity.

A. Simplified dynamics near the critical point: Connection to the conformable derivative framework

We begin with the full conformable GL-equation (65) obtained from minimising the deformed free energy functional. This nonlinear second-order differential equation governs the thermal evolution of the order parameter $\psi(T)$, incorporating both nonlinear saturation ($|\psi|^2\psi$) and conformable kinetic effects through temperature-dependent coefficients.

Near the critical point, the system undergoes a continuous phase transition. In this regime, we make the following approximations to simplify the analysis: (i) *We assume that $\psi(T)$ is real-valued and slowly varying.* This allows us to treat ψ and its derivatives in a simplified manner. Moreover, we (ii) *neglect higher-order kinetic corrections.* The second-order derivative term and the prefactor corrections from conformable calculus are of higher order and can be dropped close to T_c. This reduces the second-order equation to a first-order approximation. Finally, we (iii) *retain only the dominant balance between the kinetic and the potential terms.* Taking into account these approximations, the *reduced equation* can be cast as

$$\frac{d\psi}{dT} = a(T)\psi - b|\psi|^2\psi, \tag{68}$$

where $a(T)$ is defined in Eq. (53) and $b > 0$ ensures the stability of the ordered phase. Note that the modified GL-equation introduced here describes the temperature-dependent behaviour of the order parameter near the critical point in the sense that the time-dependence is replaced by temperature-dependence to emphasise that the conformable operator acts on the thermal variable T rather than on real time t.

B. Emergence of critical scaling from conformable GL relaxation with temperature-dependent kinetics

While the standard GL-equation near the critical temperature T_c yields a regular (non-singular) behaviour for the order parameter $\psi(T)$, a phenomenological equation capable of reproducing power-law critical behaviour can be obtained by introducing a thermally deformed kinetic coefficient into the GL-like relaxation framework. This modification effectively incorporates the temperature-dependent slowing down of dynamics near T_c into the evolution of the order parameter.

As before, we generalise the time derivative to the form

$$\frac{d\psi}{dT} \to D_T^{(\mu_\psi)}\psi(T) := T^{1-\mu_\psi}\frac{d\psi}{dT}, \tag{69}$$

which reflects the modified temperature response of the system in the conformable thermodynamic formulation.

At the heart of the conformable dynamics approach lies a generalised time-dependent GL-type equation that governs the thermal evolution of the order parameter $\psi(T)$ under non-equilibrium conditions. This equation introduces a thermally modulated kinetic response and is given by

$$\Gamma(T)T^{1-\mu_\psi}\frac{d\psi}{dT} = -\frac{\delta\mathcal{F}}{\delta\psi}, \tag{70}$$

where $\Gamma(T)$ is a temperature-dependent kinetic coefficient encoding the critical slowing down, μ_ψ is the conformable deformation parameter, and $\mathcal{F}[\psi]$ is the free energy functional. Relation (70) is not merely a phenomenological extension but constitutes the cornerstone of the proposed non-equilibrium conformable dynamics framework. It encapsulates the scale-sensitive kinetic behaviour characteristic of critical systems, embedding a power-law response directly into the governing evolution law. Equation (70) allows the derivation of critical exponents such as β from first principles within a thermodynamically consistent and geometrically interpretable formalism, thereby bridging equilibrium mean-field results with more general, deformation-induced scaling behaviours.

We start with the standard GL-potential (neglecting gradient terms)

$$\mathcal{F}[\psi] = \int \left[a_0(T_c - T)|\psi|^2 + \frac{b}{2}|\psi|^4\right] dT, \tag{71}$$

which produces the variational derivative

$$\frac{\delta\mathcal{F}}{\delta\psi} = a_0(T_c - T)\psi + b|\psi|^2\psi. \tag{72}$$

Substituting this expression into Eq. 70 yields

$$\Gamma(T)T^{1-\mu_\psi}\frac{d\psi}{dT} = -a_0(T_c - T)\psi - b|\psi|^2\psi. \tag{73}$$

Near the critical point, $T \to T_c$, the order parameter $\psi(T)$ becomes small, allowing us to neglect the nonlinear term in expression (73). The relaxation equation then reduces to the linearised form

$$\Gamma(T)T^{1-\mu_\psi}\frac{d\psi}{dT} \approx -a_0(T_c - T)\psi. \tag{74}$$

Rewriting this in differential form yields the form

$$\frac{d\psi}{\psi} = -\frac{a_0}{\Gamma(T)}T^{\mu_\psi - 1}(T_c - T)dT, \tag{75}$$

which will be used to extract the scaling behaviour of the order parameter close to the critical temperature.

To obtain the desired critical scaling behaviour, we now propose that the kinetic coefficient $\Gamma(T)$ vanishes near T_c, reflecting the critical slowing down. Specifically, we take

$$\Gamma(T) = \frac{1}{\gamma}(T_c - T)^2, \tag{76}$$

where $\mu_\psi \in (0, 1]$ is a conformable deformation parameter, and where γ is a dimensional constant. Substituting back, we obtain the phenomenological differential equation

$$\frac{d\psi}{\psi} = -\gamma T^{\mu_\psi - 1}(T_c - T)^{-1}dT. \tag{77}$$

This equation directly yields the critical behaviour

$$\int \frac{d\psi}{\psi} = -\gamma T^{\mu_\psi - 1}\int \frac{dT}{(T_c - T)}. \tag{78}$$

Integration produces the result

$$\ln\psi(T) = \gamma T_c^{\mu - 1}\ln|T - T_c| + \text{const.} \tag{79}$$

From this we conclude that

$$\psi(T) \sim (T_c - T)^{\beta(\mu)}, \text{ with } \beta(\mu) = \gamma T_c^{\mu_\psi - 1}, \tag{80}$$

where we approximated $T^{\mu_\psi - 1} \approx T_c^{\mu_\psi - 1}$ near $T = T_c$. Thus, the critical exponent β emerges naturally from a GL-type relaxation model with thermally modulated kinetics.

Here, the scaling exponent $\beta = \frac{1}{2}$ emerges in the equilibrium case (in the adiabatic limit $\frac{d\psi}{dT} \to 0$ or at least for the stationarity condition dominated by the free energy balance as detailed in section VII) and appendix B), corresponding to the local, undeformed limit ($\alpha = 1$). The case $\beta > \frac{1}{2}$ arises when the conformable deformation ($\mu = 2\alpha < 1$) introduces a generalised thermodynamic response. The general form for the critical exponent $\beta(\mu)$ can be viewed as a function encoding microscopic memory or a fractal behaviour in temperature evolution.

Deviations from the mean-field behaviour occur when (i) **thermal fluctuations** near T_c become significant (especially in $d < 4$), requiring renormalisation group treatment; (ii) **nonlocal effects** or memory are present, as modelled here via conformable derivatives; (iii) the system is **driven out of equilibrium** or experiences dissipation or long-time correlations. These effects modify the scaling balance in the GL-equation and yield $\beta > \frac{1}{2}$, consistent with experimental observations in real superconductors—which often exhibit a critical behaviour closer to the 3D XY universality class ($\beta \approx 0.35$) [2, 46, 77, 78].

C. Emergence of penetration-depth scaling exponent from thermally deformed conformable dynamics

The London penetration depth $\lambda_L(T)$ characterises the distance over which magnetic fields decay inside a superconductor. Near the critical temperature T_c, $\lambda_L(T)$ diverges as superconductivity breaks down [8, 71]. In analogy with critical phenomena, this divergence can be modelled by a deformed dynamical relation using conformable derivatives.

We begin with the assumption that $\lambda_L(T)$ obeys a conformable differential equation of the form

$$D_T^{(\mu)} \lambda_L(T) = -\frac{\gamma \lambda_L(T)}{T_c - T}, \tag{81}$$

where $\mu \in (0, 1]$ is the order of the deformation and γ is a positive scaling constant. Equation (81) states that the relative temperature derivative of λ_L, modulated by $T^{1-\mu}$, is proportional to the inverse distance from the critical point T_c. It encapsulates both the scaling nature of the divergence and possible memory/fractal effects through μ.

We proceed by first substituting the definition of the conformable derivative,

$$T^{1-\mu} \frac{d\lambda_L}{dT} = -\frac{\gamma \lambda_L(T)}{T_c - T}, \tag{82}$$

such that

$$\frac{d\lambda_L}{\lambda_L} = -\frac{\gamma T^{\mu-1}}{T_c - T} dT. \tag{83}$$

Near the critical point, $T \approx T_c$, we use the approximation $T^{\mu-1} \approx T_c^{\mu-1}$, yielding

$$\frac{d\lambda_L}{\lambda_L} \approx -\frac{\gamma T_c^{\mu-1}}{T_c - T} dT. \tag{84}$$

Integration on both sides produces

$$\ln \lambda_L(T) = -\gamma T_c^{\mu-1} \ln |T - T_c| + \text{const.} \tag{85}$$

Thus we obtain the scaling law

$$\lambda_L(T) = \frac{B}{|T - T_c|^\alpha}, \text{ with } \alpha = \gamma T_c^{\mu-1}. \tag{86}$$

This result shows that the penetration depth diverges as a power-law near T_c, with an exponent that depends on the conformable order μ, the critical temperature T_c, and the coupling constant γ.

To include a possible asymmetry in the behaviour above and below T_c, we generalise the penetration depth to a piecewise form with different exponents, in a analogy to our procedure with C_V above,

$$\lambda_L(T) = \begin{cases} B_1 |T - T_c|^{-\alpha_1}, & T < T_c, \\ B_2 |T - T_c|^{-\alpha_2}, & T \geq T_c, \end{cases} \tag{87}$$

where $\alpha_i = \gamma_i T_c^{\mu_i - 1}$.

Finally, to prevent a divergence and ensure continuity near $T = T_c$, we introduce a regularisation parameter $\epsilon > 0$, resulting in the practical model used in the fit procedure below,

$$\lambda_L(T) = \frac{B_i}{(|T - T_c| + \epsilon)^{\alpha_i}}, \tag{88}$$

for $i = 1, 2$ above and below T_c, see Eq. (87). This regularised conformable model reproduces the observed smooth divergence and captures the asymmetric critical behaviour seen in (finite) superconducting materials such as niobium, as shown below.

Connection between penetration depth and order parameter

The London penetration depth $\lambda_L(T)$ characterises the distance over which an external magnetic field decays exponentially inside a superconductor. It provides a direct measure of the material's ability to expel a magnetic flux via the Meissner effect. In GL-theory, the superconducting state is described by the complex order parameter $\psi(T)$, whose squared magnitude $|\psi(T)|^2$ corresponds to the density of the superconducting carriers. This connection leads to a fundamental relation between the order parameter and the electromagnetic response of the superconductor [66, 79].

From the London equation, which relates the supercurrent $\mathbf{J}_s$ to the vector potential $\mathbf{A}$, we have [66, 80]

$$\nabla \times \mathbf{J}_s = -\frac{n_s e^2}{m}\mathbf{B}, \tag{89}$$

where n_s is the superconducting carrier density, e is the elementary charge, m is the effective mass, and $\mathbf{B}$ is the magnetic field. The London penetration depth is then given by the expression

$$\lambda_L^{-2}(T) = \frac{\mu_0 n_s(T) e^2}{m}, \tag{90}$$

where μ_0 is the vacuum permeability.

In the GL-framework, the carrier density is proportional to the square of the order parameter, $n_s(T) \sim |\psi(T)|^2$. Substituting into the expression for $\lambda_L^{-2}(T)$, we obtain the key relation

$$\lambda_L^{-2}(T) \propto |\psi(T)|^2. \tag{91}$$

This equation shows that, as the temperature approaches its critical value T_c, the order parameter $\psi(T) \to 0$, and consequently $\lambda_L(T) \to \infty$. Physically, this reflects the loss of superconductivity and the full penetration of magnetic fields into the material. Hence, the temperature dependence of $\lambda_L(T)$ provides a direct experimental probe of the order parameter's behaviour and is commonly used to extract critical exponents and validate theoretical models.

Near the critical temperature T_c, we have shown from the conformable GL-equation that the superconducting order parameter behaves as

$$|\psi(T)| \sim \left(1 - \frac{T}{T_c}\right)^{\beta}, \text{ for } T < T_c, \tag{92}$$

where the critical exponent β depends on the conformable deformation parameter α. This reduces to the standard mean-field behaviour with $\beta = \frac{1}{2}$ when $\alpha = 1$, recovering the classical GL-result [66]. Thus, near the critical temperature T_c, the superconducting order parameter vanishes in power-law form. Moreover, the London penetration length according to relation (91) scales as

$$\lambda_L(T) \sim (T_c - T)^{-\beta}. \tag{93}$$

Thus the penetration depth diverges as the temperature approaches T_c from below. This scaling relation is a direct consequence of the GL-formalism and can be used to extract the critical exponent β from experimental measurements of $\lambda_L(T)$ near the superconducting transition.

D. Conformable scaling of the specific heat near T_c: Heat capacity jump

In classical thermodynamics of phase transitions, the specific heat near the critical temperature T_c typically scales as [2, 3]

$$C(T) \sim |T - T_c|^{-\alpha}, \tag{94}$$

where α is the associated critical exponent. In mean-field theory, $\alpha = 0$, corresponding to a finite jump at T_c [1, 79]. However, in more generalised or fluctuation-driven models, α may take nonzero values and describe a divergence [46, 77, 81].

To generalise the scaling behaviour (94) within the conformable derivative framework, we showed that the specific heat satisfies a deformed differential equation of the form of Eq. (25), such that we find the scaling form

$$C(T) \sim |T - T_c|^{-\alpha}, \text{ with } \alpha = \kappa T_c^{\mu_C - 1}. \tag{95}$$

Thus the critical exponent α is no longer a universal constant but depends explicitly on the deformation parameter μ_C and the critical temperature T_c. The conformable framework thus naturally introduces a generalised scaling law for the specific heat.

The generalised conformable framework enables direct fitting of experimental data from superconducting systems using critical scaling laws derived from deformed thermodynamic equations, as shown in the next section. Specifically, the temperature dependence of key observables can be analysed as follows: the order parameter $\psi(T)$ may be extracted from angle-resolved photoemission spectroscopy (ARPES), tunnelling spectroscopy, or other coherence-sensitive measurements; the London penetration depth $\lambda_L(T)$ is accessible through microwave cavity perturbation, magnetic susceptibility, or muon spin rotation techniques; and the specific heat $C(T)$ is measured via calorimetry near the superconducting transition. Each observable exhibits a specific power-law scaling near the critical temperature T_c, governed by exponents that emerge naturally from the conformable derivative formulation.

Summarising these results, we note that when solving the time-dependent or inhomogeneous problem in the context of GL-theory, derivative terms must be kept and may dominate if $\psi(T)$ varies sharply. However, for static, near-equilibrium (mean-field) behaviour, they contribute only to subleading corrections. The Landau-based equation is an equilibrium condition: it comes from minimising a free energy, assuming time-independence, homogeneity, and absence of fluctuations. The conformable dynamic equation, instead, describes a non-equilibrium evolution, in which the system may be relaxing toward equilibrium with scale-sensitive kinetics due to $\mu_\psi \neq 1$. Therefore, the deformed equation gives rise to a modified critical exponent, which reduces to $\beta = 1/2$ only when $\mu_\psi = 1$, i.e., when the dynamics are classical. The exponent $\beta = 1/2$ emerges from equilibrium minimisation

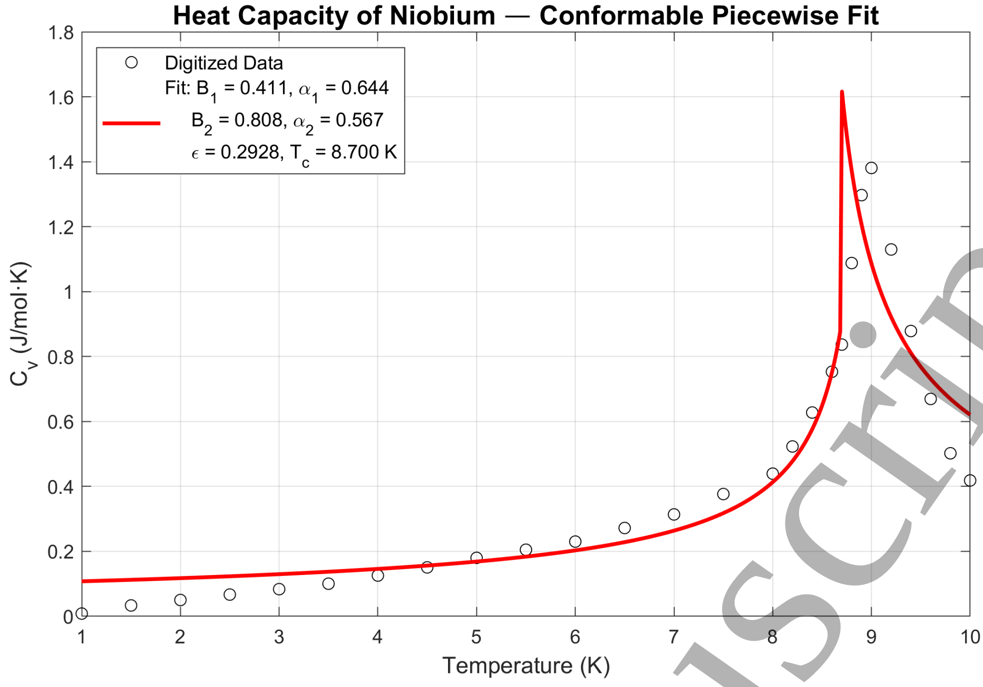

Figure 1: Heat capacity of niobium fitted by the smoothed conformable piecewise model (31) with $C_V(T) = B_i(|T-T_c|+\epsilon)^{-\alpha_i}$. Fitted parameters with 95% confidence intervals: $B_1 = 0.411 \pm 0.023$, $\alpha_1 = 0.644 \pm 0.031$ ($T < T_c$); and $B_2 = 0.808 \pm 0.041$, $\alpha_2 = 0.567 \pm 0.028$ ($T > T_c$). We used the offset parameter $\epsilon = 0.293 \pm 0.015$ to ensure regularisation near the divergence at the critical temperature $T_c = 8.700 \pm 0.005$ K. Data from [82]. The fitting employed weighted least-squares with weights proportional to $1/\sigma_i^2$ where σ_i are experimental uncertainties. Small deviations away from the immediate critical region reflect finite-size effects, experimental uncertainties, and crossover contributions not explicitly included in the minimal conformable model.

of the GL-free energy, while $\beta = \gamma T_c^{\mu_\psi - 1}$ results from integrating the conformable dynamic evolution equation, which accounts for non-equilibrium scaling governed by the conformable derivative. They agree only when the deformation vanishes.

IX. NUMERICAL FITTING AND DATA ANALYSIS

To test the predictive power of the conformable formalism, we apply it to experimental data for the superconducting transition of niobium.

A. Heat capacity of niobium superconductor

Figure 1 shows the experimental heat capacity data for niobium near its superconducting transition, along with a piecewise conformable fit that captures the critical behaviour on both sides of the transition temperature $T_c = 8.7$ K. The fitted parameters are provided in the caption. For the fit we used the regularised model (31). While the fit is not perfect, it demonstrates good agreement with the data over a wider interval around the critical temperature.

B. Penetration depth and conformable scaling

The temperature dependence of the London penetration depth $\lambda_L(T)$ provides key informations about the superconducting coherence and the fluctuation behaviour near the critical temperature T_c. Analogous to the divergence observed in the specific heat, the penetration depth exhibits a critical-like behaviour and can be modelled using the conformable scaling framework.

We adopt the piecewise model (87) with the regularised form (88), where $\epsilon > 0$ is the smoothing parameter that ensures a regularised behaviour at the transition point. The conformable derivative framework again justifies this scaling law via the modified differential relation (81) such that $\lambda_L(T) \sim |T - T_c|^{-\gamma T_c^{\mu-1}}$.

The temperature dependence of the normalised London penetration depth for niobium is shown in Fig. 2. The data are well described by the conformable piecewise model with distinct scaling exponents below and above the critical temperature. The fitted parameters are listed in the caption of Fig. 2.

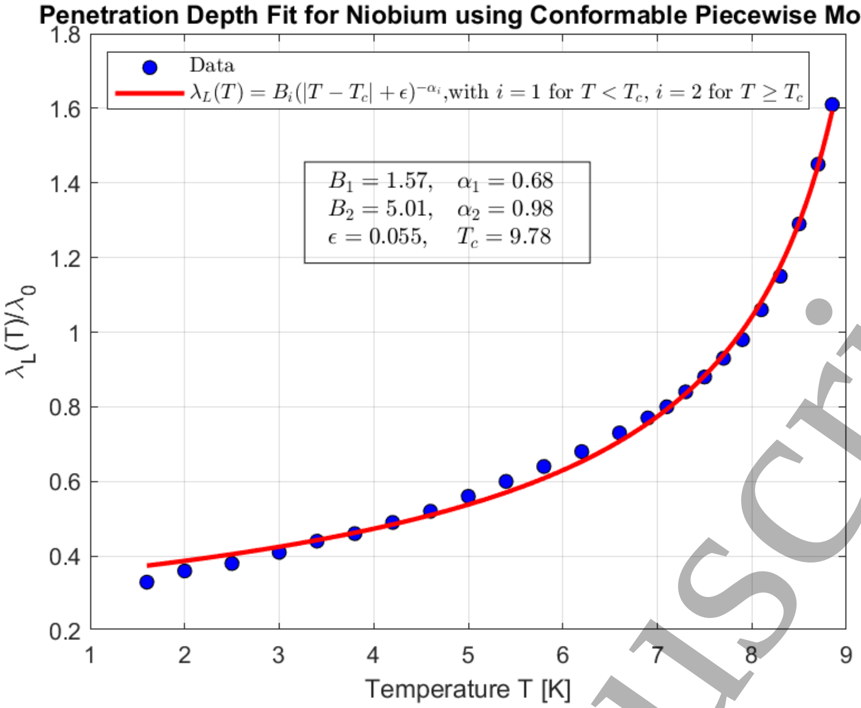

Figure 2: Fit of the normalised London penetration depth $\lambda_L(T)/\lambda_0$, where λ_0 denotes the London penetration depth extrapolated to zero temperature, serving as a normalisation constant for comparison between theory and experiment. The data for niobium are fitted using the piecewise conformable model (87) with the regularised form $\lambda_L(T) = B_i(|T-T_c|+\epsilon)^{-\alpha_i}$, as given in Eq. (88). The fitted parameters are $B_1 = 1.57$, $\alpha_1 = 0.68$ for $T < T_c$ and $B_2 = 5.01$, $\alpha_2 = 0.98$ for $T \geq T_c$, with $\epsilon = 0.055$ and $T_c = 9.78$ K. Experimental data are taken from [83]. Small deviations away from the immediate critical region reflect finite-size effects, experimental uncertainties, and crossover contributions not explicitly included in the minimal conformable model.

C. Coherence length from $H_{c2}(T)$ in niobium

To further validate our theoretical predictions, we turn to experimental data on niobium and extract the temperature-dependent coherence length using the upper critical field $H_{c2}(T)$.

1. *Theoretical background*

In the GL-framework, the upper critical magnetic field $H_{c2}(T)$ is related to the superconducting coherence length $\xi(T)$ by the expression [66, 79, 84]

$$H_{c2}(T) = \frac{\Phi_0}{2\pi\xi^2(T)}, \tag{96}$$

where $\Phi_0 = h/2e \approx 2.07 \times 10^{-15}$ Wb is the magnetic flux quantum. This inverse-square relation permits the extraction of $\xi(T)$ from experimental measurements of $H_{c2}(T)$.

Near the critical temperature T_c, the coherence length diverges according to the power law (44). Such a scaling behaviour arises naturally when the GL-framework is extended to include fractional or conformable spatial derivatives. These generalised operators account for non-locality, fractality, and memory in the order parameter field, yielding a power-law divergence of the correlation length and critical exponents beyond mean-field values [14, 15, 85–87]. Similar approaches using fractional or stochastic generalisations of the GL-equation have also been explored in dynamical settings [88]. To see this, consider the modified GL-free energy with the conformable derivative

$$|D_x^{(\mu)}\psi(x)|^2 = x^{2(1-\mu)}\left(\frac{d\psi}{dx}\right)^2, \tag{97}$$

which replaces the standard gradient term $|\nabla\psi|^2$. The corresponding GL-equation in one dimension becomes

$$a_0(T-T_c)\psi + b|\psi|^2\psi - \frac{1}{2m}\left[x^{2(1-\mu)}\frac{d^2\psi}{dx^2} + (1-\mu)x^{1-2\mu}\frac{d\psi}{dx}\right] = 0. \tag{98}$$

Assuming a linearised form near T_c and a trial solution $\psi(x) = \psi_0 e^{-x/\xi}$, we compute the derivatives

$$\frac{d\psi}{dx} = -\frac{\psi_0}{\xi}e^{-x/\xi}, \tag{99}$$

$$\frac{d^2\psi}{dx^2} = \frac{\psi_0}{\xi^2}e^{-x/\xi}. \tag{100}$$

Substituting these into the kinetic term (97) produces

$$\text{Kinetic term} \approx \frac{1}{2m}\left[\frac{x^{2(1-\mu)}}{\xi^2} + \frac{(1-\mu)x^{1-2\mu}}{\xi}\right]\psi(x). \tag{101}$$

Balancing with the linear term $a_0(T-T_c)\psi$ in relation (98) we find

$$a_0(T-T_c) \sim \frac{1}{2m}\left[\frac{x^{2(1-\mu)}}{\xi^2} + \frac{(1-\mu)x^{1-2\mu}}{\xi}\right]. \tag{102}$$

Let $\tau = T_c - T > 0$ be the reduced temperature. The modified GL-equation involves two characteristic contributions: the standard term $T_1(\tau,\xi) \propto \xi^{-2}$, associated with the spatial gradient energy, and the conformable (or fractional) correction $T_2(\tau,\xi) \propto \tau$, which encodes temperature-dependent scaling. The relative magnitude of these two terms determines the dominant physical regime. The crossover point is obtained when both contributions are of comparable strength,

$$T_1(\tau_*,\xi_*) = T_2(\tau_*,\xi_*), \tag{103}$$

which defines a characteristic temperature scale τ_* and a corresponding coherence length ξ_*. In practice, τ_* marks the boundary between the mean-field region and the conformable-dominated region, and ξ_* is the coherence length at this crossover.

As $T \to T_c$, the reduced temperature becomes very small ($\tau \ll \tau_*$) and the coherence length grows ($\xi \gg \xi_*$). In this regime, the conformable correction dominates, leading to the scaling law

$$\xi(T) \sim (T_c - T)^{-1}. \tag{104}$$

In the opposite regime, defined by $\tau \gg \tau_*$ (equivalently $\xi \ll \xi_*$), the standard GL-term prevails, and the classical mean-field result is recovered,

$$\xi(T) \sim (T_c - T)^{-1/2}. \tag{105}$$

Therefore, (τ_*,ξ_*) act as crossover parameters separating two asymptotic domains: a critical region governed by conformable scaling and a mean-field region governed by conventional GL-behaviour. Thus, the power-law divergence (44) is recovered with an effective exponent ν that may vary depending on which term dominates and the value of the conformable parameter μ. In fractional generalisations of the GL-theory, one may therefore consider replacing the usual spatial gradient term $|\nabla\psi|^2$ by this deformed expression involving the conformable derivative, leading to a modified scaling law for $\xi(T)$.

The experimental data for $H_{c2}(T)$ were extracted from Fig. 4 of the classical work by Williamson [89], which presents the behaviour of the upper critical field for niobium. The experimental points (open circles) were digitised using the *WebPlotDigitizer* tool.

The original figure displays the data in a dimensionless form, with the vertical axis representing the ratio $-H_{c2}^2/(dH_{c2}/dt)_{t=1}$ as a function of the reduced temperature $t = T/T_c$. To reconstruct the actual values of $H_{c2}(T)$ in physical units (Tesla), we assumed a functional dependence near the critical transition of the form

$$H_{c2}(T) = H_{c2}(0)\left(1 - \frac{T}{T_c}\right)^n, \tag{106}$$

where $H_{c2}(0)$ denotes the extrapolated upper critical field at zero temperature and n is a fitting exponent. The critical temperature was fixed at $T_c = 9.2$ K, consistent with values for pure niobium and in line with the parameters adopted in literature [66]. After conversion, the data were fitted with the empirical model (106), enabling the subsequent extraction of the temperature dependence of the coherence length $\xi(T)$ through inversion of relation (96), i.e.,

$$\xi(T) = \sqrt{\frac{\Phi_0}{2\pi H_{c2}(T)}}. \tag{107}$$

This procedure provides an experimental foundation to assess the validity of theoretical models for the temperature dependence of the coherence length.

We note that here we claim that the use of a conformable derivative in the GL-framework is motivated by the need to capture mesoscopic and spatially inhomogeneous effects that are not adequately described by the standard theory. In particular, real superconducting samples often exhibit intrinsic disorder, nonuniform pinning landscapes, or microstructural granularity, leading to effective nonlocal interactions and spatially varying coherence. The conformable derivative introduces a scale-dependent deformation of the spatial gradient, which can mimic the impact of fractal-like geometries or finite-size scaling in constrained domains. This modification allows for a more flexible description of the kinetic term and leads to a generalised coherence length scaling, making the theory more compatible with experimental deviations from classical mean-field behaviour.

2. *Results and analysis*

Figure 3 displays the digitised data and the fitted model for the coherence length. The excellent agreement indicates the suitability of the power-law model in this temperature range. The coherence length $\xi(T)$ describes the spatial extent over which the superconducting order parameter ψ varies appreciably. As $T \to T_c$, $\xi(T) \to \infty$, and A sets the overall magnitude of ξ for temperatures below T_c. This coefficient A encodes material-specific properties, including microscopic interactions, effective mass of Cooper pairs, and strength of the pairing potential.

The extracted critical exponent $\nu \approx 0.329$ is notably smaller than the classical GL-mean-field prediction of $\nu = 1/2$. Several factors may contribute to this deviation:

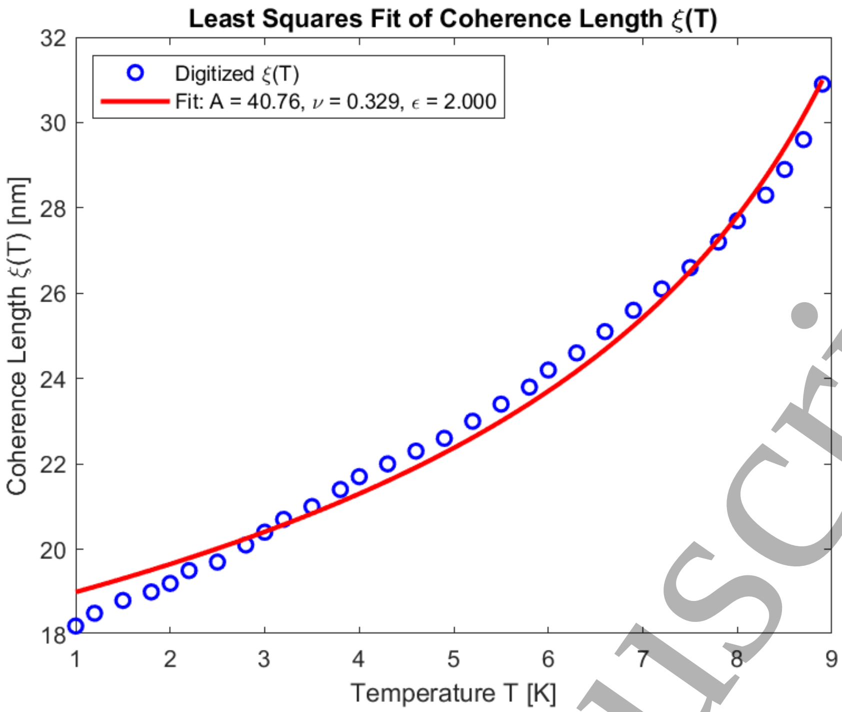

Figure 3: Extracted coherence length $\xi(T)$ (blue dots) and fitted model (red line) using $\xi(T) = A(T_c - T)^{-\nu}$ with $\nu = 0.329$ and $A = 40.76$. Experimental data adapted from [89], see text. Small deviations away from the immediate critical region reflect finite-size effects, experimental uncertainties, and crossover contributions not explicitly included in the minimal conformable model.

(i) A limited resolution or experimental uncertainty in the high-temperature region ($T \to T_c$); (ii) residual disorder or impurity-induced scattering effects not incorporated into the ideal GL-framework; (iii) memory effects—either thermal or spatial—captured through conformable (fractional-order) derivatives; (iv) effective averaging over anisotropic coherence lengths in polycrystalline or textured samples.

From the expression

$$\xi(T) = A(T_c - T)^{-\nu}, \tag{108}$$

we deduce the units of the coefficient A by dimensional analysis. Since $\xi(T)$ is given in nanometres (nm), $(T_c - T)$ in Kelvin (K), and ν is dimensionless, A must have units of $\mathrm{nm} \cdot \mathrm{K}^{\nu}$. Given the fitted exponent $\nu \approx 0.329$ and $A = 40.76$, this means that

$$\xi(T) \approx \frac{40.76}{(T_c - T)^{0.329}} \text{ [in nm]}. \tag{109}$$

To express A in SI units (metres), we convert according to

$$A_{\mathrm{SI}} = 40.76 \times 10^{-9}\ \mathrm{m} \cdot \mathrm{K}^{0.329} = 4.076 \times 10^{-8}\ \mathrm{m} \cdot \mathrm{K}^{0.329}. \tag{110}$$

Such a reduced exponent is qualitatively compatible with generalised GL-models that incorporate non-integer-order differential structures.

For clean elemental superconductors such as niobium, coherence lengths on the order of 30-40 nm are typical at intermediate temperatures. The value $A = 40.76\ \mathrm{nm} \cdot \mathrm{K}^{\nu}$ is thus physically reasonable and consistent with known properties of niobium. It represents the critical amplitude of the coherence length near the critical temperature T_c. Its physical meaning and units depend on how the data was scaled in the fitting process. A larger A implies that $\xi(T)$ becomes large faster as $T \to T_c$, for a fixed ν. This can be influenced by material purity, anisotropy or also electron-phonon coupling strength.

D. Quality of fits and origin of deviations

In Figs. 1 to 3, experimental data for the heat capacity and the London penetration depth are fitted using the parameters B_1, B_2, α_1, and α_2, which characterise the leading scaling behaviour within the conformable GL-framework. While the overall agreement captures the dominant critical trends, small but noticeable deviations between the data and the fitted curves are indeed present and merit discussion.

First, it is important to emphasise that the conformable formulation is designed to model the critical scaling regime, rather than the full temperature range. Outside the immediate vicinity of T_c, additional physical effects, such as crossover to mean-field behaviour, non-critical background contributions, and material-specific microscopic details are expected to contribute and are not explicitly included in the present minimal model. These effects naturally lead to systematic deviations from the asymptotic scaling window.

Second, both heat-capacity and penetration-depth measurements are known to be sensitive to experimental uncertainties, including sample inhomogeneity, finite-size effects, and resolution limitations close to the transition temperature. In particular, the rounding of singular behaviour near T_c may obscure the ideal critical form, producing small discrepancies even when the underlying scaling is correctly captured.

Third, the conformable GL-approach deliberately retains a compact parameter set to preserve analytical transparency and thermodynamic consistency. Subleading corrections to scaling, logarithmic terms, or additional crossover parameters may further improve the fit quality, yet at the cost of introducing non-universal parameters that would obscure the physical interpretation of the deformation indices. The present fits therefore represent a controlled compromise between descriptive accuracy and theoretical clarity.

In this sense the adequacy of the fitting procedure should be judged not solely by pointwise agreement, but by the general ability of the model to reproduce the correct scaling trends and yield consistent deformation parameters across different observables.

Within this criterion, the fits shown in Figs. 1 to 3 provide quite good, stable and reproducible estimates of the effective indices α_1 and α_2, thus supporting the physical relevance of the conformable framework despite the presence of minor deviations.

We further note that the fitting procedure employed in the preparation of Figs. 1 to 3 is based on a piecewise conformable analysis around the critical temperature, allowing for distinct effective scaling indices above and below T_c. This approach reflects the known asymmetry of fluctuation mechanisms in the ordered and disordered phases and avoids imposing an artificial global fit across crossover regimes. The parameters B_1, B_2, α_1, and α_2 were obtained from stable nonlinear regressions over restricted near-critical windows, and were verified to be robust under moderate variations of the fitting range. Residual deviations therefore primarily reflect crossover effects and experimental rounding rather than deficiencies of the scaling framework itself.

It is worth emphasising that the observed deviations in these figures are not merely numerical artifacts but are physically consistent with the interpretation of the deformation parameter μ as an effective coarse-grained index. From a microscopic perspective, these deviations reflect the influence of disorder, finite-size effects, and heterogeneous relaxation pathways that are known to generate asymmetric scaling and crossover behaviour near the critical point. From a theoretical standpoint, this is fully consistent with RG-expectations, where different pre-asymptotic regimes and effective exponents emerge depending on the dominant RG-trajectories. In the conformable framework, these effects are incorporated directly at the operator and functional levels, allowing the model to capture the leading scaling behaviour while naturally delimiting its controlled domain of applicability.

In the specific case of niobium, the remaining deviations observed in the heat-capacity peak and in the temperature dependence of the London penetration depth are also consistent with well-known microscopic effects not explicitly resolved within a minimal GL-description. As reviewed by Kröger [90], niobium is a strong-coupling superconductor in which electron-phonon interactions and surface-related phonon modes lead to renormalisations of the electronic density of states and effective masses near T_c. Such effects naturally induce asymmetric scaling and rounding behaviour in thermodynamic observables. Within the conformable approach, the deformation parameters should therefore be interpreted as effective indices summarising the cumulative impact of these microscopic renormalisations on the near-critical scaling behaviour, rather than as a substitute for a fully microscopic strong-coupling theory.

X. CONCLUSIONS AND OUTLOOK

In this work, we developed and applied the conformable derivative framework to describe critical phenomena near continuous phase transitions. By deforming the standard derivative operator through a temperature- or space-dependent scaling, we derived modified evolution equations for key thermodynamic observables, including the heat capacity, magnetisation, susceptibility, and correlation length. These equations yield consistent power-law behaviours near the critical point, with critical exponents expressed analytically in terms of conformable parameters. The formalism preserves dimensional consistency and remains compatible with equilibrium thermodynamics, while offering greater flexibility than standard approaches. In the limiting case $\mu \to 1$, the conformable derivative reduces to the standard derivative, and the classical GL-theory with mean-field critical exponents is recovered.

In particular, the observed deviations from the mean-field exponents are explained as consequences of conformable deformation, capturing finite-size effects, spatial granularity, and critical slowing down. The flexibility of the deformation parameter enables the modelling of crossover behaviours between distinct universality classes, potentially capturing nontrivial fixed-point trajectories. The conformable approach thus provides a meaningful interpolation between classical mean-field theory and more general statistical frameworks, such as nonextensive thermodynamics.

We demonstrated the relevance of the conformal framework by fitting experimental data of the superconducting transition of niobium. Good fit qualities could be achieved over relatively broad intervals around the critical temperature for the heat capacity, the London penetration depth, and the coherence length. For this purpose, an asymmetric formulation was proposed based on the conformal GL-formulation, in which different model parameters appear below and above the critical temperature.

Contrary to the impression that the present formulation merely restates known mean-field scaling, our results highlight several nontrivial and new physical insights:

(i) Dynamical origin of static exponents. By combining the conformable derivative with the critical kinetic coefficient $\Gamma(T) \sim |T - T_c|^{z\nu}$, the static exponents emerge directly from the relaxation dynamics, rather than being postulated. For instance, the order parameter equation yields the form $\Gamma(T)T^{1-\mu_\psi}\frac{d\psi}{dT} \simeq -a_0(T_c - T)\psi$, leading to $\psi(T) \propto (T_c - T)^{\beta(\mu_\psi)}$, where β is determined by (μ_ψ, Γ). This mechanism is absent in simple reparametrisations of the temperature.

(ii) Cross-observable constraints. Equation (111) establishes the internal consistency among the conformable parameters associated with different observables. Specifically, the exponents governing the temperature dependence of $\{C_V, |M|, \chi, \xi\}$ are not independent but related by the classical scaling identities expressed in terms of the deformation indices μ_X, namely,

$$\mu_C^{-1} = 2\mu_M^{-1} + \mu_\chi^{-1} - 2\mu_\xi^{-1}. \tag{111}$$

Hence, once μ_C is determined experimentally, the remaining parameters $\{\mu_M, \mu_\chi, \mu_\xi\}$ are constrained by this relation. This feature defines a set of cross-observable tests for the conformable scaling hypothesis, providing a stringent criterion for internal consistency across thermodynamic observables.

(iii) Distinct experimental signatures. The prediction that $T^{\mu_X} d\ln X/dT$ has a constant slope in $\ln|T - T_c|$ provides a direct discriminant from conventional fits: the slope is observable-dependent and determined by microscopic couplings, not just by a universal change of variables.

(iv) Physical interpretation of μ. The conformable index μ captures finite-size effects, disorder, and nontrivial memory of thermal fluctuations, bridging kinetic slowing down and effective fractal dimensionality. This extends beyond a notational reformulation.

A significant advantage of the conformable framework lies in its analytical tractability. Unlike renormalisation group methods, which often rely on asymptotic expansions and perturbative schemes, the conformable model yields closed-form expressions that can be directly fitted to experimental data. This was demonstrated through application to superconducting phase transitions in niobium, where the model successfully captured the asymmetric scaling of specific heat and London penetration depth near the critical temperature. The extracted exponents are consistent with deviations from mean-field theory and suggest the presence of mesoscopic effects, spatial inhomogeneities, and dynamics characterised by memory effects.

The geometric interpretation of the conformable framework developed herein is linked to deformations in the thermal or spatial metric [14, 19, 75] and offers insight into how fractal geometry and nonlocal responses might influence critical behaviour. In parallel, conformable symmetry approaches have also found applications in nuclear structure and critical point symmetries [91], illustrating the broader relevance of such deformations across diverse physical systems. In particular, the framework is general and can be adapted to a broad range of critical systems, including percolation thresholds, spin glasses, disordered media, and quantum phase transitions. Its generality makes it a valuable tool for exploring universality in complex systems, particularly in non-equilibrium or nonextensive regimes.

Beyond phenomenological applications, several open questions remain as to the rigorous mathematical formulation of the conformable framework. These include the development of a well-defined operator semigroup structure, the spectral theory of deformed differential operators, and the formulation of variational principles compatible with conformable dynamics [20]. Addressing these foundational issues may reveal deeper connections between the conformable approach and the broader structure of mathematical physics.

The possible extension of the conformal framework to quantum phase transitions, non-equilibrium critical dynamics, and systems with complex topology will be discussed in the future. These applications require further theoretical development and experimental validation.

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Declarations

Conflicts of interest/Competing interests:

The authors declare no conflicts of interest related to this work.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGpt in order to improve the English. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

Appendix A: Dimensional consistency

The mathematical structure of the conformable derivative suggests a deeper geometric meaning. In this section, we explore interpretations based on fractal metrics and spatial deformation. Specifically, we analyse the dimensional consistency of the primary equations and critical exponents derived using the conformable derivative framework.

1. Conformable derivative units

The conformable derivative with respect to temperature is defined as

$$D_T^{(\mu)} f(T) = T^{1-\mu} \frac{df}{dT}. \tag{A1}$$

Given that temperature has units of Kelvin, $[T] = \mathrm{K}$ we see that $\frac{df}{dT}$ has units of $[f]/\mathrm{K}$. Therefore

$$[D_T^{(\mu)} f] = [f] \cdot \mathrm{K}^{-\mu}. \tag{A2}$$

This confirms that the conformable derivative scales as expected, preserving consistent units when applied to temperature-dependent quantities.

2. Critical exponent dimensionality

The critical exponents α, β, γ, and ν are defined through

$$\alpha = \kappa T_c^{\mu_C - 1}, \quad \beta = \gamma T_c^{\mu_M - 1}, \\ \gamma = \lambda T_c^{\mu_\chi - 1}, \quad \nu = \rho T_c^{\mu_\xi - 1}. \tag{A3}$$

Since exponents such as α appear in power-law expressions of the form $C(T) \sim |T - T_c|^{-\alpha}$, they must be dimensionless. For this to hold, κ must have units of $\mathrm{K}^{1-\mu_C}$, and similarly for the other exponents. Hence, with this choice all critical exponents are indeed dimensionless, satisfying the necessary conditions for physical consistency.

3. Modified Ginzburg–Landau equation

The modified GL-equation derived using the conformable kinetic term reads

$$a(T)\psi + b|\psi|^2\psi - \frac{1}{2m} T^{2(1-\alpha)} \frac{d^2\psi}{dT^2} - \frac{1-\alpha}{m} T^{1-2\alpha} \frac{d\psi}{dT} = 0. \tag{A4}$$

Now let ψ have the units $[\psi] = \mathrm{K}^{-1/2}$. Then (i) $a(T) = a_0(T - T_c)$ implies $[a_0] = \mathrm{K}^{-2}$; (ii) to match units, the coefficient b must have the units $[b] = \mathrm{K}^{-5/2}$; (iii) the kinetic terms involve second and first derivatives, and with the conformable powers of T, they retain the same dimension as ψ/K, ensuring consistency. Thus, all terms in the equation have matching units.

4. Scaling of the heat capacity

The heat capacity scales as

$$C(T) \sim |T - T_c|^{-\alpha}, \quad [C] = \frac{E}{T}. \tag{A5}$$

For dimensional consistency, α must be dimensionless, which is satisfied given that $\alpha = \kappa T_c^{\mu_C - 1}$.

5. Penetration depth and order parameter

The London penetration depth scales as

$$\lambda_L(T) \sim (T_c - T)^{-\zeta}, \quad \zeta = \beta/2. \tag{A6}$$

Assuming that $[\lambda_L] =$ length, and that ζ is dimensionless, the units are correctly preserved.

In summary, all core expressions and critical exponent definitions derived from the conformable derivative framework are dimensionally consistent. This ensures the robustness of the theoretical framework and its applicability to thermodynamical systems such as superconducting transitions.

Appendix B: Variational derivation of the modified Ginzburg-Landau equation

We start from the conformable free energy functional

$$\mathcal{F}_\alpha[\psi] = \int \left[a(T)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{2m} \left| T_\alpha \psi(T) \right|^2 \right] dT. \tag{B1}$$

The conformable derivative with respect to temperature reads

$$T_\alpha \psi(T) = T^{1-\alpha} \frac{d\psi}{dT}. \tag{B2}$$

Then, the kinetic term takes on the form

$$\left| T_\alpha \psi(T) \right|^2 = T^{2(1-\alpha)} \left| \frac{d\psi}{dT} \right|^2, \tag{B3}$$

and the full functional reads

$$\mathcal{F}_\alpha[\psi] = \int \left[a(T)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{2m} T^{2(1-\alpha)} \left| \frac{d\psi}{dT} \right|^2 \right] dT. \tag{B4}$$

To minimise the free energy, we compute the variation with respect to $\psi^*(T)$,

$$\delta\mathcal{F}_\alpha = \int \left[a(T)\delta|\psi|^2 + b|\psi|^2\delta|\psi|^2 + \frac{1}{2m} T^{2(1-\alpha)} \left(\frac{d\psi}{dT}\frac{d(\delta\psi^*)}{dT} + \frac{d\psi^*}{dT}\frac{d(\delta\psi)}{dT} \right) \right] dT. \tag{B5}$$

Using integration by parts and assuming vanishing boundary terms, we find

$$\int T^{2(1-\alpha)} \frac{d\psi}{dT} \frac{d(\delta\psi^*)}{dT} dT = -\int \delta\psi^* \left[\frac{d}{dT} \left(T^{2(1-\alpha)} \frac{d\psi}{dT} \right) \right] dT. \tag{B6}$$

Therefore, setting $\delta\mathcal{F}_\alpha = 0$, the associated Euler–Lagrange equation becomes

$$a(T)\psi + b|\psi|^2\psi - \frac{1}{2m} \frac{d}{dT} \left(T^{2(1-\alpha)} \frac{d\psi}{dT} \right) = 0. \tag{B7}$$

Expanding the total derivative, we obtain

$$a(T)\psi + b|\psi|^2\psi - \frac{1}{2m} T^{2(1-\alpha)} \frac{d^2\psi}{dT^2} - \frac{(1-\alpha)}{m} T^{1-2\alpha} \frac{d\psi}{dT} = 0. \tag{B8}$$

Final form of in powers of $1 - T/T_c$

Substituting the ansatz

$$\psi(T) = \psi_0 \left(1 - \frac{T}{T_c} \right)^\beta \tag{B9}$$

into the full equation, we obtain

$$a_0(T - T_c)\psi_0 \left(1 - \frac{T}{T_c} \right)^\beta + b\psi_0^3 \left(1 - \frac{T}{T_c} \right)^{3\beta} - \frac{1}{2m} T^{2(1-\alpha)} \frac{\beta(\beta-1)\psi_0}{T_c^2} \left(1 - \frac{T}{T_c} \right)^{\beta-2} + \frac{(1-\alpha)}{m} T^{1-2\alpha} \frac{\beta\psi_0}{T_c} \left(1 - \frac{T}{T_c} \right)^{\beta-1} = 0. \tag{B10}$$

Matching powers of $(1 - T/T_c)$, we recover $\beta = 1/2$ as the critical exponent.

Now, we analyse the behaviour of $\psi(T)$ near the critical temperature T_c. To this end, consider again the nonlinear temperature-deformed GL-type equation

$$a(T)\psi + b|\psi|^2\psi - \frac{1}{2m} T^{2(1-\alpha)} \frac{d^2\psi}{dT^2} - \frac{1-\alpha}{m} T^{1-2\alpha} \frac{d\psi}{dT} = 0, \tag{B11}$$

with $a(T) = a_0(T - T_c)$. We again assume the ansatz

$$\psi(T) = \psi_0 \left(1 - \frac{T}{T_c} \right)^\beta = \psi_0 \varepsilon^\beta, \text{ where } \varepsilon = 1 - \frac{T}{T_c}. \tag{B12}$$

Then, forming the derivatives with respect to T, we find

$$\begin{aligned} \frac{d\varepsilon}{dT} &= -\frac{1}{T_c}, \\ \frac{d\psi}{dT} &= -\frac{\beta\psi_0}{T_c} \varepsilon^{\beta-1} \\ \frac{d^2\psi}{dT^2} &= \frac{\beta(\beta-1)\psi_0}{T_c^2} \varepsilon^{\beta-2}. \end{aligned} \tag{B13}$$

We can then identify the different terms. Starting with the linear term,

$$a_0(T - T_c)\psi = -a_0 T_c \psi_0 \varepsilon^{\beta+1}; \tag{B14}$$

then the nonlinear term

$$b|\psi|^2\psi = b\psi_0^3 \varepsilon^{3\beta}; \tag{B15}$$

the term with the second derivative,

$$-\frac{1}{2m} T^{2(1-\alpha)} \frac{\beta(\beta-1)\psi_0}{T_c^2} \varepsilon^{\beta-2}; \tag{B16}$$

and the term with the first-order derivative,

$$\frac{(1-\alpha)\beta\psi_0}{mT_c} T^{1-2\alpha} \varepsilon^{\beta-1}. \tag{B17}$$

We see that $\varepsilon = 1 - T/T_c$ enters these different terms with a different exponent. Seeking a consistent balance, we suppose that the nonlinear term $\propto \varepsilon^{3\beta}$ and the linear term $\propto \varepsilon^{\beta+1}$ dominate. We then equate their powers: $3\beta = \beta + 1$, implying $\beta = 1/2$. This is consistent with standard Landau theory and a typical critical exponent in second-order phase transitions. This means that in the four terms above, we have the orders

$$\begin{aligned} \varepsilon^{\beta-2} &= \varepsilon^{-3/2} \\ \varepsilon^{\beta-1} &= \varepsilon^{-1/2} \\ \varepsilon^{3\beta} &= \varepsilon^{3/2} \\ \varepsilon^{\beta+1} &= \varepsilon^{3/2}, \end{aligned} \tag{B18}$$

i.e., the derivative terms diverge as $\varepsilon \to 0$. However, despite their divergence, the derivative terms describe kinetic effects (gradients of ψ). Near $T = T_c$, the order parameter becomes small and smooth, $\psi \to 0$, and its variation is slow. Finally the mean-field and equilibrium theories focus on minimising the free energy—gradient terms are subleading. This allows us to match

$$a(T)\psi + b\psi^3 = 0 \quad \Rightarrow \quad \psi = \pm\sqrt{-\frac{a(T)}{b}}. \tag{B19}$$

From this we conclude that

$$\psi(T) = \psi_0 \left(1 - \frac{T}{T_c} \right)^{1/2} \tag{B20}$$

Thus, the derivative terms represent corrections to the scaling behaviour, but not to the leading-order critical exponent. This demonstrates that the ansatz (B9) satisfies the nonlinear equation near $T = T_c$, with leading order balance provided by the algebraic terms,

$$a(T)\psi + b\psi^3 = 0 \quad \Rightarrow \quad \boxed{\beta = 1/2}. \tag{B21}$$

This matches classical Landau theory and remains valid under conformable kinetic corrections. Moreover, this represents a static equilibrium solution for which the derivatives are zero (i.e., long-time equilibrium). It is important to note that our model for equilibrium and homogeneous conditions and when fluctuations are negligible, has the classical exponent $\beta = 1/2$.

Appendix C: Recovery of the relaxation equation in the undeformed limit

We now show that the classical critical exponent $\beta = \frac{1}{2}$ naturally emerges from the conformable relaxation equation in the undeformed equilibrium limit. To this end, consider the generalised relaxation model

$$\Gamma(T)T^{1-\mu_\psi}\frac{d\psi}{dT} = -a_0(T_c - T)\psi - b|\psi|^2\psi, \qquad \text{(C1)}$$

where $\Gamma(T)$ is a temperature-dependent kinetic coefficient, $\mu_\psi \in (0, 1]$ is the conformable deformation parameter, and the right-hand side arises from the variational derivative of the standard GL-free energy.

In the undeformed limit, we take

$$\mu_\psi = 1, \text{ such that } T^{1-\mu_\psi} = 1, \qquad \text{(C2)}$$

and for simplicity we assume that the kinetic coefficient is constant near T_c, i.e., $\Gamma(T) \approx \Gamma_0$. Equation (C1) then becomes

$$\Gamma_0\frac{d\psi}{dT} = -a_0(T_c - T)\psi - b|\psi|^2\psi. \qquad \text{(C3)}$$

We now look for a solution near the critical point of the form

$$\psi(T) = \psi_0(T_c - T)^\beta, \qquad \text{(C4)}$$

valid for $T < T_c$ and where ψ_0 is a constant amplitude to be determined. Computing the derivative, we have

$$\frac{d\psi}{dT} = -\beta\psi_0(T_c - T)^{\beta-1}. \qquad \text{(C5)}$$

Substituting into Eq. (C3), we obtain

$$\begin{aligned}&-\Gamma_0\beta\psi_0(T_c - T)^{\beta-1}\\&= -a_0(T_c - T)\psi_0(T_c - T)^\beta - b\psi_0^3(T_c - T)^{3\beta}. \qquad \text{(C6)}\end{aligned}$$

Dividing both sides by $\psi_0(T_c - T)^{\beta-1}$ (assuming $\psi_0 \neq 0$ and $T < T_c$),

$$-\Gamma_0\beta = -a_0(T_c - T)^2 - b\psi_0^2(T_c - T)^{2\beta+1}. \qquad \text{(C7)}$$

Now we check the leading-order behaviour as $T \to T_c^-$: The term $(T_c - T)^2$ dominates, while the nonlinear term $\propto (T_c - T)^{2\beta+1}$ is subleading if $2\beta+1 > 2$ and thus $\beta > \frac{1}{2}$; or they are comparable if $\beta = \frac{1}{2}$.

Assuming that the dominant balance occurs between the terms of order

$$-\Gamma_0\beta \approx -a_0(T_c - T)^2, \qquad \text{(C8)}$$

we see that this leads to a contradiction: the left-hand side is constant, while the right-hand side vanishes as $T \to T_c$. Therefore, this balance fails. Instead, if we assume that the dominant balance occurs between the first-derivative term and the nonlinear cubic term,

$$-\Gamma_0\beta\psi_0(T_c - T)^{\beta-1} \approx -b\psi_0^3(T_c - T)^{3\beta}, \qquad \text{(C9)}$$

and we cancel signs and factors of ψ_0; then we see that

$$\Gamma_0\beta(T_c - T)^{\beta-1} = b\psi_0^2(T_c - T)^{3\beta}. \qquad \text{(C10)}$$

Divide both sides by $(T_c - T)^{\beta-1}$ to find

$$\Gamma_0\beta = b\psi_0^2(T_c - T)^{2\beta+1}. \qquad \text{(C11)}$$

To keep both sides finite and nonzero as $T \to T_c$, we must have that

$$2\beta + 1 = 0 \quad \Rightarrow \quad \beta = -\tfrac{1}{2}, \qquad \text{(C12)}$$

which is unphysical.

Thus we try instead to match the linear and nonlinear terms (equivalent to the adiabatic limit),

$$a_0(T_c - T)\psi \sim b\psi^3 \quad \Rightarrow \quad a_0(T_c - T) \sim b\psi^2. \qquad \text{(C13)}$$

Solving for $\psi(T)$, we see that

$$\psi^2(T) \sim \frac{a_0}{b}(T_c - T), \quad \Rightarrow \quad \psi(T) \sim \sqrt{\frac{a_0}{b}}(T_c - T)^{1/2}. \qquad \text{(C14)}$$

Thus, the leading-order balance occurs not with the derivative term, but with the free-energy terms, confirming that in the long-time (quasi-equilibrium) limit, the stationary solution satisfies

$$\beta = \frac{1}{2}. \qquad \text{(C15)}$$

This demonstrates that even when starting from the generalised relaxation equation, the classical mean-field exponent $\beta = 1/2$ is recovered in the undeformed limit $\mu_\psi = 1$, under equilibrium assumptions. The derivative terms decay near T_c, and the critical scaling is governed by the balance of the potential terms in the GL-free energy.

Higher-order corrections

Derivative terms can be treated as perturbations (next-to-leading order). They modify the scaling only slightly and do not affect the critical exponent to leading order. Thus, (i) $\varepsilon^{\beta-1}$ and $\varepsilon^{\beta-2}$ diverge as $\varepsilon \to 0$ but are derivative-based; (ii) near T_c, ψ is small and smooth, so derivatives are small; (iii) the dominant balance is between $a(T)\psi$ and $b\psi^3$, yielding again $\beta = 1/2$. Thus, indeed the derivative terms are corrections to this leading-order behaviour.

Appendix D: On the claim of simple reparametrisation

For a scalar $f(T)$ with constant coefficients, we have that $D_T^{(\mu)}f = T^{1-\mu}f'(T) = df/dT'$ with $T' = T^\mu/\mu$. However, consider the two cases:

(i) *Critical kinetics.* With $\Gamma(T) \sim |T - T_c|^{z\nu}$, the relaxation law becomes $\Gamma(T(T'))\frac{dX}{dT'} = -\partial F/\partial X$. The nontrivial T-dependence of Γ survives, producing a distinct scaling.

(ii) *Thermal GL-functional.* The conformable GL-free energy includes a gradient term $\propto T^{2(1-\alpha)}|\frac{d\psi}{dT}|^2$. Under a transformation $T \mapsto T'$, Jacobian factors remain, yielding Euler-Lagrange equations [Eq. (B8)] that are not reducible to the standard GL-form.

Hence, the conformable framework only collapses to a trivial change of variables in the static, constant-coefficient case. In the presence of critical kinetics and weighted GL-terms, it carries genuine physical content.

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