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# Anomalous diffusion and fluctuations in complex systems and networks

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# Anomalous diffusion and fluctuations in complex systems and networks

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## ABSTRACT

Following significant advances in microscopic and macroscopic single-particle tracking and supercomputing, the theoretical investigation of fluctuations and anomalous dynamics in complex systems is currently of high interest. Stochastic processes and their generalizations represent an important tool for the statistical description of such systems. Modeling random walks and stochastic processes in complex systems, including complex networks and graphs, requires an interdisciplinary approach due to the different applications in various fields, such as physics, biology, chemistry, engineering, computer science, and economy. Various studies of active and passive tracer diffusion, for instance, in biological cells and in heterogeneous and porous media showed that the underlying structure of the environment has a strong effect on the particle movement, leading to anomalous dynamics due to the constrained particle motion or the variation of the local diffusion coefficient and the potential energy function. Moreover, determining optimal search strategies is central in diverse fields, from physics to computer science, from biology to robotics. In particular, random search strategies have been widely observed for animal foraging, in reaction pathways in DNA-binding proteins, in intracellular transport, etc. Furthermore, it has been shown that the resetting of the searcher to its initial position can improve the search strategy by appropriate optimal resetting rate, which results in minimizing the mean first-passage time. This Editorial is meant to serve as an Introduction to this Focus Issue in the form of a mini-review of the field.

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The study of anomalous diffusion and its characteristic fluctuations has been a prominent field of research in the study of dynamic phenomena in a variety of complex systems such as soft- and bio-matter or networks. Unlike normal diffusion, the paradigm of classical Brownian motion characterized by a linear time-dependence of the mean squared displacement and a Gaussian probability density function, anomalous diffusion exhibits a power-law time-dependence of the mean squared displacement and often a non-Gaussian or non-Markovian process, featuring ergodicity breaking and aging, as observed in many systems far from equilibrium. Different models of anomalous dynamics have been developed for the description of various stochastic processes, depending on the physical nature of the system under

consideration. This editorial provides a brief overview of anomalous diffusion and summarizes the contribution in this focus issue.

## I. INTRODUCTION

The theoretical investigation of the fluctuations and stochastic dynamics in complex systems has been of interest for decades. The modeling of random walks in complex systems, including complex networks and graphs, requires an interdisciplinary approach. An example is the simple random walk model of Brownian motion

observed in the movement of pollen granules in water.<sup>1,2</sup> Normal-diffusive behavior underlying this Brownian motion was first described by Fick<sup>3</sup> through Fick's laws, and it was later applied to statistical time series analysis by the astronomer and statistician Thorvald Nicolai Thiele,<sup>4</sup> as well as by the famous mathematician Louis Bachelier<sup>5</sup> for the analysis of the dynamic behavior of stock prices at the Paris stock exchange. Later, Einstein,<sup>6</sup> von Smoluchowski,<sup>7</sup> and Sutherland<sup>8</sup> developed a theory of Brownian motion in terms of the diffusion equation. Thus, it is evident that one single model can be used for the description of vastly different phenomena in physics, mathematical statistics, economy, and geophysics.

The one-dimensional Brownian motion can also be described in terms of the standard Langevin equation,<sup>9–11</sup>

$$m\ddot{x}(t) + m\gamma_0\dot{x}(t) + \frac{dV(x)}{dx} = \xi(t), \quad \dot{x}(t) = v(t), \quad (1)$$

where  $x(t)$  and  $v(t)$  are the particle's displacement and velocity coordinates, respectively,  $V(x)$  is an external potential, while  $\xi(t)$  is a stochastic force (the "noise"), which is Gaussian with zero mean, correlation  $\langle \xi(t)\xi(t') \rangle = 2mk_B T\gamma_0\delta(t-t')$ , and thermal energy  $k_B T$ . Here,  $\langle \cdot \rangle$  means ensemble averaging over many trajectories. The mean squared displacement (MSD)  $\langle x^2(t) \rangle$  obtained from the Langevin equation (1) behaves ballistically at short times,  $\langle x^2(t) \rangle \simeq t^2$  due to inertial effects, while it crosses over to a linear dependence on time,  $\langle x^2(t) \rangle \simeq t$ , in the long time limit.

In the overdamped limit, when the inertial term can be neglected, i.e., when the friction term  $m\gamma_0\dot{x}(t)$  is dominant, the Langevin equation has the form

$$m\gamma_0\dot{x}(t) + \frac{dV(x)}{dx} = \xi(t), \quad \dot{x}(t) = v(t). \quad (2)$$

In the absence of an external potential, it reduces to

$$m\gamma_0\dot{x}(t) = \xi(t), \quad \dot{x}(t) = v(t). \quad (3)$$

The variance  $\sigma^2(t) = \langle x^2(t) \rangle - \langle x(t) \rangle^2$  for this process reads

$$\sigma^2(t) = \frac{2k_B T}{m\gamma_0}t, \quad (4)$$

which means normal diffusion with diffusion coefficient  $D = k_B T/(m\gamma_0)$  whose dimension is  $[D] = \text{length}^2/\text{time}$ .

As we are dealing with a Gaussian process, from the Langevin equation (2), one can derive the corresponding equation for the probability density function (PDF), which is the standard Fokker–Planck equation,<sup>10,12–14</sup>

$$\frac{\partial}{\partial t}P_0(x, t) = L_{\text{FP}}P_0(x, t), \quad (5)$$

with the Fokker–Planck operator

$$L_{\text{FP}} \equiv \left[ \frac{\partial}{\partial x}V(x) + D\frac{\partial^2}{\partial x^2} \right]. \quad (6)$$

In the absence of an external potential ( $V(x) = \text{const}$ ), the solution of the diffusion equation for the initial condition  $P(x, 0) = \delta(x - x_0)$

is the Gaussian PDF,

$$P_0(x, t) = \frac{1}{\sqrt{4\pi Dt}}e^{-(x-x_0)^2/(4Dt)}, \quad (7)$$

and thus, the variance is again of the form  $\sigma^2(t) = 2Dt$ .

Brownian motion can also be described using the standard random walk model. The walker jumps at each time step  $t = 0, \Delta t, 2\Delta t, \dots, n\Delta t$ , to one of the nearest neighbor sites in a randomly selected direction, covering the distance  $\Delta x$ , which we call the "lattice constant." For the one-dimensional random walk, the walker jumps left or right with the same probability. Thus, the PDF to find the walker at position  $j$  at time  $t + \Delta t$  in dependence of the population of the two adjacent sites  $j \pm 1$  at time  $t$  can be determined from the master equation<sup>15</sup>

$$W_j(t + \Delta t) = \frac{1}{2}W_{j-1}(t) + \frac{1}{2}W_{j+1}(t), \quad (8)$$

which means that the walker at time  $t + \Delta t$  can arrive at the site  $j$  with equal probability  $1/2$  from the left, from the site  $j - 1$ , and from the right, from the site  $j + 1$ . In the continuum limit  $\Delta x \rightarrow 0, \Delta t \rightarrow 0$ , by Taylor expansion in the master equation, and by defining the diffusion coefficient as  $D = \lim_{\Delta x \rightarrow 0, \Delta t \rightarrow 0} [\Delta x]^2 / (2\Delta t)$ , one arrives at the standard diffusion equation for the Brownian motion.<sup>15,16</sup> For the case when the probability of the particle to jump left or right is not equal and depends on the walker's position, such as in the presence of external force, by using the master equation, in the continuum limit, one can derive the standard Fokker–Planck equation (5).

"Anomalous is normal"<sup>17</sup> is a famous statement based on the experimental observation related to diffusive transport in complex and heterogeneous systems. Specifically, contrary to the well-known normal-diffusive transport in which the mean squared displacement grows linearly in time,  $x^2(t) \simeq t$ , anomalous transport is characterized by a power-law dependence of the MSD on time,  $\langle x^2(t) \rangle \simeq t^\alpha$  with  $\alpha \neq 1$ .<sup>15,16</sup> The anomalous diffusion exponent  $\alpha$  defines different diffusive regimes: subdiffusion for  $0 < \alpha < 1$ , normal diffusion for  $\alpha = 1$ , and superdiffusion for  $\alpha > 1$ . For  $\alpha = 2$ , it is ballistic motion, while for  $\alpha > 2$ , it is sometimes called hyperdiffusion. There are also so-called ultraslow-diffusive processes for which the MSD has a logarithmic dependence in time,  $\langle x^2(t) \rangle \simeq \log t$  or a more complicated behavior of the form  $\langle x^2(t) \rangle \simeq \log^\nu t$ ,  $\nu > 0$ , as in the Sinai-diffusion for  $\nu = 4$ .<sup>18–21</sup>

In this Focus Issue, the contributing papers can be categorized into several groups:

- random walks and anomalous diffusion modeling;<sup>22–32</sup>
- non-ergodicity, non-Gaussianity, and aging;<sup>33,34</sup>
- heterogeneous and/or bounded systems;<sup>35–43</sup>
- first-passage processes, random search, and stochastic resetting;<sup>44–47</sup>
- random walks on complex networks;<sup>48–52</sup>
- applications in physics, biology, chemistry, and economy;<sup>53–60</sup>
- data assimilation techniques for (anomalous) diffusion and fluctuations;<sup>61,62</sup> and
- anomalous dynamics in quantum systems.<sup>63,64</sup>

We have provided a brief primer on various stochastic pathways to anomalous diffusion and other stochastic modeling techniques.

Along the way, we connect to the papers in this Focus Issue. At the end of the paper, we give a summary and future directions in the field.

## II. RANDOM WALKS AND ANOMALOUS DIFFUSION

Different approaches to anomalous diffusion have been developed.<sup>15,65,66</sup> We explicitly mention the continuous time random walk (CTRW) model, fractional diffusion and Fokker–Planck equations (FDEs and FFPEs), fractional and generalized Langevin equations (FLEs and GLEs), fractional Brownian motion (FBM), heterogeneous diffusion and telegraphers equations, to name but a few. Generally, all these models lead to identical scaling for the MSD; however, the corresponding processes are fundamentally different. Therefore, when choosing a particular model for the description of experimental and simulations data, additionally to the MSD, the time-averaged MSD, PDF, higher order moments, velocity auto-correlation function, etc., should be analyzed as well. Some models exhibit multi-scale time behavior, which makes them suitable for the description of different diffusive regimes and characteristic crossover dynamics in complex systems.

There are many other stochastic models of anomalous diffusion and fluctuations in complex systems. Here, we mention the quenched trap model,<sup>16,67</sup> the annealed transit time model,<sup>68</sup> mobile–immobile diffusion models of anomalous diffusion,<sup>69,70</sup> models of anomalous transport of self-propelled particles,<sup>71,72</sup> fractional Laplace motion,<sup>73,74</sup> multifractional Brownian motion (MFBM),<sup>75,76</sup> and memory<sup>77</sup> and incremental<sup>78</sup> MFBMs. These are not discussed in this work. Anomalous diffusion in chaotic,<sup>79</sup> random,<sup>80</sup> and deterministic<sup>81</sup> dynamical systems are not discussed as well.

### A. Continuous time random walks

We start analyzing the PDF  $W(x, t)$  of a particle at position  $x$  at time  $t$  in the CTRW model. The PDF in Fourier–Laplace space (the Fourier transform is defined by  $\mathcal{F}[f(x)] = \tilde{f}(\kappa) = \int_{-\infty}^{\infty} f(x)e^{i\kappa x} dx$ , while the Laplace transform by  $\mathcal{L}[f(t)] = \hat{f}(s) = \int_0^{\infty} f(t)e^{-st} dt$ ) is given by the Montroll–Weiss equation<sup>15,82</sup>

$$\tilde{W}(\kappa, s) = \frac{1 - \hat{\psi}(s)}{s} \frac{\tilde{W}(\kappa, 0)}{1 - \tilde{\Psi}(\kappa, s)}, \quad (9)$$

where  $\tilde{\Psi}(\kappa, s)$  is the joint PDF of jumps and waiting times,

$$\lambda(x) = \int_0^{\infty} \Psi(x, t) dt \quad (10)$$

is the jump length PDF,

$$\psi(t) = \int_0^{\infty} \Psi(x, t) dx \quad (11)$$

is the waiting time PDF, and  $\tilde{W}(\kappa, 0)$  is the Fourier transform of the initial condition  $W(x, 0)$ . If the jump length and the waiting time are independent random variables, then we assume the product form  $\Psi(x, t) = \lambda(x)\psi(t)$ , and thus,  $\tilde{\Psi}(\kappa, s) = \tilde{\lambda}(\kappa)\hat{\psi}(s)$ .

The corresponding diffusion (Fokker–Planck) equation for the decoupled CTRW model with Gaussian jump PDF reads<sup>83</sup>

$$\frac{\partial}{\partial t} W(x, t) = D \int_0^t \eta(t - t') \frac{\partial^2}{\partial x^2} W(x, t') dt', \quad (12)$$

where the memory kernel  $\eta(t)$  is related to the waiting time PDF in the Laplace domain through  $\hat{\psi}(s) = [1 + 1/\hat{\eta}(s)]^{-1}$ . The dimension of the diffusion coefficient depends on the form of the memory kernel. The PDF to find the particle at position  $x$  at time  $t$  is given by<sup>83,84</sup>

$$W(x, t) = \sum_n P_0(x, n)h_n(t), \quad (13)$$

where  $P_0(n, t)$  is the PDF (7) to find the particle at  $x$  after  $n$  steps for the standard diffusion equation, while  $h_n(t)$  is the PDF to make exactly  $n$  steps up to time  $t$ . This function is connected to the waiting time PDF in the Laplace space as<sup>83</sup>

$$\hat{h}_n(s) = \frac{1 - \hat{\psi}(s)}{s} \hat{\psi}^n(s) \approx \frac{\hat{\phi}(s)}{s} e^{-n\hat{\phi}(s)}, \quad (14)$$

where  $\hat{\phi}(s) = 1 - \hat{\psi}(s)$ . The random number of steps  $n$  performed in this CTRW process has the role of the “operational time.” The MSD is a function of the mean number of steps  $\langle x^2(t) \rangle \simeq \langle n(t) \rangle$ . This CTRW process in the continuum approximation can be described in terms of the coupled Langevin equations<sup>85</sup> (see also Ref. 86 for the case in the presence of an external force)

$$\begin{aligned} \dot{x}(u) &= \sqrt{2D}\xi(u), \\ \dot{t}(u) &= \zeta(u), \end{aligned} \quad (15)$$

where  $\xi(u) = \frac{dB(u)}{du}$  represents white Gaussian noise with zero mean  $\langle \xi(u) \rangle = 0$  and correlation  $\langle \xi(u)\xi(u') \rangle = \delta(u - u')$ ,  $B(u)$  is the standard Brownian motion (Wiener process), and thus,  $x(u) = \sqrt{2D} \int_0^u \xi(u') du'$ , while  $\zeta(u)$  is a completely one-sided Lévy stable noise. The inverse process  $S(t)$  of the one-sided Lévy stable process  $t(u)$  with the characteristic function  $\langle e^{-st(u)} \rangle = e^{-\hat{\Psi}(s)u}$ , where  $\hat{\Psi}(s)$  is a “characteristic exponent” given by  $S(t) = \inf\{u > 0 : t(u) > t\}$ ; i.e., it represents a distribution of first-passage times for the operational time  $u$  given the laboratory time  $t$ . The processes  $B(u)$  and  $S(t)$  are independent. From the coupled Langevin equations (15), one concludes that  $x(t)$  is parameterized in terms of the number of steps  $u$  (in the CTRW model, it corresponds to  $n$ ). The connection of the operational time to the lab time  $t$  is given by  $t(u) = \int_0^u \tau(u') du'$ , where  $\tau(u)$  is the total of individual waiting times  $\tau$  for each step. By use of this subordination approach, the solution of the generalized diffusion equation (12) can be found from<sup>15,83,87–89</sup>

$$W(x, t) = \int_0^{\infty} P_0(x, u)h(u, t) du. \quad (16)$$

Here,  $P_0(x, u)$  is the Gaussian PDF (7), and  $h(u, t)$  is the subordination function, which in the Laplace space is defined by

$$\hat{h}(u, s) = \frac{1}{s\hat{\eta}(s)} e^{-\frac{u}{\hat{\eta}(s)}}, \quad (17)$$

from where it follows that

$$\hat{W}(x, s) = \int_0^\infty P_0(x, u) \hat{h}(u, s) du = \frac{1}{s \hat{\eta}(s)} \hat{P}_0 \left( x, \frac{1}{\hat{\eta}(s)} \right). \quad (18)$$

This relation directly follows from the PDF obtained from the CTRW model (13), for  $n \rightarrow u, h_n(t) \rightarrow h(u, t)$  and  $\sum_n \rightarrow \int du$ .

For a power-law waiting time PDF  $\psi(t) \sim \tau^{-1}(t/\tau)^{-\alpha-1}$ ,  $0 < \alpha < 1$  [i.e.,  $\eta(t) = (t/\tau)^{\alpha-1}/\Gamma(\alpha)$ ], this approach leads to the time fractional diffusion equation with a solution in terms of the Fox  $H$ -function  $H_{p,q}^{m,n}(z)$ ,<sup>15,90</sup>

$$W(x, t) = \frac{1}{2|x|} H_{1,1}^{1,0} \left[ \frac{|x|}{\sqrt{D_\alpha t^\alpha}} \middle| \begin{matrix} (1, \alpha/2) \\ (1, 1) \end{matrix} \right], \quad (19)$$

for which the subordination function is a completely one-sided Lévy  $\alpha$ -stable PDF,

$$\begin{aligned} h(u, t) &= \mathcal{L}^{-1} \left[ \tau^{\alpha-1} s^{\alpha-1} e^{-u\tau^\alpha s^\alpha} \right] \\ &= \frac{t/\tau}{\alpha(u/\tau)^{1/\alpha}} L_\alpha \left( \frac{t/\tau}{(u/\tau)^{1/\alpha}} \right). \end{aligned} \quad (20)$$

The corresponding MSD has the power-law dependence

$$\langle x^2(t) \rangle = \frac{2D_\alpha t^\alpha}{\Gamma(1+\alpha)} \quad (21)$$

on time, which is a signature of subdiffusion. Here, the diffusion coefficient has dimension  $[D_\alpha] = \text{length}^2/\text{time}^\alpha$ . In this case,  $\hat{\Psi}(s) = s^\alpha$ ,  $0 < \alpha < 1$ ; the noise  $\zeta(u)$  is a one-sided  $\alpha$ -stable Lévy noise with the stable index  $0 < \alpha < 1$ , while  $\mathcal{S}(t)$  is called the inverse-time  $\alpha$ -stable subordinator.<sup>86,88</sup>

In the case of a Lévy distribution of jumps,  $\lambda(x) \simeq |x|^{-\beta-1}$ , with Lévy index  $1 < \beta < 2$ , the corresponding diffusion equation becomes<sup>91</sup>

$$\frac{\partial}{\partial t} W(x, t) = D_{\eta,\beta} \int_0^t \eta(t-t') \frac{\partial^\beta}{\partial |x|^\beta} W(x, t') dt', \quad (22)$$

where  $D_{\eta,\beta}$  is the generalized diffusion coefficient whose dimension depends on the form of the memory kernel  $\eta(t)$ , and the operator  $\mathcal{F} \left[ \frac{\partial^\beta}{\partial |x|^\beta} f(x) \right] = -|k|^\beta \tilde{f}(k)$  is the Riesz derivative of order  $\beta$ . Equation (22) describes a random walk with long-tailed jumps (Lévy flights) and long-tailed waiting times represented by the memory kernel  $\gamma(t)$ . In the case of Lévy flights, the second moment diverges due to the very long jumps; instead, the fractional order moments,  $\langle x^q(t) \rangle$ ,  $0 < q < \beta < 2$ , can be calculated, and the rescaling can be used  $\langle x^q(t) \rangle^{2/q}$ . In the limit of  $\beta = 1$ , the jump lengths are distributed according to the Cauchy law. For a power-law memory kernel  $\eta(t) = (t/\tau)^{\alpha-1}/\Gamma(\alpha)$ , the MSD is  $\langle x^q(t) \rangle^{2/q} \simeq t^{2\alpha/\beta}$ , which is subdiffusive for  $2\alpha/\beta < 1$ , normal diffusive for  $2\alpha/\beta = 1$ , and superdiffusive for  $2\alpha/\beta > 1$ .<sup>92</sup> This means that in the system, there is competition between long jumps and long waiting times.<sup>93</sup>

Since in the Lévy flight model, the long jumps happen instantaneously, a more realistic model of Lévy walks has been introduced by limiting the velocity of the jumps. In the Lévy walk model<sup>94,95</sup> with a fixed velocity  $v$ , the waiting time and the jump length PDFs are coupled, that is, a motion for which the joint probability  $\Psi(x, t)$  is coupled and take into account the fact that only particles flying from  $x - vt$  and  $x + vt$  can reach  $x$  in time  $t$  and change the

direction of their velocities after the flight time of  $t$ ; i.e.,  $\Psi(x, t) = \frac{1}{2} \delta(|x| - vt) \psi(t)$ . Using Eq. (9), for Lévy walks, the PDF can be obtained from<sup>95,96</sup>

$$\tilde{W}(\kappa, s) = \frac{\hat{\Phi}(s + i\kappa v) + \hat{\Phi}(s - i\kappa v)}{2 - [\hat{\psi}(s + i\kappa v) + \hat{\psi}(s - i\kappa v)]} \tilde{W}(\kappa, 0), \quad (23)$$

where  $\hat{\Phi}(s) = [1 - \hat{\psi}(s)]/s$ ; i.e.,  $\Phi(t) = 1 - \int_0^t \psi(t') dt'$  is the survival probability.

Recently, Araújo and Pagnini<sup>22</sup> studied the properties of Lévy flight PDFs not only in the tails of the PDF but also in the bulk of the PDF. They use the Kramers–Moyal expansion and the Pawula theorem to show that the process remains non-self-similar for a timescale longer than the Markovian time-lag of at least one order of magnitude, which means that for Lévy flights, the Markovianity time-lag is not the only timescale of the process; another and longer timescale controls the transition to the familiar power-law regime in the final diffusive limit, whose magnitude is independent of the Lévy index. Such an expansion method is important in the analysis of the transition from the bulk to the tails of the PDF and could be used in the investigation of rare fluctuations, for example, in chemical reactions, in optimal search strategy, etc.

In this direction, Bassanoni *et al.*<sup>23</sup> studied rare events in the extreme value statistics of Lévy flights, Lévy walks, and the Lévy–Lorentz gas by using the big-jump principle.<sup>97</sup> The Lévy–Lorentz gas, first introduced by Barkai *et al.*,<sup>98</sup> is a special kind of a CTRW such that the walker travels in a fixed disordered environment composed of scattered points on a line. The distance between two successive scattered points follows a PDF with a power-law tail, and upon each collision, the walker performs a flight, to the left or to the right with equal probability, to the nearest scattered point. They determined the analytical forms of the displacement PDF for all three considered processes. Particularly, for the Lévy–Lorentz gas, they showed that the topology of the disordered lattice along which the walker moves induces memory effects in its dynamics. Such models can be of interest for various processes either in chemistry for the estimation of reaction thresholds or in climatology for earthquake risk assessment, in finance for portfolio management, in ecology for the collective behavior of species, etc.

In the work by Teuerle,<sup>24</sup> a new model of Lévy walks with rests was introduced, which also shows an anomalous diffusive behavior. It is a coupled CTRW process such that every waiting time of the walker is a sum of two independent positive random variables with at least one of them strongly coupled with the respective jump. By asymptotic analysis, the author investigated the impact of these resting periods on the overall walker’s dynamics. Such resting events in Lévy walks could be of interest in modeling different phenomena in animal foraging and for optimal search strategies, or in finance in analysis of the income dynamics.

## B. Generalized stochastic and kinetic models

Let us consider a particle of mass  $m$  in a viscoelastic bath with correlated thermal noise  $\xi(t)$ . Moreover, the particle is placed under an external potential  $V(x)$ . The generalized Langevin equation

(GLE) for the particle can then be written as<sup>99,100</sup>

$$m\ddot{x}(t) + m \int_0^t \gamma(t-t')\dot{x}(t')dt' + \frac{dV(x)}{dx} = \xi(t). \quad (24)$$

Here,  $\gamma(t)$  is the generalized friction kernel, which correlates the velocity of the particle at different times. This can be understood as the non-Markovian response of the particles in the fluid (the “memory”). The generalized fluctuation–dissipation theorem for the GLE (24) is given by<sup>99,100</sup>

$$\langle \xi(t)\xi(t') \rangle = k_B T m \gamma(|t-t'|). \quad (25)$$

If this relation satisfies the noise,  $\xi(t)$  is often called an internal noise, while if it does not hold, it is “external noise.” Evidently, when  $\gamma(t-t') = 2\gamma\delta(t-t')$ , we recover the standard Langevin equation with a constant friction  $\gamma$ . In the case of a free particle  $V(x) = 0$  and for a power-law memory kernel  $\gamma(t) = \gamma_\alpha t^{\alpha-2} / \Gamma(\alpha-1)$ ,  $1 < \alpha < 2$ , the MSD in the long time limit becomes<sup>101</sup>

$$\langle x^2(t) \rangle \simeq t^{2-\alpha}, \quad (26)$$

which is subdiffusion. The case of a harmonic potential  $V(x) = m\omega^2 x^2 / 2$  and large friction force (or high damping) has been used in description of data related to movement within proteins,<sup>102,103</sup> due to the liquid environment of proteins, the friction forces are usually very high, while the movement is confined to a short range, which can be well approximated by harmonic potential.<sup>104</sup>

Closely related to the GLE is the fractional Brownian motion (FBM), a self-similar Gaussian process with stationary, power-law correlated increments, which was first introduced by Kolmogorov<sup>105</sup> in 1940. In 1968, Mandelbrot and Van Ness<sup>106</sup> introduced FBM in terms of a stochastic integral, which can also be given by the following Langevin equation:

$$\dot{x}(t) = \xi(t), \quad (27)$$

where  $\xi(t)$  is the “fractional Gaussian noise” (FGN). Such a model was used in the analysis of economic time series. FGN is a Gaussian noise with a power-law correlation

$$\langle \xi(t)\xi(t') \rangle \sim \alpha(\alpha-1)D_\alpha |t-t'|^{\alpha-2}, \quad (28)$$

which is “persistent” for  $1 < \alpha < 2$  and “antipersistent” for  $0 < \alpha < 1$ . Here,  $\alpha = 2H$ , and  $H$  is the Hurst exponent. FGN is an external noise; there is no fluctuation–dissipation relation as in the GLE. The corresponding PDF has the form

$$P(x, t) = \frac{1}{\sqrt{4\pi D_\alpha t^\alpha}} e^{-x^2 / (4D_\alpha t^\alpha)}, \quad (29)$$

and the MSD shows a power-law dependence on time,

$$\langle x^2(t) \rangle = 2D_\alpha t^\alpha. \quad (30)$$

Thus, FBM describes subdiffusion for  $0 < H < 1/2$  and superdiffusion for  $1/2 < H < 1$ . This behavior is the same as for the MSD (21) obtained from the CTRW with long-tailed waiting time PDF. However, both processes are different; the CTRW model is a non-stationary process, while FBM is Markovian in increments and Gaussian; see the PDF (29). Apart from transport phenomena, persistent FBM can also be used as a polymer model in the description of serotonergic brain fibers.<sup>107</sup>

In the work by Tóthová and Lisý,<sup>25</sup> the GLE for a tagged particle in a liquid of charged particles is studied under the influence of an external oscillating electric field. The emerging MSDs in the underdamped and overdamped regimes, obtained exactly in the case of a power-law memory kernel, exhibit anomalous diffusive behavior; the MSDs consist of a part that corresponds to the case without the external field and a part influenced by the external oscillating electric field. The authors observed hyper-ballistic diffusion, a behavior not known for the memoryless GLE. Such a model could be of interest in the investigation of anomalous diffusion in plasma or liquid electrolytes.

In the paper by Wozcsek *et al.*,<sup>26</sup> two stochastic processes of FBM with random Hurst exponents were considered. Both models show anomalous diffusion properties and long-range dependencies. One of the considered models, the so-called Riemann–Liouville FBM, is also a self-similar Gaussian process, but its increments are no longer stationary. The authors studied the second moment, the autocovariance function, the time-averaged MSD, and the ergodic properties of both processes. Such models are of interest for the analysis of data from biological experiments due to the fact that anomalous diffusion models with fixed Hurst exponents sometimes are inadequate for describing the observed phenomena.

Marinari and Oshanin<sup>27</sup> discuss discrete, in time and in space, FBM. The authors show that the corresponding discrete random walks have the same power-law covariance function as the standard FBM, and that the individual trajectories follow those of superdiffusive FBM. The obtained results are confirmed by extensive numerical simulations. The authors also pointed out that the problem of finding a discrete in space and time analog of a subdiffusive FBM is a challenging and still an open problem.

In the paper by Dybiec,<sup>28</sup> a different generalization of Brownian motion is studied. In the considered Langevin equation  $\dot{x} = -V'(x) + \eta(t)$  in the presence of an external potential  $V(x)$ , the noise  $\eta(t)$  is not a simple Gaussian white noise, but instead, it is an Ornstein–Uhlenbeck noise, that is, a noise that itself is a random process satisfying the Langevin equation for the Ornstein–Uhlenbeck process,  $\dot{\eta}(t) = -\rho\eta(t) + \rho\sqrt{2D}\xi(t)$ , where  $\xi(t)$  is Gaussian white noise. The author analyzed the transition from unimodality to bimodality of the stationary states induced by the Ornstein–Uhlenbeck noise in the presence of static, single-well, power-law potentials.

The standard diffusion process with a Gaussian PDF has one unphysical characteristic. There is a non-zero probability to find the particle at very long distances (at infinity) from the initial position at any time  $t$ , which means that the particles can move with arbitrary and infinite velocities. To avoid this property, a finite-velocity diffusion is introduced.<sup>108,109</sup> It can be modeled in different ways. One of them is by the Langevin equation  $\dot{x}(t) = \xi(t)$  with stationary dichotomic noise  $\xi(t)$  that switches between two values  $\pm 1$  with a mean rate  $\nu$  such that  $\tau = 1/(2\nu)$  and correlation<sup>110</sup>

$$\langle \xi(t)\xi(t') \rangle = \nu^2 e^{-|t-t'|/\tau}. \quad (31)$$

Here,  $\nu > 0$  is the particle’s velocity and  $D$  is the diffusion coefficient such that  $\nu^2 \tau = D$ . In the white-noise limit  $\tau = 0$ ,  $1/\tau \rightarrow \infty$ ,

$v \rightarrow \infty$ , the diffusion coefficient  $D = v^2\tau$  is finite. The corresponding equation for the PDF is the telegrapher's equation

$$\tau \frac{\partial^2}{\partial t^2} u(x, t) + \frac{\partial}{\partial t} u(x, t) = D \frac{\partial^2}{\partial x^2} u(x, t). \quad (32)$$

The telegrapher's motion is related to Lévy walks.<sup>111,112</sup> For exponentially distributed flight time  $\psi(t) = \frac{1}{\tau} e^{-t/\tau}$  in Eq. (23), one directly obtains the telegrapher's equation (32). The corresponding MSD of the telegrapher's process is

$$\langle x^2(t) \rangle = x_0^2 + 2D [\tau (e^{-t/\tau} - 1) + t], \quad (33)$$

which describe the characteristic crossover dynamics from ballistic motion  $\langle x^2(t) \rangle \simeq t^2$  in the short time limit to normal diffusion  $\langle x^2(t) \rangle \simeq t$  in the long time limit. The telegrapher's process has been further generalized to telegrapher's processes with memory, heterogeneous telegrapher's processes, etc., in order to capture the anomalous crossover dynamics. The telegrapher's processes are related to different generalization of the run-and-tumble motion,<sup>113,114</sup> which has been used in various fields.

In the paper by Dutta *et al.*,<sup>29</sup> the dynamics of a single run-and-tumble particle on a straight line is investigated, where inertial effects are taken into consideration. The authors showed, contrary to the passive run-and-tumble particle, that due to the interplay of the inertial and active timescales, different regimes characterized by calculation of the corresponding MSD and survival probability are observed. The results of this paper could be used in the description of diffusion in active systems, such as bacterial colonies or a flock of birds.

Li *et al.*<sup>30</sup> study the transition path properties of active run-and-tumble particles for a freely diffusing motion. The authors show that the forward and backward transition path times of run-and-tumble particles in the transition region can be reduced with increasing particle speed or by decreasing the tumble rate, and that there is no symmetry breaking of the transition path time and shape of the non-equilibrium system.

The telegrapher's process perturbed by a uniformly moving external harmonic source is investigated by Pietrzak *et al.*<sup>31</sup> The authors analyze the memory effects on the telegrapher's process and show in which case the solution represents traveling or standing waves. The effects of the frequency and velocity of external sources on the corresponding processes are analyzed in detail.

In the paper by Angelani *et al.*,<sup>32</sup> a generalized kinetic equation, which emerges in run-and-tumble models, is introduced. Such a model represents a generalization of the telegrapher's and fractional diffusion equations used to model anomalous diffusion. The authors find the conditions under which the solution of the kinetic equation represents a PDF and discuss the special cases of the fractional telegrapher's equation, higher order fractional diffusion equations, etc.

### III. NONERGODICITY, NON-GAUSSIANITY, AND AGING

Ergodicity in the sense of Boltzmann, Birkhoff, and Khinchin means equivalence of the ensemble average of a physical observable and its time average if the ensemble is large enough and the measurement time is sufficiently long. In single-particle tracking experiments, time series for the individual test particle position is

obtained, and the time-averaged MSD (TAMSD) is defined as<sup>65</sup>

$$\overline{\delta^2(\Delta)} = \frac{1}{t - \Delta} \int_0^{t-\Delta} [x(t' + \Delta) - x(t')]^2 dt', \quad (34)$$

where  $t$  is the measurement time and  $\Delta$  is the lag-time such that  $\Delta \ll t$ . In order to avoid fluctuations, one can average the TAMSD over sufficiently many trajectories,  $\langle \overline{\delta^2(\Delta)} \rangle$ . If  $\langle \overline{\delta^2(\Delta)} \rangle = \langle x^2(\Delta) \rangle$ , then the process is ergodic; otherwise, the process is non-ergodic. It is well known that for a Brownian motion,  $\langle \overline{\delta^2(\Delta)} \rangle = \langle x^2(\Delta) \rangle = 2D\Delta$ , while for the CTRW process, weak ergodicity breaking occurs.<sup>115,116</sup>

Another important aspect of non-ergodic processes is the aging property,<sup>65</sup> which represents the dependence of the physical observables, for example, the MSD, on the time span between the initial time when the process started and the time when the measurement is started, the aging time. Thus, if the observation time is much shorter than the measurement time, one may observe normal diffusion for a system showing otherwise anomalous diffusion. This is in the case of the CTRW process with power-law waiting time PDF since in such processes after a long waiting time period, there is a large probability that we have to wait a long time for the next jump to occur.<sup>81,117,118</sup>

The aging properties are investigated in the paper by Brevitt *et al.*<sup>33</sup> for a deterministic dynamical system, which can display intermittent dynamics and weak chaos, leading to anomalous diffusion. Such an example is the Pomeau–Manneville map, which was shown to generate superdiffusion, a process that can be reproduced by Lévy walks. The authors found quantitative deviations between deterministic diffusion and diffusion generated by the Lévy walks. The aging effects are also analyzed, and it was shown that there are changes in the diffusive properties by variation of the aging time.

In anomalous diffusion theory, an important quantity for the analysis of the type of stochastic process is the non-Gaussianity measure, which in  $d$  dimensions is<sup>65,119</sup>

$$G(\Delta) = \frac{d}{d+2} \frac{\langle \overline{\delta^4(\Delta)} \rangle}{\langle \overline{\delta^2(\Delta)} \rangle^2} - 1, \quad (35)$$

where the fourth moment is

$$\overline{\delta^4(\Delta)} = \frac{1}{t - \Delta} \int_0^{t-\Delta} [x(t' + \Delta) - x(t')]^4 dt'. \quad (36)$$

This measure is different from zero for non-Gaussian diffusion, while for Brownian motion, it equals zero,  $G(\Delta) = 0$ .

The non-Gaussianity analyzed in the paper by Marchenko *et al.*<sup>34</sup> for a Brownian particle in a spatially periodic potential under an external time-periodic force is analyzed by using the Langevin equation approach. The authors study the excess kurtosis measuring the Gaussianity of the particle displacement distribution. It was shown that the excess kurtosis shows a very rich behavior: it has a platykurtic form at the beginning, and then it takes a leptokurtic form before it decays to zero; i.e., it has a mesokurtic form. For such a process, in the long time limit, the diffusion is Brownian, yet non-Gaussian, a phenomenon that has attracted considerable interest recently.

#### IV. HETEROGENEOUS AND/OR BOUNDED SYSTEMS

Heterogeneous diffusion processes can be described by a Langevin equation with position-dependent diffusion coefficient<sup>65,120–123</sup>

$$\dot{x}(t) = \sqrt{2D(x)}\xi(t), \quad (37)$$

where  $\xi(t)$  is the white Gaussian noise with  $\langle \xi(t) \rangle = 0$ ,  $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$ . Equation (37) requires the choice of interpretation for the multiplicative noise, among which the Itô, Stratonovich, and Hänggi–Klimontovich schemes are the most frequently used.<sup>121</sup>

The corresponding Fokker–Planck equation reads<sup>121</sup>

$$\frac{\partial}{\partial t}P(x, t) = \frac{\partial}{\partial x} \left\{ D(x)^{1-A/2} \frac{\partial}{\partial x} [D(x)^{A/2}P(x, t)] \right\}, \quad (38)$$

where  $P(x, t)$  is the PDF and  $D(x) = D_\alpha|x|^\alpha$ , where  $\alpha < 2$  is the position-dependent diffusion coefficient. The case with  $A = 1$  corresponds to the Stratonovich interpretation of the heterogeneous diffusion,  $A = 0$  to the Hänggi–Klimontovich case, and  $A = 2$  to the Itô interpretation. For  $\alpha = 0$ , one obtains the standard diffusion equation with a constant diffusion coefficient. The corresponding mean squared displacement (MSD) for the heterogeneous diffusion process behaves as  $t^\beta$ ,  $\beta = 2/(2 - \alpha)$ , which has rich diffusion realizations. Specifically, for  $1 < \alpha < 2$ , one observes superdiffusion since  $\beta > 1$ ; for  $\alpha = 1$  corresponds to ballistic motion ( $\beta = 2$ ); for  $0 < \alpha < 1$ , one finds superdiffusion ( $1 < \beta < 2$ ); for  $\alpha = 0$ , normal diffusion ( $\beta = 1$ ); and for  $\alpha < 0$ , subdiffusion. The case with  $\alpha = 2$  leads to exponential dependence of the MSD on time, which is related to geometric Brownian motion (GBM) with log-normal PDF.<sup>124</sup>

In the paper by Sau Fa,<sup>35</sup> two population growth dynamics with different interactions are described in terms of coupled Langevin equations with position-dependent drifts and position-dependent diffusion coefficients. The corresponding two-dimensional Fokker–Planck equation is derived, and the corresponding  $n$ -moments are calculated exactly. An exact solution for the PDF for Gaussian white noises is found. It is shown that interactions can modify the behaviors of the population growth models; that is, the species may collaborate and/or compete. The considered model is a two-dimensional generalization of the heterogeneous diffusion equation (38).

A simple example of a heterogeneous diffusion process with a position-dependent diffusion coefficient is the comb model. The comb geometry is one of the most simple paradigms of a two-dimensional structure, in which anomalous dynamics can be realized. It is composed of a main channel along the  $x$ -direction, the so-called backbone, and side branches, called fingers of the comb along the  $y$ -direction. In such a two-dimensional structure, the diffusion along the  $x$  direction is possible only at  $y = 0$ , which means that the longitudinal diffusion coefficient for all other values of  $y$  is zero. Therefore, diffusion along the  $x$  direction is possible only along the backbone; thus, the particle's movement in the fingers can be considered trapping events for the backbone dynamics. The resulting diffusion along the backbone is anomalous with an anomalous diffusion exponent equal to  $1/2$ .<sup>125</sup> In the paper by Lenzi *et al.*,<sup>36</sup> a comb structure is considered in which the backbone is coupled to the bulk, and sorption–desorption events as well as reactions on

the surface are taken into consideration. Various anomalous diffusive behaviors are observed, which makes the model suitable for the description of surface diffusion in complex systems.

In the paper by Rosseto *et al.*,<sup>37</sup> the electrical impedance is investigated for an adsorption–desorption diffusion process on a surface in which memory effects and heterogeneity of the environment are taken into consideration. The memory is introduced via non-local operators, while the heterogeneity of the environment is represented by a position-dependent diffusion coefficient. It is shown that the resulting diffusion processes are anomalous by calculation of the MSDs.

Another example of heterogeneous diffusion processes is the diffusion of particles in confined geometries, such as tubes and channels with varying cross sections. Such motions can be mapped onto an effective one-dimensional Fick–Jacobs equation for a diffusing particle inside a three-dimensional tube with a cross-sectional area  $s(x)$ <sup>126–128</sup>

$$\frac{\partial P(x, t)}{\partial t} = D_0 \frac{\partial}{\partial x} \left\{ s(x) \frac{\partial}{\partial x} \left[ \frac{P(x, t)}{s(x)} \right] \right\}, \quad (39)$$

where  $D_0 > 0$ . There are different generalizations and corrections of the diffusion equation for diffusion in confined channels,<sup>129–133</sup> which can also have application in the description of the transport of particles in nanopores and gel networks.

González<sup>38</sup> studies the three-dimensional diffusive transport of particles through a double-cone channel by using the Fick–Jacobs equation (39). Stochastic resetting is introduced into the system, and exact analytical expressions for the unconditional first-passage density and the mean first-passage times in the channel are obtained. The author analyzes the MFPTs for different channels and showed that in the narrow–wide–narrow double-cone channel with absorbing boundaries, a discontinuous transition for the optimal resetting rates occurs, phenomena not observed in the wide–narrow–wide double-cone channel. The usefulness of resetting in the escape of the particle through the double-cone channel was investigated as well.

By using the Fick–Jacobs equation approach, in the paper by Cieřla *et al.*,<sup>39</sup> diffusion of spherical particles, passing through a conical pore restricted by absorbing and reflecting boundaries, is investigated as a function of particle size with respect to pore width. The authors analyze the MSD, the mean, and the median of the first-passage times. By numerical simulations, the authors show that the motion through the narrowing conical channel from its wider to its narrower end is effectively subdiffusive, while in the opposite case, it is superdiffusive.

Instead of position-dependent, one can consider a time-dependent diffusion coefficient in the Langevin equation, which is also known as scaled Brownian motion (SBM),<sup>134,135</sup>

$$\dot{x}(t) = \sqrt{2D(t)}\xi(t), \quad (40)$$

where  $\xi(t)$  is again white Gaussian noise. The corresponding diffusion equation has the form

$$\frac{\partial P(x, t)}{\partial t} = D(t) \frac{\partial^2 P(x, t)}{\partial x^2}. \quad (41)$$

For a power-law diffusion coefficient of the form  $D(t) = \alpha Dt^{\alpha-1}$ ,  $0 < \alpha < 2$ , the solution becomes

$$P(x, t) = \frac{1}{\sqrt{4\pi Dt^\alpha}} e^{-(x-x_0)^2/(4Dt^\alpha)}, \quad (42)$$

while the MSD reads  $\langle x^2(t) \rangle = 2Dt^\alpha$ , which means subdiffusion for  $0 < \alpha < 1$ , normal diffusion for  $\alpha = 1$ , and superdiffusion for  $1 < \alpha < 2$ . The diffusion equation with a time-dependent diffusion coefficient was proposed as an alternative to the Richardson diffusion equation for the description of superdiffusive turbulent dispersion. It is also used in the description of the evolution of granular systems,<sup>136,137</sup> in which due to dissipative collisions, part of the kinetic energy of the granular gases is lost.<sup>138</sup>

There are many other models of heterogeneous diffusion, for example, with position- and time-dependent diffusion coefficients. An annealed approach to a heterogeneous diffusive process is the so-called diffusing-diffusivity model with a fluctuating diffusion coefficient,<sup>139-141</sup> etc.

A diffusing-diffusivity model of Brownian motion with a fluctuating diffusion coefficient on the spherical surface is used by Wu *et al.*<sup>40</sup> to describe the diffusion dynamics of a polymer in contact with a monomer bath. Starting from the coupled Langevin equations, the corresponding Fokker-Planck and Feynman-Kac equations on the spherical surface are derived. The authors present Monte Carlo simulations to confirm the analytical results. The considered model could be of interest to model transport of macromolecules, such as proteins and polysaccharides, in living cells, which naturally have a curved structure.

In the paper by Zhang *et al.*,<sup>41</sup> diffusion in non-static media is investigated. A diffusing-diffusivity model for a polymer is employed as well. The MSD and kurtosis are analyzed via the subordination approach. Normal diffusion and anomalous diffusion are observed, as well as an exponential spreading of the MSD. The authors find that the Brownian yet non-Gaussian diffusion behavior of the center of mass of the polymer is related to the variation of the polymer size and the expansion/contraction of the medium.

SBM can be generalized to scaled Lévy motion (SLM) when the noise  $\xi(t)$  is a Lévy noise. Further generalization of the SBM and SLM is the power Lévy motion, recently introduced in the papers by Eliazar.<sup>42,43</sup> It is a Markov process; however, it displays anomalous diffusive behavior. The author shows that the power Lévy motion is an infinite-variance process with very rich diffusive behavior. Brownian motion, Lévy motion, SBM, SLM, and power Brownian motion are special cases of power Lévy motion.

## V. FIRST-PASSAGE PROCESSES, RANDOM SEARCH, AND STOCHASTIC RESETTING

For the case of Brownian motion, one may analyze the first-passage process by finding the first-passage time density for a searcher starting at  $x_0$  to pass a given target at position  $x = 0$ . The problem can be solved by finding the survival probability, the probability that the particle starting at  $x = x_0 > 0$  has not reached the target located at  $x = 0$  up to time  $t$ . For Brownian motion, one finds<sup>142</sup>

$$Q(x_0, t) = \int_0^{x_0} P_0(x, t) dx = \text{erf}\left(\frac{x_0}{\sqrt{4Dt}}\right), \quad (43)$$

where  $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy$  is the error function. Then, the first-passage time density,  $\wp_{\text{fp}}(t)$ , can be derived as the negative derivative of the survival probability in time,

$$\wp_{\text{fp}}(t) = -\frac{d}{dt} Q(x_0, t), \quad (44)$$

which for Brownian search becomes<sup>142</sup>

$$\wp_{\text{fp}}(t) = \frac{|x_0|}{\sqrt{4\pi Dt^3}} e^{-x_0^2/(4Dt)}. \quad (45)$$

This is the Lévy-Smirnov distribution with power-law decay  $\wp_{\text{fp}} \simeq t^{-3/2}$  in the long time limit. From the survival probability, one can calculate the mean first-passage time (MFPT),  $\langle T \rangle = -\int_0^\infty t \frac{\partial Q(x_0, t)}{\partial t} dt$ , which for Brownian motion in the semi-infinite domain is infinite.

Similar to the first-passage problem is the first-arrival problem. We consider one-dimensional Brownian search described by the standard diffusion equation with a  $\delta$ -sink of strength  $\wp_{\text{fa}}(t)$ ,<sup>143,144</sup>

$$\frac{\partial}{\partial t} P(x, t) = D \frac{\partial^2}{\partial x^2} P(x, t) - \wp_{\text{fa}}(t) \delta(x), \quad (46)$$

where the initial condition is  $P(x, t = 0) = \delta(x)$ . In this equation, the  $\delta$ -sink means that the random searcher positioned at  $x = x_0$  at the beginning will be removed when it arrives (for the first time) at  $x = 0$ ; that is,  $P(x = 0, t) = 0$ . Thus,  $\wp_{\text{fa}}(t)$  represents the first-arrival time density (FATD). Integrating Eq. (46) with respect to  $x$  from  $-\infty$  to  $\infty$ , we obtain the expression of FATD, which reads

$$\wp_{\text{fa}}(t) = -\frac{d}{dt} \int_{-\infty}^\infty P(x, t) dx = -\frac{d}{dt} S(t). \quad (47)$$

It is the negative time derivative of the survival probability  $S(t) = \int_{-\infty}^\infty P(x, t) dx$ . The FATD then becomes

$$\wp_{\text{fa}}(t) = \mathcal{L}^{-1} \left[ e^{-s^{1/2}|x_0|/\sqrt{D}} \right] = \frac{|x_0|}{\sqrt{4\pi Dt^3}} e^{-x_0^2/(4Dt)}, \quad (48)$$

which coincides with the first-passage time density (45) since the particle does not perform jump-like motion and the first-passage time will be the same as the first-arrival/hitting time; see also Ref. 145.

This situation changes if one considers Lévy flight search. In such a case, the first-passage time will differ from the first hitting time since the searcher may overshoot the target due to the long jumps. The FPTD for Lévy flights behaves as  $\wp_{\text{fp}}(t) \simeq t^{-3/2}$ , while the FATD as  $\wp_{\text{fa}}(t) \simeq t^{-2+1/\beta}$ , where  $1 < \beta < 2$  is the Lévy index.

In search processes, it is important that the MFPT for the searcher to find a given target is finite and optimal. Since, as we mentioned before, the MFPT of a Brownian searcher to hit the target at the origin is infinite, one should introduce some mechanism, which makes the MFPT finite. Such a mechanism is stochastic resetting, a process that interrupts the particle motion after a random time interval  $\tau$ , and the particle is reset to its initial position. The simplest mechanism of stochastic resetting is Poissonian resetting with resetting time PDF  $p(\tau) = r e^{-r\tau}$ . The PDF  $P_r(x, t|x_0)$  of the diffusion process with resetting can be expressed via the PDF  $P(x, t)$  of

the diffusion process in the absence of resetting by using the renewal equation<sup>146</sup>

$$P_r(x, t|x_0) = e^{-rt}P(x, t) + \int_0^t r e^{-r\tau} P(x, \tau) d\tau. \quad (49)$$

That is, resetting events to the initial position  $x_0 > 0$  renews the process at a rate  $r$ . Between two consecutive renewal events, the particle undergoes free diffusion. The first term on the right-hand side of the equation corresponds to the case when there is no resetting event up to time  $t$ , while the second term describes multiple resetting events. Taking the Laplace transformation of Eq. (49), one concludes that the PDF of the process with resetting can be calculated from the PDF in the absence of resetting in Laplace space via

$$\hat{P}_r(x, s|x_0) = \frac{s+r}{s} \hat{P}(x, s+r), \quad (50)$$

where  $\hat{P}(x, s)$  is the Laplace transform of the PDF. Therefore, for simple Brownian motion (normal diffusion), which has the Gaussian PDF (7), under stochastic resetting in the long time limit, the particle approaches a non-equilibrium stationary state (NESS), which is a Laplacian distribution,<sup>146</sup>

$$P_r^{st}(x) = \lim_{t \rightarrow \infty} P_r(x, t|x_0) = \frac{\alpha_0}{2} e^{-\alpha_0|x-x_0|}. \quad (51)$$

Here,  $\alpha_0 = \sqrt{r/D}$  is the inverse length scale, i.e., the typical distance diffused by the particle between resets. Thus, in the presence of resetting, the MSD is given by

$$\langle x^2(t) \rangle = x_0^2 + \frac{2D}{r} (1 - e^{-rt}) \sim \begin{cases} x_0^2 + 2Dt & \text{for } t \ll 1/r, \\ x_0^2 + 2D/r & \text{for } t \gg 1/r. \end{cases} \quad (52)$$

The transition to the NESS is far from trivial, as observed for the simple Brownian motion under resetting, where a second order dynamical phase transition occurs.<sup>146</sup> This transition to the NESS can be analyzed by employing large deviation theory. The renewal equation approach can be used in analysis of various (anomalous) diffusion processes under resetting.<sup>147</sup>

The topic of stochastic processes under resetting has been considered one of the most popular topics in non-equilibrium statistical physics in the last 15 years. The influence of stochastic resetting on particle dynamics has been investigated in different diffusion models, and various resetting mechanisms have been introduced.

In the paper by Radice *et al.*,<sup>44</sup> stochastic resetting for jump processes, such as Lévy flights, is studied. The authors are interested in the minimization of the mean number of jumps required to cross a given barrier or threshold, and in the optimal resetting strategy, which makes the considered model potentially applicable to designing experiments with an optimal random search protocol in biology, finance, and ecology.

Related to the random search problem, in the paper by Singh *et al.*,<sup>45</sup> a pair of independent searchers, which compete for a target under resetting, is considered. The authors find that restarts of searchers tend to enhance the search efficiency of an already efficient searcher, a result independent of the identity of individual searchers or the specific details of the distribution of restart times. When a subsystem restarts, i.e., when only one of a pair of searchers is subject to restarts while the other evolves in an unperturbed manner, the

authors observed that the search probability exhibits a non-monotonic dependence on the restart rate. The results are demonstrated for a pair of run-and-tumble and for Brownian searchers.

Biswas *et al.*<sup>46</sup> consider a GLE, describing the transport of particles in inhomogeneous or viscoelastic environments, under stochastic resetting, by using the renewal formalism. The authors show that resetting induces a stationary state in the system and also allows one to harness the timescales arising from the associated memory effect. The general results for the position variance and the correlation function are applied to the Jeffreys viscoelastic fluid model in which various timescales and intermittent plateaus in the transient phase before the system relaxes to the steady state are observed.

In the paper by Di Bello *et al.*,<sup>47</sup> a symmetric  $\alpha$ -stable Lévy process and a Brownian motion in  $d$ -dimensions under partial resetting<sup>148–150</sup> are considered. Partial resetting means that at random moments in time, a stochastic process is multiplied by a factor between 0 and 1, thus approaching but not reaching the resetting position; i.e., the instantaneous jump in the stochastic process is proportional to the current value of the process. The authors showed that, while approaching a NESS, a dynamical phase transition is observed for Brownian motion, but not for Lévy flights. Such models can be used for modeling income dynamics in different insurance models and in population dynamics sedimentation.

## VI. RANDOM WALKS ON COMPLEX NETWORKS

Random walks in complex networks have been intensively studied in the past years<sup>151,152</sup> due to their wide range of applications in science and engineering. A fundamental quantity related to random walks is the above-mentioned first-passage time,<sup>153</sup> which is the expected time required for a random walker starting from a source point to a given target point. The MFPT is defined as the average of first-passage times over all source nodes in the network, which is a useful tool to analyze the behavior of random walks, including target search. Analysis of the first-passage properties on networks is of great interest in studying animal foraging, population dynamics, epidemic spreading, etc. There are different generalizations of random walks on complex networks in which fractional dynamics,<sup>154</sup> super-fast spreading,<sup>155</sup> stochastic resetting,<sup>156–159</sup> as well as quantum dynamics<sup>160</sup> were considered.

In the paper by Yuan *et al.*,<sup>48</sup> the first-passage properties of a biased random walk on bundled networks, obtained by a base and fiber structure, are investigated. The authors calculate the MFPT, the mean-trapping time, the global-MFPT, as well as the stationary distribution. Since the global-MFPT is an indicator for how efficiently network transport occurs, the results in the paper could contribute to designing network structures with preferred MFPTs.

Gao *et al.*<sup>49</sup> study the transport search efficiency of a biased random walk on a class of scale-free trees. The authors analyze the global-MFPT and find the interval of system parameters for which the transport efficiency can be improved. These results can be of interest in finding the optimal search strategy on scale-free networks.

Contrary to Markovian random walks, non-Markovian random walks on networks are rarely studied. In the paper by Guerrero-Estrada,<sup>50</sup> a random walk model with long-range memory on arbitrary finite networks is investigated. Different network

and graph structures are analyzed, such as rings, complete graphs, Watts–Strogatz, and Barabási–Albert networks, as well as Barbell and comb-like graphs. The authors show that the stationary state is the same as in the absence of memory, while the relaxation toward the stationary state is anomalous in the form of an inverse power-law.

Another interesting topic for research is the influence of stochastic resetting in the efficiency of the random search on complex networks. In the paper by Michelitsch *et al.*,<sup>51</sup> Markovian and non-Markovian random walks on Watts–Strogatz and Barabási–Albert random graphs under stochastic resetting are considered. The ergodic properties of the system are also analyzed, and the MFPTs are calculated for different resetting mechanisms.

In the paper by Li *et al.*,<sup>52</sup> a random walk with stochastic resetting on scale-free flowers is investigated. The authors find a relation between the MFPTs in the presence and absence of resetting, and identified the optimal resetting probability, which minimizes the MFPTs and improves the search efficiency. This study is also a valuable work in optimizing the search strategy and opens new questions related to random walk processes on complex networks.

## VII. APPLICATIONS IN PHYSICS, BIOLOGY, CHEMISTRY, AND ECONOMY

The investigation of anomalous dynamics and fluctuations in complex systems and networks is of high interest in many fields. Their applications are manifold: starting from physics for the description of charge carrier motion in amorphous semiconductors,<sup>161</sup> porous material<sup>162</sup> and in surface transport at solid–liquid interfaces,<sup>163</sup> in geophysics for a description of anomalous transport in a fractured porous rock,<sup>164–167</sup> in biology for the analysis of passive motion of lipid granules or telomeres in living cells,<sup>168</sup> in gene expression<sup>169</sup> and movement ecology,<sup>170–172</sup> in chemistry for the investigation of barrier crossing in chemical reactions,<sup>173</sup> in economy to describe the income dynamics,<sup>174</sup> to name a few. Some further examples of applications of such models are presented below.

Singh and Granek<sup>53</sup> investigate the subdiffusive motion of chromosomal loci in cells and provide an explanation why the anomalous diffusion exponents in both normal and ATP-depleted cells are identical. Due to the fractal nature of chromatin, the authors study the dynamics of elastic fractal networks influenced by active forces (monopolar and dipolar) by calculation of the MSD of a network bead, both analytically and by extensive numerical simulations. The authors show that contrary to force monopoles, force dipoles do not induce superdiffusion–subdiffusion crossover dynamics, but instead in the long time limit, the MSD saturates. The obtained results are applied to the motion of chromosomal loci in bacteria and yeast cell chromatin, and it is shown that the interplay of thermal, monopolar, and dipolar forces in chromatin is typically dominated by the active monopolar and thermal forces.

In the paper by Polanowski and Sikorski,<sup>54</sup> a two-dimensional coarse-grained model of anomalous motion of liquid molecules in the presence of obstacles in crowded heterogeneous environments is studied when the movement of liquid molecules is highly correlated. The diffusion problem is investigated as a combination of bond and site percolation. By computer simulations, the authors show that, for

the same obstacle concentration, the liquid molecules in the model with blocked sites are much more mobile, and that the dynamics of the system are related to the morphology of the system. The present work could be of interest in the investigation of molecular transport in disordered systems.

In the paper by Maryshev and Klimenko,<sup>55</sup> a two-dimensional periodic flow of a viscous fluid under an external driving force is investigated. The authors report a characteristic subdiffusion- to normal diffusion-crossover dynamics for passive tracers in a plain periodical flow induced by Gaussian noise, and that on an intermediate timescale, subdiffusion is also observed. A periodic pattern of asymmetric vortices arises when the amplitude of the external force exceeds a critical value. This work is of interest for the analysis of anomalous diffusive transport in complex and turbulent flows.

Sibatov *et al.*<sup>56</sup> introduce a fractional generalization of the leaky integrate-and-fire model, which can be used for modeling neuronal dynamics in spiking networks. The authors evaluate the statistical properties of such a generalized model to a flickering input voltage pulse flow, which is characterized by a fractional Poisson process, and develop a microscale transistor using a network of single-walled carbon nanotubes with an electrolyte gate. As a result, the corresponding system exhibits anomalous dynamics.

In the paper by Nandi *et al.*,<sup>57</sup> noise propagation in a biochemical network is investigated, where due to the inherent randomness in gene expression and fluctuations of the environment, intrinsic and extrinsic noise occur. The authors analyze the influence of noise on information transmission in a cascaded network and its optimization. Knowing how the interplay between intrinsic and extrinsic noise changes the network ability to adapt and maintain robustness in cellular response, the present work will be of interest in the study of biological signal transduction.

Ageeva and Goldobin<sup>58</sup> analyze the effect of asymmetric Cauchy noise on the macroscopic dynamics of sin-coupled oscillators. The Cauchy noise is a special case of a Lévy noise when the Lévy index is equal to 1. The authors study the effect of individual oscillator frequencies entrainment by global oscillations. The analysis performed in this work could be of interest for neuronal populations, in which asymmetric Cauchy noise can be important.

In the paper by Sales *et al.*,<sup>59</sup> the survival probability for a billiard system with a hole on its boundary is studied for different positions of the hole. The authors show that the survival probability follows an exponential time dependence when the hole is positioned partially or entirely over large stability islands in phase space, while it follows a stretched exponential decay for holes placed in predominantly chaotic regions. Moreover, when the hole is placed partially or entirely over a stability island, the survival probability exhibits scaling invariance with respect to the size of the exit.

In the paper by Wand *et al.*,<sup>60</sup> an extension of an agent-based insurance model is considered, that is, a multiplicative stochastic gamble with reward  $r$  and cost  $c$ . It is shown that in the long run, all agents have a profit from cooperating through insurance. By variation of the reward and cost and allowing cooperation between nearest neighbors in the lattice network, it is shown that the model has a tendency to keep rich agents rich and poor agents poor; i.e., wealth inequalities persist for a long duration. The present study could be of interest in the analysis of many different socio-economic phenomena,

such as opinion dynamics and optimal human decision making.

### VIII. DATA ASSIMILATION TECHNIQUES FOR (ANOMALOUS) DIFFUSION AND FLUCTUATIONS

As it was already presented in Secs. I–VII, there are a number of different approaches to anomalous diffusion. Many different models, which describe different physical processes, frequently give the same or similar results for the MSD. Therefore, it is important to identify the corresponding stochastic process from measurements of individual trajectories and determine the anomalous diffusion exponent and/or the diffusion coefficient. More precisely, these tasks correspond to the classification of the process and the regression of the process parameters. Typically, this has been done using statistical observables and quantification of the differences between the models, which can be a challenging task due to the stochastic nature of the models, the limited and noisy experimental data, the computational costs, etc. In this direction, different approaches have been proposed and developed, starting from the classical approaches, such as  $p$ -variation tests<sup>175</sup> and the method of single trajectory power spectral density,<sup>176</sup> to the modern approaches, such as Bayesian inference,<sup>177</sup> Bayesian deep learning techniques,<sup>178</sup> a stochastic weight averaging Gaussian technique,<sup>179,180</sup> and other machine-learning techniques.<sup>181–187</sup> Due to the huge interest in the characterization of anomalous diffusive processes, the so-called Anomalous Diffusion (AnDi) challenge has been organized,<sup>188,189</sup> and it has been shown that the machine-learning-based approaches achieved superior performance.

By using the sample autocovariance function and the empirical anomaly measure, Grzesiek *et al.*<sup>61</sup> discuss the problem of distinguishing between two models, FBM with a constant and random Hurst exponents. The authors introduce a testing procedure to differentiate between the two models and apply the considered methodology in two different real-world datasets: (i) financial market for analysis of the time series corresponding to the daily data of a Bitcoin closing price and (ii) single particle tracking data from biological experiments.

In the paper by Balcerek *et al.*,<sup>62</sup> a model of correlated two-dimensional Brownian motion is studied by analysis of various statistical properties, such as velocity autocorrelation function, MSD, and turning angles generated by consecutive increments. The theoretical results are supported by numerical simulations and are applied to different real-world datasets: (i) financial data of the main US market indices, the Dow Jones Industrial Average, and the Standard and Poor's 500, and (ii) data of two-dimensional trajectories of polystyrene beads in water. Additionally, the authors show that the considered model can also be extended to trajectories with correlations that change over time, which are relevant for diffusion processes with temporal or spatial heterogeneity.

### IX. ANOMALOUS DYNAMICS IN QUANTUM SYSTEMS

After establishing anomalous dynamics in classical systems and their theoretical modeling, similar approaches have more recently been considered in quantum systems. Starting from Lévy quantum processes described by the space fractional Schrödinger

equation,<sup>190</sup> the time fractional Schrödinger equation,<sup>191</sup> which takes into account memory effects caused by the environment, was also introduced. Time fractional Schrödinger equations have been used to model real physical systems, which belong to a large class of non-Markovian, non-unitary quantum dynamics.

As an example, a time fractional Schrödinger equation has been derived considering quantum motion in two-dimensional comb-like structures. The anomalous dynamics is observed as a result of a projection of the two-dimensional quantum motion in comb to the one-dimensional configuration space. The anomalous character of the corresponding process can be described by the fact that the particle, which moves along the main backbone channel, can get trapped in the fingers of the comb. In the paper by Iomin,<sup>63</sup> such quantum motion in a two-dimensional comb structure is considered. Analytical solutions for the Green functions are obtained for both conservative and periodically driven in time Hamiltonian systems. The subordination approach [see Eq. (16)], which reflects the Lévy stable processes in quantum systems,<sup>192</sup> is applied for the analysis of the Green functions.

In the paper by Vertessen *et al.*,<sup>64</sup> a generalization of the Caldeira–Leggett model is considered for analyzing quantum diffusion. The authors investigate the asymptotic behavior of the MSD and observed various anomalous diffusive regimes, such as ballistic, sub-ballistic, and super-ballistic behavior for short times, as well as saturation, subdiffusion, and superdiffusion for long times.

### X. SUMMARY AND FUTURE DIRECTIONS

The field of anomalous dynamics in complex systems is a very active area of research due to its emergence in a multitude of systems. As collected in this Focus Issue, various theoretical models and concepts are being developed to capture the anomalous character of stochastic processes in disordered media. Furthermore, the fast development of machine-learning and deep learning techniques ensures the correct decoding of anomalous diffusion trajectories obtained in different experimental and molecular dynamics setups.

Future studies will be related to the further development of optimal search protocols, including non-Markovian random searches on networks. Generalization of the existing models in higher dimensions is still rarely explored. The mechanism of stochastic resetting in the investigation of different non-Markovian search processes and the thermodynamic cost of the resetting is less explored as well, and new experimental setups are needed for confirmation of the theoretical results. The study of non-Markovian dynamics in open quantum systems is a topic of potential interest for future investigations. The further development of machine-learning techniques to decode anomalous dynamics will be a central topic as well.

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## AUTHOR DECLARATIONS

## Conflict of Interest

The authors have no conflicts to disclose.

## DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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