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PAPER

Exactly solvable diffusions from space-time transformations

Costantino Di Bello[1], Édgar Roldán[2] and Ralf Metzler[1,3,*]

[1] University of Potsdam, Institute of Physics & Astronomy, 14476 Potsdam, Germany
[2] ICTP—The Abdus Salam International Centre for Theoretical Physics, 34151 Trieste, Italy
[3] Asia Pacific Centre for Theoretical Physics, Pohang 37673, Republic of Korea
[*] Author to whom any correspondence should be addressed.

E-mail: rmetzler@uni-potsdam.de, costantinodibello@hotmail.it and edgar@ictp.it

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Abstract

We consider a general one-dimensional overdamped diffusion model described by the Itô stochastic differential equation (SDE) $\mathrm{d}X_t = \mu(X_t,t)\mathrm{d}t + \sigma(X_t,t)\mathrm{d}W_t$, where W_t is the standard Wiener process. We obtain a specific condition that μ and σ must fulfil in order to be able to solve the SDE via mapping the generic process, using a suitable space-time transformation, onto the simpler Wiener process. By taking advantage of this transformation, we obtain the propagator in the case of open, reflecting, and absorbing *time-dependent* boundary conditions for a large class of diffusion processes. In particular, this allows us to derive the first-passage time statistics of such a large class of models, some of which were so far unknown. While our results are valid for a wide range of non-autonomous, non-linear and non-homogeneous processes, we illustrate applications in stochastic thermodynamics by focusing on the propagator and the first-passage-time statistics of isoentropic processes that were previously realised in the laboratory by Brownian particles trapped with optical tweezers.

1. Introduction

The effort to understand the fundamental nature of reality is a timeless pursuit that has fascinated humanity for centuries. One of the first proofs in support of the atomistic theory of matter is the famous paper by Einstein on Brownian motion [1]. Contemporarily, Sutherland [2], Smoluchowski [3], and Langevin [4] all contributed towards the physical theory of Brownian motion. The successive formalisation by Wiener [5] established the foundations of non-equilibrium statistical mechanics. The partial differential equations describing stochastic motion were developed independently by Fokker and Planck [6, 7] and by Kolmogorov [8] (see also [9, 10]). Indeed, in modern literature they either go under the name of *forward and backward Fokker–Planck* equations, or *first and second Kolmogorov* equations. An alternative mathematical formulation of diffusion processes is via stochastic differential equations (SDEs), firstly developed by Itô [11]. Nevertheless, stochastic processes have been applied not only to the Brownian motion but have found numerous applications in various fields. For instance, to study the motion of passive molecules [12–15] and actively transported particles [16, 17] in biological cells, lipids in membranes [18, 19], animal motion [20], active matter [21], geophysics [22], condensed matter [23], financial markets [24], or disease spreading [25]. One of the most significant random variables for stochastic models is the first-passage time (FPT), a positive random variable defined as the first time at which a stochastic walker reaches a threshold or exits from a certain region of space [26, 27]. To calculate the full distribution of FPTs even in simple geometries is a formidable task, that can be solved by, e.g methods such as Newton series, spectral methods or self-consistent approaches [28–30]. One of the most prominent examples is the Kramers problem [31, 32], widely used to model the activation rate in chemical reactions. Applications of the FPT are extensive and a complete literature report would be a formidable task. As a few examples, we mention applications to animal foraging [33], to model the disease spreading of infections [34, 35], or in finance [36, 37] where information regarding the FPT is fundamental to determine actions such as buying or selling. The FPT also provides valuable information regarding the extreme values of random processes [38] and on observables in non-equilibrium statistical physics [39]. A recent survey on applications is available here [40].

In this paper, we address the problem of solving stochastic differential equations (SDEs) (or Fokker–Planck equations (FPEs)) and FPT problems via a space-time coordinate transformation. Further details on the physical interpretation of this transformation are presented in section 3. Nevertheless, as a similar idea is roughly outlined in some old works, we first provide a literature overview of these references. Probably the first author to have the idea of space-time transformation was Kolmogorov [8]. In section 17 of his seminal work, Kolmogorov discussed a couple of examples of FPEs that can be solved by a change of variables, i.e. transforming $x \to x'$ and $t \to t'$. Some 25 years later, Cherkasov [41], another Russian mathematician, following the same idea of variable change, discovered a sufficient and necessary condition to map a generic FPE onto the FPE of the Wiener process. Nevertheless, Cherkasov's work showed an error in the proof, which was later corrected by Shirokov [42]. Interestingly, the paper [41] was nearly forgotten. As Ricciardi, an Italian mathematician, wrote in his paper [43] '*apparently Cherkasov's work did not receive much attention in the Western world*'. In [43] the author basically reproduces the results of [41], yet with a clearer notation. Later on, Ricciardi and collaborators [44] mentioned, without proof, that a similar mapping technique could be adapted to solve FPT problems. A similar idea can be found in [45]. We credit the authors of [44] for their intuition, although, to the authors' knowledge, this idea has never been used by any author to solve FPT problems.

The aim of our paper is to both present some of the results of the aforementioned papers in a simpler way and both to further develop some of the ideas and to obtain the FPT density (FPTD) analytically for a large class of diffusion models. Therefore, throughout the text, we will expose a mixture of known and novel results. We hope that this paper will serve as a key reference to the physics community for these valuable yet arguably forgotten transformation techniques. As a particular example of application, we discuss (i) the isoentropic protocol, an important diffusion process for stochastic thermodynamics and related experiments [46, 47]; and (ii) the stochastic Gompertz model, used in population dynamics (see [48] for a review). The paper is organised as follows: in section 2 we outline, in four different subsections, the main results of the paper. In sections 3 and 4 we provide an intuitive explanation of our technique, respectively for the propagator and the FPT. In section 5 we focus on specific applications of our theory, and section 6 details some connections with existing results. Our conclusions are drawn in section 7.

2. Summary of the main results

2.1. Solution of the SDE and propagator

We start by defining the problem and the most important quantities needed for presenting the results. We consider generic diffusion processes described by the one-dimensional Itô SDE [49]

$$\mathrm{d}X_t = \mu\left(X_t, t\right)\mathrm{d}t + \sigma\left(X_t, t\right)\mathrm{d}W_t, \tag{1}$$

with initial condition $X_{t_0} = x_0$, where W_t is the standard Wiener process.

Let us introduce the following function, which will be crucial later on,

$$\mathcal{C}\left(x,t\right) \equiv \frac{1}{\sigma\left(x,t\right)}\frac{\partial\sigma\left(x,t\right)}{\partial t} + \sigma\left(x,t\right)\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial\sigma\left(x,t\right)}{\partial x} - \frac{\mu\left(x,t\right)}{\sigma\left(x,t\right)}\right). \tag{2}$$

Here and below, we call $\mathcal{C}(x,t)$ the *Cherkasov function*. Notice that $\mathcal{C}(t)$ has physical dimensions of inverse time. Although the function $\mathcal{C}$ never appears in the works of Cherkasov, we decided to use the notion 'Cherkasov function' in Cherkasov's honour. As we show later in appendix A, if $\mathcal{C}(x,t) \equiv \mathcal{C}(t)$ is solely a function of time, i.e.

$$\frac{\partial\mathcal{C}\left(x,t\right)}{\partial x} = 0, \tag{3}$$

then it is possible to obtain an exact analytical solution of the SDE (1) and an exact expression for the propagator of the process. More precisely, when equation (3) holds, then it is possible to find two real *deterministic* functions $\psi(x,t)$ (equation (8)) and $\tau(t)$ (equation (9)), both invertible, that enable one to solve explicitly the SDE (1) as

$$X_t = \psi^{-1}\left(W_{\tau(t)}, t\right); \tag{4}$$

in other words, $W_{\tau(t)} = \psi(X_t, t)$, i.e. the time reparametrisation of the Wiener process equals the function ψ evaluated along the process X_t. Note that the time change $t \to \tau$ is not a random-time transformation but a deterministic one-to-one mapping, yet $W_{\tau(t)}$ is a martingale [50]. Equivalently to (1), we can say that the propagator of the process, $P(x,t|x_0,t_0)\mathrm{d}x \equiv \mathrm{Prob}\{x < X_t = x + \mathrm{d}x | X_{t_0} = x_0\}$ fulfils the FPE

$$\frac{\partial P(x,t|x_0,t_0)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,t)P(x,t|x_0,t_0)\right) - \frac{\partial}{\partial x}\left(\mu(x,t)P(x,t|x_0,t_0)\right), \tag{5}$$

with delta initial condition $\lim_{t\to t_0} P(x,t|x_0,t_0) = \delta(x-x_0)$ and with some appropriate boundary conditions (BC). We remind the reader that the probability flux associated with the FPE (5) obeying $\partial_t P(x,t|x_0,t_0) = -\partial_x j(x,t|x_0,0)$ is defined as

$$j(x,t|x_0,t_0) \equiv \mu(x,t)P(x,t|x_0,t_0) - \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2(x,t)P(x,t|x_0,t_0)\right). \tag{6}$$

Property (4) implies that the propagator of the process for natural (open) boundary conditions ($\lim_{|x|\to\infty} P(x,t|x_0,t_0) = 0$) reads

$$P(x,t|x_0,t_0) = \frac{\partial\psi(x,t|x_0,t_0)}{\partial x}\frac{1}{\sqrt{2\pi\tau(t|t_0)}}\exp\left(-\frac{\psi^2(x,t|x_0,t_0)}{2\tau(t|t_0)}\right). \tag{7}$$

The functions $\psi(x,t|x_0,t_0)$ and $\tau(t|t_0)$, which have dimensions of square root of time and time, respectively, are given by

$$\psi(x,t|x_0,t_0) \equiv \exp\left(\int_{t_0}^{t}\mathcal{C}(s)\,\mathrm{d}s\right)\int_{x_0}^{x}\frac{1}{\sigma(x',t)}\mathrm{d}x' + \int_{t_0}^{t}\mathrm{d}s\left(\frac{1}{2}\frac{\partial\sigma}{\partial x}\bigg|_{(x_0,s)} - \frac{\mu(x_0,s)}{\sigma(x_0,s)}\right)\exp\left(\int_{t_0}^{s}\mathcal{C}(s')\,\mathrm{d}s'\right) \tag{8}$$

and

$$\tau(t|t_0) \equiv \int_{t_0}^{t}\exp\left(2\int_{t_0}^{s}\mathcal{C}(s')\,\mathrm{d}s'\right)\mathrm{d}s. \tag{9}$$

Both functions ψ and τ depend parametrically on x and t and their initial conditions x_0 and t_0 (through the functions μ, σ, as well as $\mathcal{C}$ given by equation (2)); however, we will omit this dependence in the notation throughout the manuscript whenever there is no risk of confusion. The proof of formulae (4), (7)–(9) is shown in appendix A; they are the first main result of the paper. We obtained formulae (7)–(9) with a significantly simpler proof as compared to the original work [41], while (4) is a novel result of this paper. Condition (3) was also found in [45]. It remains unclear, to the authors' knowledge (and an interesting avenue for future research), whether condition (3) can be understood from intuitive physical arguments.

The class of Wiener transformable SDEs and their propagators is constrained by fulfilling equation (3). Nonetheless, there are examples of SDEs that, despite not fulfilling (3) are solvable analytically [51–54]. We noticed that the cases discussed in the aforementioned papers are equivalent to the so-called inhomogeneous geometric Brownian motion (IGBM) (see also [55]), which is discussed in section 2.4. Moreover, within the class defined by (3), not all the SDEs are amenable to analytical solution for their FPTD. In the next section we outline for which problems we can obtain exact closed forms for the FPTD.

2.2. FPTD

Following the discussion of the previous subsection, consider a generic stochastic process described by equation (1) starting at $X_{t_0} = x_0$ and with the boundary $a(t)$ with $a(t_0) > x_0$, we define the random variable FPT $\mathcal{T}_X$ as

$$\mathcal{T}_X(a(t)|x_0,t_0) \equiv \inf\{t > 0 : X_t > a(t) | X_{t_0} = x_0\}. \tag{10}$$

In simple words this represents the first time that the stochastic process X_t, which started below $a(t)$, crosses the boundary. We specify that the boundary $a(t)$ may be any time-dependent yet deterministic function of time t. To avoid problems in this definition (10) we further assume that $a(t)$ is a continuous function. The definition in the case $a(t_0) < x_0$ is analogous. The analytical result we obtained on the FPTD, besides expression (3), further requires another condition, namely, that the FPT of the transformed process W_τ to reach the transformed threshold $\psi(a(t),t)$ is analytically solvable. An example class for which this is possible is when

$$\psi(a(t),t) = \nu\tau(t) + a_0, \tag{11}$$

with $v, a_0 \in \mathbb{R}$ being constants, that is, ψ evaluated at the boundary depends linearly on τ. We denote by $\wp(a(t), t|x_0, t_0)\mathrm{d}t \equiv \mathrm{Prob}\{t < \mathcal{T}_X(a(t)|x_0, t_0) < t + \mathrm{d}t\}$ the FPTD. Assuming that (3) and (11) hold, it reads

$$\wp(a(t), t|x_0, t_0) = \frac{|a_0|}{\sqrt{2\pi\tau(t)^3}} \exp\left(-\frac{(a_0 + v\tau(t))^2}{2\tau(t)}\right)\left(\frac{\mathrm{d}\tau(t)}{\mathrm{d}t}\right). \tag{12}$$

Formulae (11) and (12) are the second main result of the paper. We stress the fact that formula (12) is a generalisation of several results already known in the literature, such as those reported in [56, 57]. Further results for absorbing and reflecting boundaries are given in the next subsection.

2.3. Propagators for absorbing and reflecting boundary conditions

If the two conditions (3) and (11) are met, further results are available for the two cases with (i) *absorbing boundary condition* $P_a(a(t), t|x_0, t_0) = 0$; and (ii) *reflecting boundary condition* (see appendix C for a proof)

$$j_r(a(t), t|x_0, t_0) = a'(t) P_r(a(t), t|x_0, t_0), \tag{13}$$

where j_r represents the probability flux at the time-dependent position $a(t)$ of the boundary. If conditions (3) and (11) are valid, the two propagators for absorbing and reflecting boundary, respectively P_a and P_r, are known analytically and read

$$P_a(x, t|x_0, t_0) = \frac{\partial\psi}{\partial x}\frac{1}{\sqrt{2\pi\tau}}\left[\exp\left(-\frac{\psi^2}{2\tau}\right) - \exp\left(-2a_0 v - \frac{(\psi - 2a_0)^2}{2\tau}\right)\right] \tag{14}$$

and

$$P_r(x, t|x_0, t_0) = \frac{\partial\psi}{\partial x}\left\{\frac{1}{\sqrt{2\pi\tau}}\left[\exp\left(-\frac{\psi^2}{2\tau}\right) + \exp\left(-2a_0 v - \frac{(\psi - 2a_0)^2}{2\tau}\right)\right]\right.$$
$$\left. - v\exp(-2v(\psi - a_0 - v\tau))\,\mathrm{erfc}\left(\frac{\psi - 2a_0 - 2v\tau}{\sqrt{2\tau}}\right)\right\}, \tag{15}$$

where erfc is the complementary error function. To make the equations shorter, we omitted the explicit dependencies of the functions ψ and τ. Equations (14) and (15) are the third main result of the paper. The proof of the results for these two cases is found in appendix C. We verified that these formulae solve the FPE with the appropriate boundary condition with Mathematica.

2.4. Solvable yet non-transformable SDEs and second Cherkasov condition

As stated at the end of section 2.1, there are some SDEs that are solvable even though they do not fulfil condition (3). These cases are discussed in [51–55] and interestingly they are all equivalent to the so-called *IGBM* [54]. The SDE for IGBM reads

$$\mathrm{d}X_t = (\alpha(t) X_t + \beta(t))\,\mathrm{d}t + \bar{\sigma}(t) X_t \mathrm{d}W_t, \tag{16}$$

where α, β, and $\bar{\sigma}$ may be any functions of time. Thus, the only difference with respect to GBM is the presence of $\beta(t)\mathrm{d}t$ in the SDE (16). Therefore, we may argue that there is a second class of SDEs that, even though they cannot be transformed into the Wiener process, are instead mappable onto IGBM The coefficients of such SDEs, while not fulfilling (3), satisfy the condition

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial \mathcal{C}(x,t)}{\partial x}\right)^{-1}\frac{\partial}{\partial x}\left(\sigma(x,t)\frac{\partial \mathcal{C}(x,t)}{\partial x}\right)\right] = 0, \tag{17}$$

where $\mathcal{C}$ was defined in (2). Moreover, the transformation ψ that maps the original process onto IGBM reads

$$\psi(x,t) = \left(\sigma(x,t)\frac{\partial \mathcal{C}(x,t)}{\partial x}\right)^{-1} \tag{18}$$

We here do not report the solution of equation (16) as it is available in standard textbooks [93]. The proof of these results is provided in appendix B.

3. Interpretation of the results: mapping a stochastic process onto the Wiener process

In this section we provide a more physical intuition on the meaning of the main formulae (4), (7)–(9). A graphical representation of our discussion is shown in Figure 1. We will not expose the proof here, this is available in appendix A. Let us start again from the SDE (1), for which the coefficients $\mu(x,t)$ and $\sigma(x,t)$ go under several names: in physics $\mu(x,t)$ represents a deterministic force acting on the particle, while $\sigma(x,t)\mathrm{d}W_t$ is the noise term due to thermal fluctuations. It is related to the diffusivity of the process via $D(x,t)=\sigma^2(x,t)/2$. Other popular names, especially in financial literature, for μ and σ are respectively the *drift* and the *volatility* (e.g see [50]).

As we show in appendix A, if and only if condition (3) holds, i.e. if and only if $\mathcal{C}(t)$ is solely a function of time, then it is possible to define a new process $Y_t=\psi(X_t,t)$ (for simplicity we do not state the dependencies on x_0 and t_0, as it is obvious) that the simpler SDE describes

$$\mathrm{d}Y_t=\sigma_Y(t)\,\mathrm{d}W_t, \tag{19}$$

where the volatility $\sigma_Y(t)$ of the new process is a function of time, only, and with $\psi(x,t)$ being strictly increasing, thus invertible, with respect to the first variable. For simplicity, without loss of generality, we assume that the initial condition of the new process is zero, $Y_{t_0}=\psi(x_0,t_0)\equiv 0$. The explicit form of $\psi(x,t)$ based on these features is given in (8) and derived in appendix A. This means that Y_t is a time-transformed version of a Wiener process. Therefore, we can define the new time variable

$$\tau\equiv\int_{t_0}^{t}\sigma_Y^2(s)\,\mathrm{d}s, \tag{20}$$

such that the reindexed process Y_τ is just a Wiener process,

$$Y_\tau=W_\tau, \tag{21}$$

with initial condition $Y_{\tau=0}=0$. In other words, if and only if condition (3) holds, there exists a special framework of coordinates, in which the stochastic motion is perceived as a simple Brownian motion. Thus, the connection between X_t and the Brownian motion W_t is

$$W_{\tau(t)}=\psi\left(X_t,t\right), \tag{22}$$

as stated in equation (4). Moreover, since the function $\psi(x,t)$ is invertible (see appendix A) it is possible to write

$$X_t=\psi^{-1}\left(W_{\tau(t)},t\right), \tag{23}$$

from which it is possible to obtain the correlations of the process as shown for the specific example discussed in section 5.1.

We now consider the propagator $P_W(y,\tau|0,0)\equiv P_Y(y,\tau)$ of the process W_τ. It is connected to the sought-after propagator $P(x,t|x_0,t_0)$ via the following change of measure (for consistency between the left and right hand sides we explicitly include the dependencies on x_0,t_0)

$$P(x,t|x_0,t_0)=\frac{\partial\psi\left(x,t|x_0,t_0\right)}{\partial x}P_W\left(y,\tau\right), \tag{24}$$

where we note that this formula relies on the fact that $\partial\psi/\partial x>0$. Using the fact that $P_W(y,\tau)$ is the propagator of a Brownian motion and rewriting the right hand side of equation (24) in terms of the original variables (x,t), we find

$$\begin{aligned} P(x,t|x_0,t_0)&=\frac{\partial\psi\left(x,t|x_0,t_0\right)}{\partial x}\frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{y^2}{2\tau}\right) \\ &=\frac{\partial\psi\left(x,t|x_0,t_0\right)}{\partial x}\frac{1}{\sqrt{2\pi\tau\left(t|t_0\right)}}\exp\left(-\frac{\psi^2\left(x,t|x_0,t_0\right)}{2\tau\left(t|t_0\right)}\right). \end{aligned} \tag{25}$$

The explicit forms of $\psi(x,t|x_0,t_0)$ and $\tau(t|t_0)$ are shown in equations (8) and (9).

A graphical representation of our discussion is provided in figure 1 where we also highlight how the FPT for a specific boundary varies after the space-time transformations. We conclude this section with the following observation: the idea of using space-time transformations is in fact reminiscent of the Cameron–Martin–Girsanov theorem [58]. This theorem describes how an SDE can be simplified via a

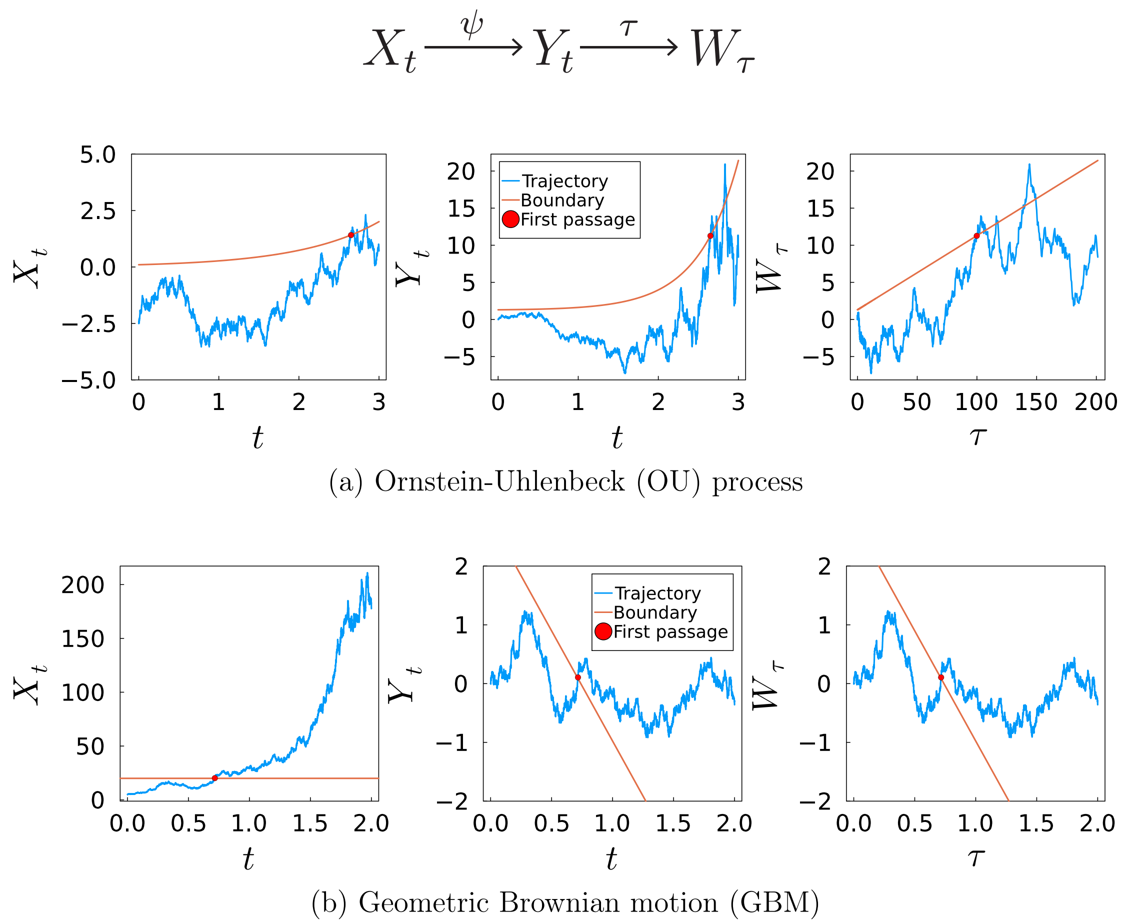

Figure 1. Examples of spatio-temporal transformations of stochastic processes (top) that enable an analytical treatment of their propagator and FPT statisticsusing the method of the current pape, along with illustrations for two examples: (a) Ornstein–Uhlenbeck (OU) process ($\mathrm{d}X_t = -\mu X_t \mathrm{d}t + \sigma \mathrm{d}W_t$, blue line) and FPT to reach a time-dependent boundary (red line). (b) Geometric Brownian motion (GBM) ($\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$, blue line) and FPT to reach a constant boundary. In the left panels we show the original processes X_t, in the central panels the space-transformed processes $Y_t = \psi(X_t, t)$, while in the right panels the space-time-transformed processes W_τ (Wiener) are shown. In (a) we considered a time-dependent boundary that grows exponentially with time $a(t) = a(0)\mathrm{e}^{\mu t}$, while in (b) we have a constant boundary. In both cases, after the transformation, the boundary becomes linear in time. Note that for GBM, the time transformation is trivially $\tau = t$ (cf (9)). The moment of the first passage is highlighted by the red circle in all panels.

change of measure, after which the transformed process becomes a martingale. While the Cameron–Martin–Girsanov theorem has found prominent applications in diverse fields such as finance (e.g the celebrated Black–Scholes formalism [59]), it does not provide a clear-cut criterion on which an SDE is amenable to exact analytical expressions for their propagators and FPTDs.

4. Which FPT problems are solvable?

We now turn our attention to the FPT problem. The question we are trying to address is: supposing that condition (3) is fulfilled, is it possible to obtain the FPTD? Why do we also require condition (11)? First of all, supposing that (3) is fulfilled, we could transform the stochastic process X_t into the Wiener process, and then consider the FPT problem for a Wiener process. In mathematical terms the idea is the following,

$$\begin{aligned}\inf\{t>0: X_t > a|X_{t_0} = x_0\} &= \inf\{t>0: Y_t > \psi(a,t)\,|Y_{t_0} = 0\} \\ &= \inf\left\{t(\tau)>0: Y_{t(\tau)} > \psi(a,t(\tau))\,|Y_{t(0)} = 0\right\} \\ &= \tau^{-1}\left(\inf\{\tau>0: W_\tau > \psi(a,\tau)\,|W_0 = 0\}\right),\end{aligned} \tag{26}$$

where in the first equality we used that the space-transformation ψ is monotonically increasing, in the second we expressed everything in terms of $t \equiv \tau^{-1}$ since the function τ is invertible (see appendix A), and in the third one we expressed everything in terms of τ. We also used the fact that $Y_\tau = W_\tau$. By denoting the FPT of a Wiener process with $\mathcal{T}_\mathrm{W}$, we thus proved that

$$\mathcal{T}_\mathrm{W}(\psi(a,t)) = \tau(\mathcal{T}_X(a|x_0,t_0)), \tag{27}$$

where on the right hand side we dropped the dependence on initial position and time as both are 0.

Therefore the FPT problem of the original process X_t for the barrier $a(t)$ is mapped onto the FPT of the Wiener process for the barrier $\psi(a(t),t)$ (refer to figure 1 for a geometric intuition). An explicit pedagogical explanation of this transformation is available in appendix D for the case of the Ornstein–Uhlenbeck process (OUP). We remind the reader that the function $\tau(t|t_0)$ defined in (9) is strictly increasing. In terms of probability distributions, equation (27) reads

$$\wp_X(a(t),t|x_0,t_0) = \wp_W(\psi(a(t),t),\tau(t))\frac{\mathrm{d}\tau(t|t_0)}{\mathrm{d}t}, \tag{28}$$

where $\wp_W$ is the FPTD of the Wiener process. Analytical results for $\wp$ are available only for a few limited cases.

The first-passage or first-crossing problem for the Wiener process to a time-dependent yet deterministic boundary is a long-standing open question in mathematics, several authors tried approached this problem with different analytical and computational methods [44, 45, 60–63]. Nevertheless, complete analytical results are available only for certain types of boundaries, such as those with linear dependencies [64], as stated in equation (11). Either way, even though the transformed boundary is non-linear, transforming the original stochastic process into the Wiener process is advantageous to find the FPTD, since in this case approximation schemes are available [45, 65]. Due to technical difficulties, we here do not report the case of boundaries with a nonlinear relation between ψ and τ, which we reserve for a future publication. Substituting into equation (28) the explicit form of the FPTD of the Wiener process (Wald or Lévy–Smirnov distribution) for this transformed linear boundary $\psi(a(t),t)$, we finally complete the proof of equation (12) (see [66, 67] for the derivation of the FPTD of the Wiener process).

Analogously, the same argument applies to propagators with absorbing and reflecting boundaries. As the FPTD, they are well known analytically only for certain boundaries. Further details on the proof of formulae (14) and (15) can be found in appendix C. In appendix D it is shown how the FPTD of an OUP for a constant boundary can be mapped onto the FPTD of the Wiener process for a square-root boundary. We refer to [68] for analytical results on square-root boundaries.

5. A few examples of application

The main results of this paper, equations (4), (7)–(9), (12), (14), and (15) can provide several results for many stochastic processes that so far were believed to be impossible to tackle analytically. As specific simple examples, we selected the (i) OUP with time-dependent stiffness and temperature, and (ii) the isoentropic protocol. We proceed with a discussion of these examples.

5.1. Non-autonomous OUP

Recent experiments explored heat engines, in which colloidal particles subject to a time-dependent temperature are confined by optical tweezers with time-dependent stiffness [69, 70]. The overdamped Langevin equation describing the fluctuating motion of the position of the particle is

$$\mathrm{d}X_t = -\frac{\kappa(t)}{\gamma}X_t\mathrm{d}t + \sqrt{\frac{2k_\mathrm{B}T(t)}{\gamma}}\mathrm{d}W_t, \tag{29}$$

where γ is the friction coefficient, $\kappa(t)$ is the trap stiffness, and $T(t)$ is the temperature. Both $\kappa(t)$ and $T(t)$ have a specific time dependence as they change during the cycle. Despite the caveat issued in [71] for the explicit time dependence of these parameters, we stress that the solution (7) is valid for *any* protocol $\kappa(t)$, $T(t)$ driving equation (29), as we discuss below. In fact, the Cherkasov function (2) associated with this non-autonomous process reads

$$\mathcal{C}(t) = \frac{1}{2}\frac{\mathrm{d}\ln(T(t))}{\mathrm{d}t} + \frac{\kappa(t)}{\gamma}, \tag{30}$$

which clearly fulfils condition (3). The integral of the Cherkasov function reads

$$\int_0^t \mathcal{C}(s)\,\mathrm{d}s = \ln\left(\sqrt{\frac{T(t)}{T(0)}}\right) + \Omega(t), \tag{31}$$

where we introduced the time-integrated corner frequency as

$$\Omega(t) \equiv \int_0^t \frac{\kappa(s)}{\gamma}\mathrm{d}s, \tag{32}$$

which is a dimensionless quantity. We can compute the form of ψ and τ using equations (8) and (9), yielding

$$\psi(x,t|x_0) = \sqrt{\frac{\gamma}{2k_\mathrm{B}T(0)}}\left(x\mathrm{e}^{\Omega(t)} - x_0\right) \tag{33}$$

and

$$\tau(t) = \frac{1}{T(0)}\int_0^t T(s)\,\mathrm{e}^{2\Omega(s)}\mathrm{d}s. \tag{34}$$

Thus, using expression (7), the propagator becomes

$$\begin{aligned} P(x,t|x_0,0) &= \sqrt{\frac{\gamma}{2k_\mathrm{B}T(0)}}\frac{1}{\sqrt{2\pi\int_0^t \frac{T(s)}{T(0)}\exp\left(-2\left(\Omega(t)-\Omega(s)\right)\right)\mathrm{d}s}} \\ &\quad\times\exp\left(-\frac{\gamma}{2k_\mathrm{B}T(0)}\frac{\left(x-x_0\mathrm{e}^{-\Omega(t)}\right)^2}{2\int_0^t\frac{T(s)}{T(0)}\mathrm{e}^{-2(\Omega(t)-\Omega(s))}\mathrm{d}s}\right). \end{aligned} \tag{35}$$

Furthermore, as mentioned before, we can use formula (23) to compute the correlations of the process. For this aim we need the inverse function of $\psi(x,t)$, which, in this case, reads

$$\psi^{-1}(x,t|x_0) = \left(\sqrt{\frac{2k_BT(0)}{\gamma}}x + x_0\right)\mathrm{e}^{-\Omega(t)}. \tag{36}$$

Therefore,

$$X_t = \left(\sqrt{\frac{2k_BT(0)}{\gamma}}W_{\tau(t)} + x_0\right)\mathrm{e}^{-\Omega(t)} \tag{37}$$

and

$$\langle X_tX_{t'}\rangle = \left\langle\frac{2k_BT(0)}{\gamma}W_{\tau(t)}W_{\tau(t')} + x_0^2\right\rangle\mathrm{e}^{-\Omega(t)-\Omega(t')}, \tag{38}$$

where we already removed all vanishing expectations. Keeping in mind that both τ and Ω are strictly increasing functions, we get

$$\begin{aligned} \langle X_tX_{t'}\rangle &= x_0^2\mathrm{e}^{-\Omega(t)-\Omega(t')} + \frac{2k_BT(0)}{\gamma}\tau(\min(t,t'))\,\mathrm{e}^{-2\Omega(\min(t,t'))} \\ &= x_0^2\mathrm{e}^{-\Omega(t)-\Omega(t')} + \frac{2k_BT(0)}{\gamma}\int_0^{\min(t,t')}\frac{T(s)}{T(0)}\mathrm{e}^{-2[\Omega(\min(t,t'))-\Omega(s)]}\mathrm{d}s. \end{aligned} \tag{39}$$

Let us now turn to the FPTD. According to equation (11) it is possible to compute it when the transformed boundary is linear in τ. Condition (11) can be reformulated in the equivalent form

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right|_{x=a(t)} = \text{const}. \tag{40}$$

In this case, $\mathrm{d}\psi/d\tau$ reads

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right|_{x=a(t)} = \left.\frac{\partial\psi}{\partial t}\right|_{x=a(t)}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{\frac{\gamma}{2k_\mathrm{B}T(0)}}\frac{T(0)}{T(t)}\frac{\kappa(t)}{\gamma}\mathrm{e}^{-\Omega(t)}a(t). \tag{41}$$

In general, this expression is not independent on time, except in the two cases

$$a(t) \propto \frac{T(t)}{\kappa(t)}\mathrm{e}^{\Omega(t)} \quad \text{and} \quad a(t) = 0. \tag{42}$$

Interestingly, it is always possible to compute the FPT for the origin, which is the point of symmetry of the potential. The FPTD for $a = 0$ reads

$$\wp(0,t|x_0) = \sqrt{\frac{\gamma}{2k_BT(0)}}\frac{T(t)}{T(0)}\frac{|x_0|\mathrm{e}^{2\Omega(t)}}{\sqrt{2\pi\left(\int_0^t \frac{T(s)}{T(0)}\mathrm{e}^{2\Omega(s)}\mathrm{d}s\right)^3}}\exp\left(-\frac{\gamma}{2k_BT(0)}\frac{x_0^2}{2\int_0^t \frac{T(s)}{T(0)}\mathrm{e}^{2\Omega(s)}\mathrm{d}s}\right).$$

The exact FPTD for the OUP is available analytically in literature only in the case of constant parameters [72, 73] (or see [57] for a review). Thus expression (43) also constitutes a novel result of this paper. Having at hand the explicit form of the FPTD can enhance the quantitative study of feedback-control protocols, such as the ones in [74–78]. In these references the authors study overdamped diffusions subjected to an information-like control feedback, i.e. whenever the particle reaches a specific threshold the stiffness of the potential is instantaneously set to another value in order to maximise the work extraction. We illustrate results (35) and (43) in the next subsection for the isoentropic protocol.

5.2. Isoentropic protocol

The so-called isoentropic (or pseudo-adiabatic) protocol [46, 47] was introduced in the field of stochastic thermodynamics as a building block of Carnot-type cycles in colloidal heat engines [70]. For overdamped Langevin dynamics in a time-dependent harmonic potential and time-dependent temperature, the protocol consists in having both temperature and stiffness explicitly time-dependent while keeping their ratio constant. In mathematical terms,

$$\frac{T(t)}{\kappa(t)} = \frac{T(0)}{\kappa(0)}. \tag{43}$$

Such a protocol ensures that, if the initial condition is equilibrium, the PDF associated with the particle position (and hence also its Shannon entropy) is conserved in time. This is why such a protocol is called adiabatic and isoentropic in the literature [46, 47]. Below we provide further insights into these features with our analytical formalism. Imposing this relationship between $T(t)$ and $\kappa(t)$ we get a simplified formula for $\tau(t)$,

$$\tau(t) = \frac{\gamma}{2\kappa(0)}\int_0^t 2\Omega'(s)\mathrm{e}^{2\Omega(s)}\mathrm{d}s = \frac{\gamma}{2\kappa(0)}\left(\mathrm{e}^{2\Omega(t)}-1\right), \tag{44}$$

hence the propagator (7) takes the simplified form

$$P(x,t|x_0,0) = \sqrt{\frac{\kappa(0)}{k_BT(0)}}\frac{1}{\sqrt{2\pi\left(1-\mathrm{e}^{-2\Omega(t)}\right)}}\exp\left(-\frac{\kappa(0)}{k_BT(0)}\frac{\left(x-x_0\mathrm{e}^{-\Omega(t)}\right)^2}{2\left(1-\mathrm{e}^{-2\Omega(t)}\right)}\right). \tag{45}$$

The FPTD (12) also has a simplified form, namely,

$$\wp(0,t|x_0) = \sqrt{\frac{\kappa(0)}{k_BT(0)}}\frac{2\kappa(t)}{\gamma}\frac{|x_0|\mathrm{e}^{2\Omega(t)}}{\sqrt{2\pi\left(\mathrm{e}^{2\Omega(t)}-1\right)^3}}\exp\left(-\frac{\kappa(0)}{k_BT(0)}\frac{x_0^2}{2\left(\mathrm{e}^{2\Omega(t)}-1\right)}\right). \tag{46}$$

While the propagator (45) was known in literature [79], the FPT statistics, such as the FPTD (46) are, to our knowledge, so far unknown for adiabatic protocols. An excellent agreement between results (45) and (46) and numerical simulations based on realisations of the Langevin equation (29) is demonstrated in figure 2.

We conclude this subsection with an observation concerning the average over initial positions. Clearly, if the system starts at equilibrium, the form of the probability distribution is preserved over time. More explicitly, if we denote with $\rho(x,t)$ the solution of the FPE (5) with initial condition

$$\rho(x,0) = \sqrt{\frac{\kappa(0)}{k_\mathrm{B}T(0)}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\kappa(0)}{k_\mathrm{B}T(0)}\frac{x^2}{2}\right), \tag{47}$$

it is possible to obtain $\rho(x,t)$ via a convolution of the initial condition with the propagator (45),

$$\rho(x,t) = \int_{-\infty}^{\infty}\rho(x_0,0)P(x,t|x_0)\mathrm{d}x_0, \tag{48}$$

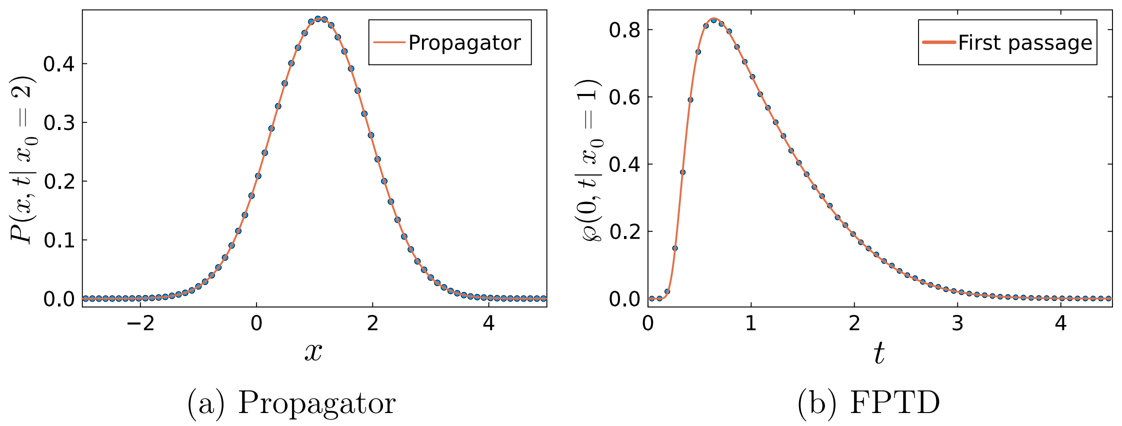

Figure 2. Comparison between analytical results (solid lines) (45) for the propagator and (46) for the FPTD versus simulations (dots). In both cases the numerical simulations were based on the Euler–Maruyama scheme of the Langevin equation (29). In panel (a) we generated 2×10^6 trajectories with parameters $x_0 = 2$, $\gamma = 1$, $\kappa(t) = 0.1 + t$, and $\kappa(0)/k_B T(0) = 1$. In panel (b) we generated 10^6 trajectories with the parameters $x_0 = 1$, $\gamma = 1$, $\kappa(t) = 0.1 + t$, and $\kappa(0)/k_B T(0) = 1$. Excellent agreement is observed.

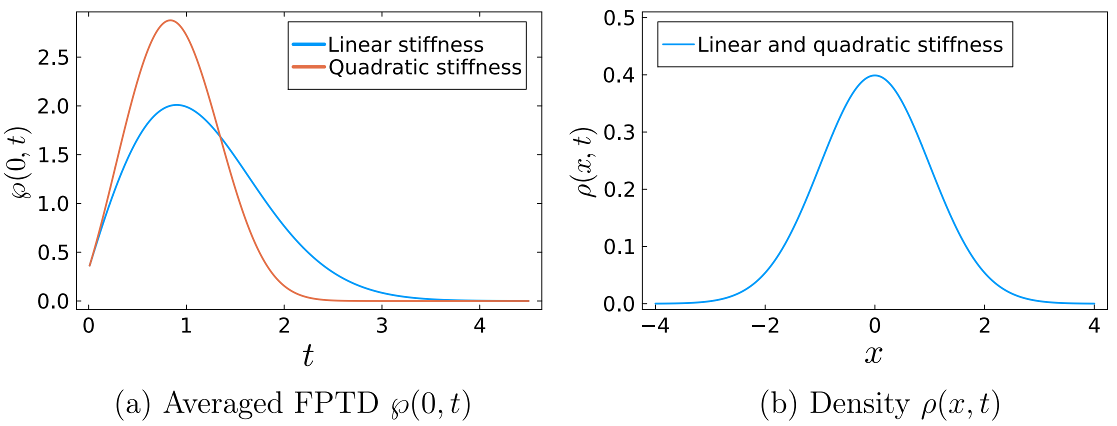

Figure 3. (a) Comparison of the FPTD averaged over initial positions for two different choices of the stiffness protocol $\kappa(t)$. For the blue curve we used the linear function $\kappa(t) = T(t) = 0.1 + t$, while for the orange curve we used the quadratic form $\kappa(t) = T(t) = 0.1 + t + t^2$. In both cases we use $\gamma = 1$. (b) Density $\rho(x, t)$, which is equal for the two cases of linear and quadratic stiffness.

which clearly shows that $\rho(x, t) = \rho(x, 0)$, hence the form of the distribution does not change with time for any choice of the protocols $\kappa(t)$ and $T(t)$. Nevertheless, the FPTD averaged over initial positions is not protocol-independent and reads

$$\wp(0, t) \equiv \int_{-\infty}^{+\infty} \rho(x_0, 0)\, \wp(0, t|x_0)\, \mathrm{d}x_0 = \frac{2}{\pi} \frac{\kappa(t)}{\gamma} \frac{1}{\sqrt{\mathrm{e}^{2\Omega(t)} - 1}}. \tag{49}$$

This is highlighted in figure 3. This example shows that knowing only the distribution of the process may not be enough to infer any specific property of the FPTD.

5.3. Stochastic Gompertz model

An important process for population dynamics is the stochastic Gompertz model proposed in [80] and studied further in [48, 54]. The SDE describing the process reads

$$\mathrm{d}X_t = -\alpha(t) X_t \ln\left(\frac{X_t}{K(t)}\right) \mathrm{d}t + \beta(t) X_t \mathrm{d}W_t. \tag{50}$$

The deterministic version of the model (without noise) was introduced two centuries ago [81] to express the law of human mortality, and it was more recently applied to the modelling of cellular growth of tumours [82]. In the existing literature [48, 54, 80] the parameters α, K, and β were considered as constants. Here, instead, with the help of our formalism, we consider them as arbitrary functions of time. Note that if

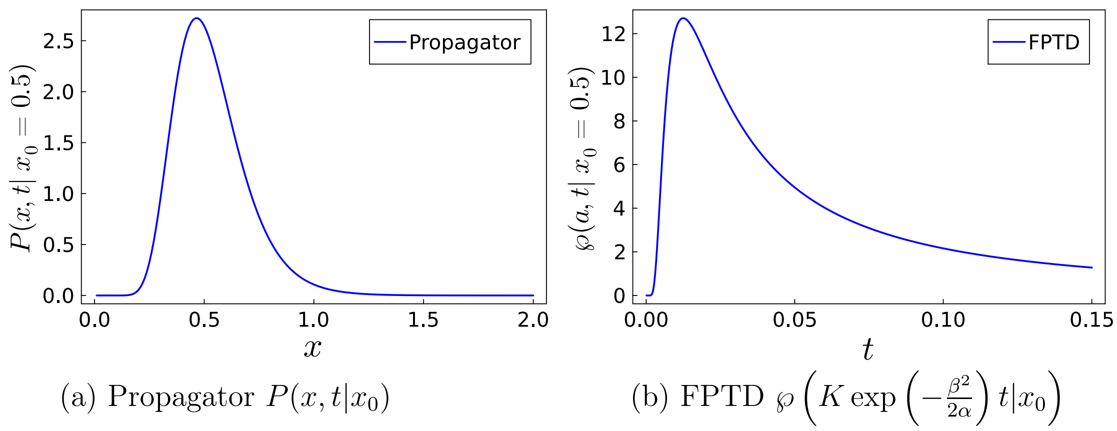

Figure 4. (a) Propagator (54) and (b) FPTD (58) for the stochastic Gompertz model (50) with constant parameters. The parameters are $\alpha = \beta = K = 1$ and $x_0 = 0.5$. The propagator is plotted for $t = 0.1$.

$\alpha(t) > 0$, the drift term in (50) is positive for $X_t < K(t)$ and negative for $X_t > K(t)$, while it vanishes for $X_t = K(t)$. Therefore, $K(t)$ represents a carrying capacity, which might depend on time if the resources of the environment are not constant, while $\alpha(t)$ represents either a birth or a mortality rate. The stochastic motion starts at some $X_0 = x_0 > 0$ and it is confined to be positive at all times.

Let us now have a look at the form of the propagator. First, using equation (2) the Cherkasov function reads

$$\mathcal{C}(t) = \frac{\mathrm{d}\ln\beta(t)}{\mathrm{d}t} + \alpha(t), \tag{51}$$

which, being independent of x, allows us to write the propagator without explicitly solving the FPE, thus avoiding the calculations in [80]. The transforming function $\psi(x,t)$ can be obtained via relation (8), producing

$$\begin{aligned}\psi(x,t|x_0) = {} & \frac{1}{\beta(0)}\exp\left(\int_0^t \alpha(t')\,\mathrm{d}t'\right)\ln\left(\frac{x}{x_0}\right) \\ & + \int_0^t \left(\frac{\beta(t')}{2} + \frac{\alpha(t')\ln\left(\frac{x_0}{K(t')}\right)}{\beta(t')}\right)\frac{\beta(t')}{\beta(0)}\exp\left(\int_0^{t'}\alpha(s)\,\mathrm{d}s\right)\mathrm{d}t';\end{aligned} \tag{52}$$

while $\tau(t)$, using (9), reads

$$\tau(t) = \frac{1}{\beta^2(0)}\int_0^t \beta^2(t')\exp\left(2\int_0^{t'}\alpha(s)\,\mathrm{d}s\right)\mathrm{d}t'. \tag{53}$$

Here we set $t_0 = 0$ everywhere. The propagator is then obtained from plugging the forms of ψ and τ into equation (7). In the simplest case of constant parameters the propagator then reads

$$P(x,t|x_0) = \frac{1}{x\sqrt{2\pi\frac{\beta^2}{2\alpha}\left(1-\mathrm{e}^{2\alpha t}\right)}}\exp\left(-\frac{\left(\ln\left(\frac{x}{K}\right) + \frac{\beta^2}{2\alpha}\left(1-\mathrm{e}^{-\alpha t}\right) - \ln\left(\frac{x_0}{K}\right)\mathrm{e}^{-\alpha t}\right)^2}{\frac{\beta^2}{2\alpha}\left(1-\mathrm{e}^{2\alpha t}\right)}\right), \tag{54}$$

which is plotted in figure 4(a). The stationary distribution can then be easily obtained as

$$P^{(s)}(x) = \frac{1}{x\sqrt{2\pi\frac{\beta^2}{2\alpha}}}\exp\left(-\frac{\left(\ln\left(\frac{x}{K}\right)+\frac{\beta^2}{2\alpha}\right)^2}{\frac{\beta^2}{2\alpha}}\right). \tag{55}$$

For the FPTD, we need to check when condition (11) is satisfied. We compute first the derivative

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right|_{x=a(t)} = \beta(0)\left[\frac{\alpha(t)}{\beta^2(t)}\ln\left(\frac{a(t)}{K(t)}\right) + \frac{1}{2}\right]\exp\left(-\int_0^t \alpha(t')\,\mathrm{d}t'\right) \tag{56}$$

and then require that it must be independent of t. A possibility is thus when $a(t)$ has the form

$$a(t) = K(t)\exp\left(-\frac{\beta^2(t)}{2\alpha(t)}\right). \tag{57}$$

For this specific choice of the boundary, considering again the case of constant parameters, the FPTD reads

$$\wp\left(K\exp\left(-\frac{\beta^2}{2\alpha}\right), t \,\middle|\, x_0\right) = \frac{\left|\frac{\beta^2}{2\alpha} + \ln\left(\frac{x_0}{K}\right)\right|}{\sqrt{2\pi\frac{\beta^2}{(2\alpha)^3}\left(e^{2\alpha t}-1\right)^3}}\exp\left(2\alpha t - \frac{\left(\frac{\beta^2}{2\alpha}+\ln\left(\frac{x_0}{K}\right)\right)^2}{2\frac{\beta^2}{2\alpha}\left(e^{2\alpha t}-1\right)}\right), \tag{58}$$

which is displayed in figure 4(b).

6. Connections with existing results

As we already discussed, our formalism can provide solutions to many non-linear SDEs. We here illustrate a class of such models satisfying condition (3). A particular choice of the coefficients μ and σ satisfying condition (3) is the product form

$$\sigma(x,t) \equiv \sigma_x(x)\,\sigma_t(t) \tag{59}$$

i.e. σ can be separated into two functions, the purely x-dependent $\sigma_x(x)$ and the purely t-dependent $\sigma_t(t)$. Then, for μ we have

$$\mu(x,t) = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\sigma^2(x,t)}{2}\right). \tag{60}$$

Interestingly, it can be shown that for this choice of the parameters, the SDE can be rewritten in the Stratonovich formalism as $dX_t = \sigma_t(t)\sigma_x(X_t)\circ dW_t$. The two equations (59) and (60) imply that both conditions (3) and (11) hold, in particular (11) holds for any boundary $a(t)$. Indeed, using expressions (59) and (60), we get that ψ and τ take the simplified expressions

$$\psi(x,t|x_0,t_0) = \frac{1}{\sigma_t(t_0)}\int_{x_0}^{x}\frac{\mathrm{d}x'}{\sigma_x(x')} \tag{61}$$

and

$$\tau(t|t_0) = \frac{1}{\sigma_t^2(t_0)}\int_{t_0}^{t}\sigma_t^2(s)\,\mathrm{d}s. \tag{62}$$

Notice that ψ does not depend on time here, therefore condition (11) holds for any boundary $a(t)$. This prompts the following observation: for all physical systems in which $\mu(x,t)$ is a potential force, i.e. $\mu(x,t) = -\frac{1}{\gamma}\frac{\partial}{\partial x}U(x,t)$, and σ^2 is related to the temperature as $\sigma^2(x,t) = 2k_{\mathrm{B}}T(x,t)/\gamma$, equation (60) implies that

$$\frac{U(x,t)}{k_{\mathrm{B}}T(x,t)} = \text{const.} \tag{63}$$

equation (63) implies that the process is purely diffusive; indeed, as stated before, the SDE reads $\mathrm{d}X_t = \sigma(x,t)\circ\mathrm{d}W_t$, thus the stationary distribution of the process is independent of space and time for the case of closed or periodic boundary conditions. Notably, our results show that the FPTD can be expressed analytically for rather generic boundary conditions for all processes for which (63) holds. Such processes were thoroughly studied by, e.g. Hänggi, Talkner and Borkovec in [32]; there the authors outline explicit results to obtain all moments of the FPT for processes with $U/k_{\mathrm{B}}T = \text{const}$, yet the full FPTD has not been investigated. Our formalism allows the determination of this entire FPTD. Moreover, processes with $U/k_{\mathrm{B}}T = \text{const}$ are a subclass of the wider class described by equations (3) and (11).

7. Conclusions

We discussed a method to obtain exact solutions of a class of time-inhomogeneous SDEs and their associated FPT problems. This method represents a valid and simpler alternative to other approaches. The technique discussed in this paper for FPTDs, while mentioned in [44], to the best of our knowledge, has been unused and largely unknown to the statistical physics community for a long time. The explicit form of FPTDs can highly enhance the quantitative study of feedback-control protocols, such as the ones in [74–78], and can provide useful insights in experiments with optical tweezers beyond those exemplified in section 5.1. To this end we identified a class of one-dimensional problems with time-dependent parameters that are amenable to exact solutions, yet extensions to higher dimensions and more complex scenarios (e.g with nonlinear time dependencies in the boundaries) are possible and will be the topic of future work. Moreover, having at hand analytical results for multidimensional models could provide interesting insights into non-Markovian processes via Markovian embedding [83]. With our method, we were able to generalise well-known results and to obtain new solutions to non-autonomous problems. We were able to find the propagators (14) and (15) in the constrained domain that could be useful for further studies in constrained random walks [84]. We only considered models with deterministic parameters, but further generalisations to models such as diffusing diffusivity [85–87, 94] or the so-called OU^2 process [88] should be rather straightforward. Within statistical physics, we believe that our results could find prominent applications, inter alia, within control theory [89], non-equilibrium calorimetry [90], or computing [91]. We believe that our work will serve as a reference also beyond the statistical physics community in other areas, in which stochastic processes and FPTs are important.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Cherkasov condition and explicit form of transforming functions

Here we will prove equations (3), (4), (7)–(9). The dynamic of the position X_t is governed by the SDE (1), which we here repeat for the convenience of the reader:

$$\mathrm{d}X_t = \mu(X_t,t)\,\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}W_t, \tag{A.1}$$

where W_t is the standard Wiener process. Consider now the transformation $Y_t = \psi(X_t,t)$; according to the Itô rule, the SDE for Y_t reads

$$\mathrm{d}Y_t = \left(\left.\frac{\partial\psi}{\partial t}\right|_{x=X_t} + \mu(X_t,t)\left.\frac{\partial\psi}{\partial x}\right|_{x=X_t} + \frac{\sigma^2(X_t,t)}{2}\left.\frac{\partial^2\psi}{\partial x^2}\right|_{x=X_t}\right)\mathrm{d}t + \sigma(X_t,t)\left.\frac{\partial\psi}{\partial x}\right|_{x=X_t}\mathrm{d}W_t. \tag{A.2}$$

In order to be mappable onto the Wiener process, both drift and volatility of the new process should be independent of the position. Therefore

$$\begin{cases} \dfrac{\partial\psi(x,t)}{\partial t} + \mu(x,t)\dfrac{\partial\psi(x,t)}{\partial x} + \dfrac{\sigma^2(x,t)}{2}\dfrac{\partial^2\psi(x,t)}{\partial x^2} = \mu_Y(t), \\ \sigma(x,t)\dfrac{\partial\psi(x,t)}{\partial x} = \sigma_Y(t), \end{cases} \tag{A.3}$$

where $\mu_Y(t)$ and $\sigma_Y(t)$ are, respectively, the new drift and variance, both solely time-dependent. We anticipate that in the derivation below it will be possible to set $\mu_Y(t) = 0$. From equation (A.3) we obtain

$$\frac{\partial\psi(x,t)}{\partial x} = \frac{\sigma_Y(t)}{\sigma(x,t)} > 0, \tag{A.4}$$

which proves that $\psi(x,t)$ is strictly increasing in the first variable x. Substituting this result into the first equation of (A.3) we get

$$\frac{\partial\psi(x,t)}{\partial t}=\sigma_Y(t)\left(\frac{1}{2}\frac{\partial\sigma(x,t)}{\partial x}-\frac{\mu(x,t)}{\sigma(x,t)}\right)+\mu_Y(t). \tag{A.5}$$

If the second partial derivatives of ψ are continuous, we know from Schwarz's theorem that $\partial^2\psi/\partial x\partial t=\partial^2\psi/\partial t\partial x$, and therefore

$$\frac{\mathrm{d}\sigma_Y}{\mathrm{d}t}\frac{1}{\sigma}-\frac{\sigma_Y}{\sigma^2}\frac{\partial\sigma}{\partial t}=\sigma_Y\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial\sigma}{\partial x}-\frac{\mu}{\sigma}\right). \tag{A.6}$$

Since there is no risk of confusion, we here drop the arguments of σ, σ_Y, and μ. From some simple manipulations we get

$$\frac{1}{\sigma_Y(t)}\frac{\mathrm{d}\sigma_Y(t)}{\mathrm{d}t}=\frac{1}{\sigma}\frac{\partial\sigma}{\partial t}+\sigma\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial\sigma}{\partial x}-\frac{\mu}{\sigma}\right)\equiv\mathcal{C}(t), \tag{A.7}$$

where the left hand side does not depend on x—thus by taking the partial derivative $\partial/\partial x$ we immediately see that the condition (3) on the Cherkasov function is indeed fulfilled. In other words, if the drift and the volatility of the new process are space-independent then the Cherkasov function fulfils condition (3), and the inverse is also true. This formally proves that condition (3) is necessary and sufficient.

Moreover, by solving the previous differential equation, we get the form

$$\sigma_Y(t)=\exp\left(\int_{t_0}^{t}\mathcal{C}(s)\,\mathrm{d}s\right) \tag{A.8}$$

of $\sigma_Y(t)$, where, since it is arbitrary, for simplicity we set $\sigma_Y(t_0)=1$. Substituting (A.8) into (A.4) and integrating over x we obtain the explicit formula of the transformation

$$\psi(x,t)=\exp\left(\int_{t_0}^{t}\mathcal{C}(s)\,\mathrm{d}s\right)\int_{x_0}^{x}\frac{1}{\sigma(x',t)}\mathrm{d}x'+\psi(x_0,t). \tag{A.9}$$

So far we have not specified the boundary condition $\psi(x_0,t)$, which is arbitrary.

Let us show that $\psi(x_0,t)$ and $\mu_Y(t)$ are related one-to-one. To see this we substitute expression (A.9) back into the first equation of the system (A.3),

$$\begin{aligned}\mu_Y(t)=\frac{\mathrm{d}\psi(x_0,t)}{\mathrm{d}t}&+\Bigg[\mathcal{C}(t)\int_{x_0}^{x}\frac{1}{\sigma(x',t)}\mathrm{d}x'-\int_{x_0}^{x}\frac{1}{\sigma^2(x',t)}\frac{\partial\sigma}{\partial t}\mathrm{d}x'\\&+\frac{\mu(x,t)}{\sigma(x,t)}-\frac{1}{2}\frac{\partial\sigma}{\partial x}\Bigg]\exp\left(\int_{t_0}^{t}\mathcal{C}(t')\,\mathrm{d}t'\right).\end{aligned} \tag{A.10}$$

We notice that the term appearing on the right hand side of this equation can be simplified by use of the definition of the Cherkasov function (2),

$$\int_{x_0}^{x}\frac{1}{\sigma^2(x',t)}\frac{\partial\sigma}{\partial t}\mathrm{d}x'+\frac{1}{2}\frac{\partial\sigma}{\partial x}-\frac{\mu(x,t)}{\sigma(x,t)}=\mathcal{C}(t)\int_{x_0}^{x}\frac{1}{\sigma(x',t)}\mathrm{d}x'-\frac{\mu(x_0,t)}{\sigma(x_0,t)}+\frac{1}{2}\frac{\partial\sigma}{\partial x}\bigg|_{(x_0,t)}, \tag{A.11}$$

which simplifies equation (A.10) for $\mu_Y(t)$,

$$\mu_Y(t)=\frac{\mathrm{d}\psi(x_0,t)}{\mathrm{d}t}+\left(\frac{\mu(x_0,t)}{\sigma(x_0,t)}-\frac{1}{2}\frac{\partial\sigma}{\partial x}\bigg|_{(x_0,t)}\right)\exp\left(\int_{t_0}^{t}\mathcal{C}(t')\,\mathrm{d}t'\right). \tag{A.12}$$

Thus we found a differential equation for $\psi(x_0,t)$. As $\psi(x_0,t)$ is arbitrary, a convenient choice is the one leading to $\mu_Y(t)=0$, yielding

$$\psi(x_0,t)=\int_{t_0}^{t}\mathrm{d}s\left(\frac{1}{2}\frac{\partial\sigma}{\partial x}\bigg|_{(x_0,s)}-\frac{\mu(x_0,s)}{\sigma(x_0,s)}\right)\exp\left(\int_{t_0}^{s}\mathcal{C}(s')\,\mathrm{d}s'\right), \tag{A.13}$$

where we set $\psi(x_0,t_0)=0$. Therefore, the final form of $\psi(x,t)$ is

$$\psi(x,t)=\exp\left(\int_{t_0}^{t}\mathcal{C}(s)\,\mathrm{d}s\right)\int_{x_0}^{x}\frac{1}{\sigma(x',t)}\mathrm{d}x' \tag{A.14}$$

$$+\int_{t_0}^{t}\mathrm{d}s\left(\frac{1}{2}\frac{\partial\sigma}{\partial x}\bigg|_{(x_0,s)}-\frac{\mu(x_0,s)}{\sigma(x_0,s)}\right)\exp\left(\int_{t_0}^{s}\mathcal{C}(s')\,\mathrm{d}s'\right).$$

This transformation maps the SDE onto $\mathrm{d}Y_t=\sigma_Y(t)\mathrm{d}W_t$ and can indeed be considered as a modification of the Lamperti transform [92].

Finally, the function reindexing time reads

$$\tau(t)=\int_{t_0}^{t}\sigma_Y^2(s)\,\mathrm{d}s=\int_{t_0}^{t}\exp\left(2\int_{t_0}^{s}\mathcal{C}(r)\,\mathrm{d}r\right)\mathrm{d}s,\tag{A.15}$$

as we set out to prove. We conclude with a final remark on $\psi(x,t)$ and $\tau(t)$. Namely, there are infinitely many transformations serving our purpose; for simplicity, we chose the specific forms (8) and (9), that vanish at the initial points x_0,t_0.

Appendix B. Extension to inhomogeneous geometric Brownian motion (IGBM)

As stated in the main text, condition (3) does not cover all SDEs that are solvable analytically. One prominent example is the IGBM which is described by the SDE

$$\mathrm{d}X_t=(\alpha X_t+\beta(t))\,\mathrm{d}t+\sigma(t)X_t\mathrm{d}W_t,\tag{B.1}$$

where the parameters α, β, and σ are generic functions of time. A complete solution to (16) is contained in the book of Mao [93]. One may argue that there must exist a *second Cherkasov condition* that whenever satisfied, it is guaranteed that the SDE is mappable onto IGBM. We now derive such a condition.

Following a procedure identical to the one described in appendix A, starting again from $\mathrm{d}X_t=\mu(X_t,t)\mathrm{d}t+\sigma(X_t,t)\mathrm{d}W_t$ and applying a generic transformation $Y_t=\psi(X_t,t)$ we end up at a new SDE for Y_t with new drift μ_Y and a new volatility σ_Y. We suppose that the transforming function $\psi(x,t)$ is sufficiently smooth, so that we can take derivatives without caring about discontinuities. We now require the forms $\mu_Y(y,t)=\alpha(t)y+\beta(t)$ and $\sigma_Y(y,t)=\zeta(t)y$. With these requirements, the system (A.3) becomes

$$\begin{cases}\dfrac{\partial\psi(x,t)}{\partial t}+\mu(x,t)\dfrac{\partial\psi(x,t)}{\partial x}+\dfrac{\sigma^2(x,t)}{2}\dfrac{\partial^2\psi(x,t)}{\partial x^2}=\alpha(t)\psi(x,t)+\beta(t),\\ \sigma(x,t)\dfrac{\partial\psi(x,t)}{\partial x}=\zeta(t)\psi(x,t),\end{cases}\tag{B.2}$$

where we substituted y with $\psi(x,t)$. From the second equation of (B.2) we get

$$\frac{\partial\psi(x,t)}{\partial x}=\zeta(t)\frac{\psi(x,t)}{\sigma(x,t)},\tag{B.3}$$

which substituted into the first equation, yields

$$\frac{\partial\psi}{\partial t}+\left[\frac{\mu}{\sigma}-\frac{1}{2}\frac{\partial\sigma}{\partial x}+\frac{\zeta}{2}\right]\zeta\psi=\alpha\psi+\beta.\tag{B.4}$$

Here, for convenience, we dropped the explicit dependence on x,t everywhere. Next, we divide by ψ and we take the derivative with respect to x,

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{\sigma}\right)+\frac{\partial}{\partial x}\left(\frac{\mu}{\sigma}-\frac{1}{2}\frac{\partial\sigma}{\partial x}\right)\zeta=\beta\frac{\partial}{\partial x}\left(\frac{1}{\psi}\right),\tag{B.5}$$

where we used the fact that $\frac{\partial}{\partial x}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial x}\right)$ and that $\frac{\partial\zeta}{\partial x}=\frac{\partial\alpha}{\partial x}=\frac{\partial\beta}{\partial x}=0$. After further manipulations we arrive at

$$\frac{1}{\sigma}\frac{\partial\zeta}{\partial t}-\frac{\zeta}{\sigma^2}\frac{\partial\sigma}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\mu}{\sigma}-\frac{1}{2}\frac{\partial\sigma}{\partial x}\right)\zeta=-\frac{\beta}{\psi}\frac{\zeta}{\sigma}.\tag{B.6}$$

We now multiply by σ and divide by ζ,

$$\frac{1}{\zeta}\frac{\partial\zeta}{\partial t}-\frac{1}{\sigma}\frac{\partial\sigma}{\partial t}+\sigma\frac{\partial}{\partial x}\left(\frac{\mu}{\sigma}-\frac{1}{2}\frac{\partial\sigma}{\partial x}\right)=-\frac{\beta}{\psi}.\tag{B.7}$$

On the left hand side we recognise the Cherkasov function (2). Thus, the previous equation can be rewritten as

$$\frac{\partial \ln \zeta}{\partial t} = \mathcal{C} - \frac{\beta}{\psi}. \tag{B.8}$$

We note that here, according to our assumption that the Cherkasov condition (3) does not hold, the function $\mathcal{C}(x,t)$ will depend both on x and t. Hence, taking a derivative with respect to x the left hand side vanishes and we get

$$\frac{\partial}{\partial x}\left(\mathcal{C} - \frac{\beta}{\psi}\right) = 0. \tag{B.9}$$

The previous expression appears like another Cherkasov condition, yet there is a dependence on β, which is an unknown parameter. We now reexpress it in terms of the original parameters μ and σ of the SDE. Using again the expression for $\partial\psi/\partial x$ we get

$$\frac{\partial \mathcal{C}}{\partial x} + \frac{\beta}{\psi}\frac{\zeta}{\sigma} = 0. \tag{B.10}$$

Further manipulating the previous expression, recalling that, by our assumption, both β and ζ do not depend on x, we get

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial \mathcal{C}}{\partial x}\right) + \zeta \frac{\partial \mathcal{C}}{\partial x} = 0, \tag{B.11}$$

from which we find the explicit form of the function $\zeta(t)$,

$$\zeta(t) = -\left(\frac{\partial \mathcal{C}}{\partial x}\right)^{-1} \frac{\partial}{\partial x}\left(\sigma \frac{\partial \mathcal{C}}{\partial x}\right), \tag{B.12}$$

and also the second Cherkasov condition

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial \mathcal{C}}{\partial x}\right)^{-1} \frac{\partial}{\partial x}\left(\sigma \frac{\partial \mathcal{C}}{\partial x}\right)\right] = 0. \tag{B.13}$$

We now turn our attention to the form of the transforming function ψ. Using equation (B.3) we obtain

$$\frac{\partial \ln \psi}{\partial x} = \frac{\zeta}{\sigma} \Rightarrow \psi(x,t) = \psi(x_0,t)\exp\left(\zeta(t)\int_{x_0}^{x} \frac{1}{\sigma(x',t)}\mathrm{d}x'\right), \tag{B.14}$$

where $\psi(x_0,t)$ is an arbitrary function of time. With the explicit form of ζ the previous expression can be conveniently rewritten as

$$\psi(x,t) = \psi(x_0,t)\,\frac{\left(\sigma \frac{\partial \mathcal{C}}{\partial x}\right)\Big|_{(x_0,t)}}{\left(\sigma \frac{\partial \mathcal{C}}{\partial x}\right)\Big|_{(x,t)}}. \tag{B.15}$$

Since the form of $\psi(x_0,t)$ is arbitrary, we can set it equal to

$$\psi(x_0,t) = \left(\left(\sigma \frac{\partial \mathcal{C}}{\partial x}\right)\Big|_{x_0,t}\right)^{-1}, \tag{B.16}$$

obtaining the result contained in the main text.

Appendix C. Derivation of absorbing and reflecting propagators

We here provide a proof for equations (14) and (15). As mentioned in the main text, the functions ψ and τ map the original stochastic process onto the Wiener process according to equation (4). Therefore, the problem can be reduced to finding the propagator of the Wiener process with either an absorbing or reflecting, time-dependent boundary. Interestingly, according to equation (11), after the space-time transformation the boundary becomes

$$a(t) \to \psi(a(t), t) = \nu\tau + a_0, \tag{C.1}$$

which is a linear function of the new time variable τ. The propagator $P_{W,a}$ of the Wiener process with absorbing linear-in-time boundary is available in standard textbooks [66, 67] and reads

$$P_{W,a}(y,\tau) = \frac{1}{\sqrt{2\pi\tau}}\left[\exp\left(-\frac{y^2}{2\tau}\right) - \exp\left(-2a_0\nu - \frac{(y-2a_0)^2}{2\tau}\right)\right]. \tag{C.2}$$

Moreover, the relation between the propagator of X_t and that of W_τ is given in equation (24); thus equation (14) can be obtained by multiplication by $\partial\psi/\partial x$ and substituting $y \to \psi(x,t)$ in equation (C.2).

For the reflecting case we reason analogously—the propagator $P_{W,r}$ of the Wiener process with reflecting boundary in $\nu\tau + a_0$ reads (see [66, 67])

$$\begin{aligned} P_{W,r}(y,\tau) = \Bigg\{ &\frac{1}{\sqrt{2\pi\tau}}\left[\exp\left(-\frac{y^2}{2\tau}\right) + \exp\left(-2a_0\nu - \frac{(y-2a_0)^2}{2\tau}\right)\right] \\ &-\nu\exp\left(-2\nu\left(y-a_0-\nu\tau\right)\right)\operatorname{erfc}\left(\frac{y-2a_0-2\nu\tau}{\sqrt{2\tau}}\right)\Bigg\}, \end{aligned} \tag{C.3}$$

in terms of the complementary error function erfc. The result, equation (15) in the main text, follows again by multiplication by $\partial\psi/\partial x$ and substituting $y \to \psi(x,t)$.

We conclude this appendix with a short remark on the appropriate reflecting boundary condition. This case is requires that the probability is conserved over time. Therefore, if $x_0 < a(t_0)$, integrating equation (5) in the time-dependent domain $[-\infty, a(t)]$, we get

$$\int_{-\infty}^{a(t)} \frac{\partial P(x,t|x_0,t_0)}{\partial t}\mathrm{d}x = -\int_{-\infty}^{a(t)} \frac{\partial j(x,t|x_0,t_0)}{\partial x}\mathrm{d}x. \tag{C.4}$$

The left hand side can be manipulated as follows

$$\begin{aligned} \int_{-\infty}^{a(t)} \frac{\partial P(x,t|x_0,t_0)}{\partial t}\mathrm{d}x = \frac{\partial}{\partial t}\int_{-\infty}^{a(t)} P(x,t|x_0,t_0)\,\mathrm{d}x \\ - \frac{\mathrm{d}a(t)}{\mathrm{d}t}P(a(t),t|x_0,t_0) = -a'(t)\,P(x,t|x_0,t_0), \end{aligned} \tag{C.5}$$

where we used the fact that $\int_{-\infty}^{a(t)} P(x,t|x_0,t_0)\mathrm{d}x = 1$. Therefore, the boundary condition for the flux is not $j(a(t),t|x_0,t_0) = 0$, but rather

$$j(a(t),t|x_0,t_0) = \frac{\mathrm{d}a(t)}{\mathrm{d}t}P(a(t),t|x_0,t_0). \tag{C.6}$$

Note that in the case of a constant boundary, i.e. $a'(t) = 0$, the last equation correctly states that the flux vanishes at the boundary. We could not find this modified boundary condition in any standard reference of the field.

Appendix D. Didactic example: Ornstein–Uhlenbeck FPT and square-root boundaries after space-time transformation

In this appendix, we show how to apply step-by-step the technique developed in the paper to the Ornstein–Uhlenbeck process (OUP), and how the FPTD of the OUP for a constant boundary coincides with the FPTD of the Wiener process for a square-root boundary. The SDE for the OUP reads

$$\mathrm{d}X_t = -\frac{\kappa}{\gamma}X_t\mathrm{d}t + \sqrt{\frac{2k_\mathrm{B}T}{\gamma}}\mathrm{d}W_t, \tag{D.1}$$

where κ is the stiffness of the harmonic potential. To be specific we consider the constant parameters and $x_0 = t_0 = 0$. From equation (2) we obtain that the Cherkasov function in this case reads

$$\mathcal{C} = \frac{\kappa}{\gamma}, \tag{D.2}$$

which does not depend either on x or t, and thus fulfils condition (3). We can thus apply our transformations; using equation (8) the space transformation reads

$$\psi(x,t) = \sqrt{\frac{\gamma}{2k_B T}} x e^{\kappa t/\gamma}, \tag{D.3}$$

while, using equation (9), the time transformation is

$$\tau(t) = \frac{\gamma}{2\kappa}\left(e^{2\kappa t/\gamma} - 1\right). \tag{D.4}$$

The propagator is immediately available substituting ψ and τ into equation (7).

Let us consider now the FPT for a constant threshold a. First, we need to understand how the boundary changes, i.e.

$$a \to \psi(a,t) = \sqrt{\frac{\gamma}{2k_B T}} a e^{\kappa t/\gamma}. \tag{D.5}$$

Second, we need to reparametrise the transformed boundary with respect to the new variable τ,

$$\psi(a,\tau) = a\sqrt{\frac{\gamma}{2k_B T}}\sqrt{\frac{2\kappa}{\gamma}\tau + 1}, \tag{D.6}$$

which is indeed a square-root function of τ. Therefore, we proved the equivalence between the FPTD we mentioned at the beginning of this appendix.

Using all the results available for the FPTD of the OU process [57, 95], we could, in principle, extend our results to square-root boundaries, as well.

ORCID iDs

Costantino Di Bello 0009-0007-6761-1411
Édgar Roldán 0000-0001-7196-8404
Ralf Metzler 0000-0002-6013-7020

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