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Anomalous and ultraslow diffusion of a particle driven by power-law-correlated and distributed-order noises

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Abstract

We study the generalised Langevin equation (GLE) approach to anomalous diffusion for a harmonic oscillator and a free particle driven by different forms of internal noises, such as power-law-correlated and distributed-order noises that fulfil generalised versions of the fluctuation-dissipation theorem. The mean squared displacement and the normalised displacement correlation function are derived for the different forms of the friction memory kernel. The corresponding overdamped GLEs for these cases are also investigated. It is shown that such models can be used to describe anomalous diffusion in complex

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media, giving rise to subdiffusion, superdiffusion, ultraslow diffusion, strong anomaly, and other complex diffusive behaviours.

Keywords: anomalous diffusion, generalized Langevin equation, mean squared displacement, correlation functions

1. Introduction

The behaviour of a test particle of mass m, that is coupled to a thermal bath of temperature, T, can be described by Newton's second law for a particle in the presence of a deterministic external potential $V(x)$ and a stochastically varying force $\xi(t)$. When the friction acting on the test particle is given by the constant value γ_0, the resulting dynamic is described by the standard Langevin equation for a Brownian particle [1–3],

$$m\ddot{x}(t) + m\gamma_0\dot{x}(t) + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = \xi(t), \quad \dot{x}(t) = v(t), \tag{1}$$

where $x(t)$ is the particle displacement and $v(t)$ is its velocity. The stochastic force, i.e. the 'noise', $\xi(t)$ is Gaussian, has zero mean, and is white such that its autocovariance is δ-correlated, $\langle\xi(t+t')\xi(t)\rangle = mk_\mathrm{B}T\gamma_0\delta(t')$ with thermal energy $k_\mathrm{B}T$. The mean-squared displacement (MSD) encoded in the stochastic equation (1) scales quadratically ('ballistically') at short times, at which it is dominated by inertial effects, and then crosses over to a linear time dependence beyond the characteristic time scale $1/\gamma_0$.

In the following, we will consider generalisations of the stochastic equation (1) that include 'memory', by which we mean non-localities in time. A simple example giving rise to memory is the Brownian harmonic oscillator ('Ornstein-Uhlenbeck process')

$$\dot{x}(t) = v(t), \quad m\dot{v} = -m\omega^2 x(t) - m\gamma_0 v(t) + \xi_v(t), \tag{2}$$

where $\xi_v(t)$ is zero-mean white Gaussian noise. The subscript v denotes that this noise appears in the equation for $\dot{v}(t)$. On purpose, we here use the phase space notation explicitly keeping the velocity $v(t)$ in the second equation. Suppose that $v(0) = 0$, so that we can integrate the second equation from $t = 0$, yielding

$$v(t) = \int_0^t \mathrm{e}^{-\gamma_0(t-t')}\left(-\omega^2 x(t') + \xi_v(t')/m\right)\mathrm{d}t'. \tag{3}$$

Substituting this result back into the first equation in (2), we find

$$\dot{x}(t) = -\int_0^t K(t')\,x(t-t')\,\mathrm{d}t' + \xi_x(t), \tag{4}$$

where we defined the memory kernel $K(t)$ and the positional friction $\xi_x(t)$,

$$K(t) = \omega^2\mathrm{e}^{-\gamma_0 t}, \quad \xi_x(t) = \frac{1}{m}\int_0^t \mathrm{e}^{-\gamma_0 t'}\xi_v(t-t')\,\mathrm{d}t'. \tag{5}$$

At sufficiently long times $t \ll 1/\gamma_0$ (or when we start the process at $t = -\infty$) the Ornstein-Uhlenbeck process reaches equilibrium and satisfies the fluctuation–dissipation theorem $\langle\xi_x(t)\xi_x(t')\rangle \sim \langle x^2\rangle_\mathrm{eq}K(|t-t'|)$ with the thermal value $\langle x^2\rangle_\mathrm{eq} = k_\mathrm{B}T/(m\omega^2)$.

From our result (4), which is characterised by an exponential memory, one thus cannot conclude that the underlying Ornstein–Uhlenbeck process is non-Markovian. However, we showed a general property: namely, when integrating out Markovian degrees of freedom, memory effects in the resulting equation for the test particle of interest emerges [2]. Indeed, much more pronounced memories, such as power-law forms, can be affected by eliminating a quasi-continuum of Markovian degrees of freedom, see, e.g. [2, 4–6].

For a general friction memory kernel $\gamma(t)$ the dynamics of a stochastic process is described in terms of the generalised Langevin equation (GLE) [2]

$$m\ddot{x}(t)+m\int_0^t \gamma(t-t')\dot{x}(t')\mathrm{d}t'+\frac{\mathrm{d}V(x)}{\mathrm{d}x}=\xi(t), \quad \dot{x}(t)=v(t). \tag{6}$$

The second fluctuation-dissipation theorem (FDT) is then valid in a thermal bath of temperature T, where fluctuations and dissipation come from the same source. The FDT relates the friction memory kernel $\gamma(t)$ with the correlation function $\xi(t)$ of the random force [2, 7, 8]. The FDT allows one to write

$$\langle \xi(t+t')\xi(t')\rangle = C(t) = k_{\mathrm{B}}T\gamma(t). \tag{7}$$

Here the friction memory kernel is assumed to satisfy [9]

$$\lim_{t\to\infty}\gamma(t)=\lim_{s\to 0} s\hat{\gamma}(s)=0, \tag{8}$$

where the hat denotes the Laplace transform of $\gamma(t)$, i.e. $\hat{\gamma}(s)=\mathscr{L}\{\gamma(t);s\}=\int_0^\infty \gamma(t)\mathrm{e}^{-st}\mathrm{d}t$. We note that when the noise is external in the sense of Klimontovich [10], i.e. the noise is not provided by a heat bath in a non-equilibrium systems, the relation (7) does not hold and $\langle\xi(t)\xi(t')\rangle = C(t-t')$ is used instead. When we again consider the harmonic oscillator and use the unit mass $m=1$ (i.e. $V(x)=\omega^2x^2/2$), then (6) can rewritten in the form [11]

$$x(t)=\langle x(t)\rangle+\int_0^t G(t-t')\xi(t')\mathrm{d}t', \quad v(t)=\langle v(t)\rangle+\int_0^t g(t-t')\xi(t')\mathrm{d}t', \tag{9}$$

where

$$\langle x(t)\rangle = x_0\left[1-\omega^2 I(t)\right]+v_0G(t), \quad \langle v(t)\rangle = v_0 g(t)-\omega^2 x_0 G(t) \tag{10}$$

are the average particle displacement velocity, respectively, given the initial conditions $x_0 = x(0)$ and $v_0 = v(0)$. Moreover, we introduced the so-called relaxation functions $G(t)$, $I(t) = \int_0^t G(\xi)\mathrm{d}\xi$ and $g(t)=\frac{\mathrm{d}G(t)}{\mathrm{d}t}$, which in the Laplace space read

$$\hat{G}(s)=\frac{1}{s^2+s\hat{\gamma}(s)+\omega^2}, \quad \hat{g}(s)=s\hat{G}(s), \quad \text{and} \quad \hat{I}(s)=s^{-1}\hat{G}(s). \tag{11}$$

These functions are used to calculate the following four fundamental quantities:

(i) the MSD [9]

$$\langle x^2(t)\rangle = 2k_{\mathrm{B}}TI(t)=2k_{\mathrm{B}}T\int_0^t G(\xi)\mathrm{d}\xi; \tag{12}$$

(ii) the diffusion coefficient $\mathcal{D}(t)=\frac{1}{2}\frac{\mathrm{d}\langle x^2(t)\rangle}{\mathrm{d}t}$ that, due to relation (12) can be expressed as

$$\mathcal{D}(t)=k_{\mathrm{B}}TG(t)=k_{\mathrm{B}}T\frac{\mathrm{d}I(t)}{\mathrm{d}t}, \tag{13}$$

compare the proof of relation (13) for the GLE in the free-force case in [8, 12];

(iii) the normalised displacement autocorrelation function (DACF), that can be experimentally measured which under the initial conditions $\langle x_0^2\rangle = k_B T/\omega^2$, $\langle x_0 v_0\rangle = 0$, and $\langle \xi(t) x_0\rangle = 0$, can be represented as [13, 14]

$$C_X = \frac{\langle x(t) x_0\rangle}{\langle x_0^2\rangle} = 1 - \omega^2 I(t) = 1 - \omega^2 \int_0^t G(\xi)\,\mathrm{d}\xi. \tag{14}$$

Different DACFs for fractional GLE are studied in [14];

(iv) the normalised velocity autocorrelation function (VACF) [9]

$$C_V(t) = \frac{\langle v(t) v_0\rangle}{\langle v_0^2\rangle} = g(t) = \frac{\mathrm{d}G(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 I(t)}{\mathrm{d}t^2}. \tag{15}$$

In the large friction limit, we neglect the inertial term $m\ddot{x}(t)$, and the resulting overdamped GLE (6) has the form

$$\int_0^t \gamma(t-t')\dot{x}(t')\,\mathrm{d}t' + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = \xi(t), \quad \dot{x}(t) = v(t). \tag{16}$$

The solution of this overdamped equation is of particular interest due to its application in modelling the anomalous dynamics of colloidal (micron-sized) test particles or the longer-time internal motion of proteins. Large friction, which appears due to the liquid environment means that the acceleration $\ddot{x}(t)$ is negligible in comparison to the effect of the friction term. Single-particle tracking of colloidal particles in an optical tweezers trap [15–19] or the internal motion of proteins can be considered as the effective motion in an harmonic potential [20, 21], the motion is described by the GLE

$$\int_0^t \gamma(t-t')\dot{x}(t')\,\mathrm{d}t' + m\omega^2 x(t) = \xi(t), \quad \dot{x}(t) = v(t). \tag{17}$$

The relaxation functions in the overdamped limit read

$$G_o(t) = \mathscr{L}^{-1}\left[\frac{1}{s\hat{\gamma}(s)+\omega^2};t\right], \; g_o(t) = \mathscr{L}^{-1}\left[\frac{s}{s\hat{\gamma}(s)+\omega^2};t\right], \text{and}$$
$$I_o(t) = \mathscr{L}^{-1}\left[\frac{s^{-1}}{s\hat{\gamma}(s)+\omega^2};t\right]. \tag{18}$$

For the case of a white Gaussian form for the noise $\xi(t)$ the GLE (6) corresponds to the classical overdamped Ornstein-Uhlenbeck process with friction coefficient γ_0. In the force-free case this further reduces to the Langevin equation (1) for a force-free Brownian particle. At times $t \gg 1/\gamma_0$ the MSD is then given by $\langle x^2(t)\rangle \sim 2[k_B T/(\gamma_0 m)]t$, and thus the diffusion coefficient becomes $\mathcal{D} = \lim_{t\to\infty}\langle x^2(t)\rangle/(2t) = k_B T/(m\gamma_0)$, whose physical dimensions are $[\mathcal{D}] = \text{length}^2\,\text{time}^{-1}$. The latter result for the diffusion coefficient of a Brownian particle in fact represents the Einstein-Smoluchowski-Sutherland relation [22–25].

Different forms for the friction memory kernel, particularly power-law forms [8, 9, 26–29] and Mittag–Leffler (ML) forms [11, 13, 14, 30, 31] have been introduced to model anomalous diffusion, for which the MSD scales non-linearly in time,

$$\langle x^2(t)\rangle = \frac{2\mathcal{D}_\mu}{\Gamma(1+\mu)} t^\mu, \tag{19}$$

where $\mathcal{D}_\mu$ is the generalised diffusion coefficient with physical dimension $[\mathcal{D}_\mu] = \mathrm{length}^2\,\mathrm{time}^{-\mu}$, and where α is the anomalous diffusion exponent. We distinguish the cases of subdiffusion $(0 < \mu < 1)$ and superdiffusion $(1 < \mu)$ [32]. Anomalous diffusion of this power-law form occurs in a multitude of systems across many scales [33–38] and it is non-universal in the sense that the MSD (19) for a given μ may emerge from a range of different anomalous stochastic processes [39–44]. We also mention that anomalous diffusion based on the GLE with crossovers to a different μ exponent or normal diffusion can be modelled in terms of tempered power-law kernels as introduced in [45] and applied to the anomalous diffusion of lipids in bilayer membranes [46]. Similar crossovers can be achieved in terms of formulations with distributed-order kernels as discussed below. We also note that non-Gaussian processes with power-law correlated noise was shown to emerge from a superstatistical approach based on the GLE [47, 48] and interactions of GLE dynamics with reflecting boundaries were analysed in [49].

In previous work [29] we considered the GLE for a free particle driven by a mixture of N independent internal white Gaussian noises

$$\xi(t) = \sum_{i=1}^{N} \alpha_i \xi_i(t), \tag{20}$$

such that each has zero mean, $\langle \xi_i(t)\xi_j(t')\rangle = 0$ and correlation

$$\langle \xi_i(t)\xi_i(t')\rangle = \delta_{ij}\zeta_i(t'-t), \tag{21}$$

where δ_{ij} is the Kronecker-δ. The correlation function of the additive noise $\xi(t)$ is then [29], see also [50],

$$\langle \xi(t)\xi(t')\rangle = \left\langle \sum_{i=1}^{N} \alpha_i \xi_i(t) \sum_{j=1}^{N} \alpha_j \xi_j(t') \right\rangle = \sum_{i=1}^{N} \alpha_i^2 \langle \xi_i(t)\xi_i(t')\rangle. \tag{22}$$

From the second FDT (7) we thus see that the noise fulfils

$$\sum_{i=1}^{N} \alpha_i^2 \zeta_i(t) = k_{\mathrm{B}} T \gamma(t), \tag{23}$$

where $\gamma(t)$ is the associated friction memory kernel. In [29] the GLE with internal noises of Dirac-δ, power-law and ML types were analysed, and various different diffusive regimes obtained. Moreover, it was shown that friction memory kernels of distributed order can be used to describe ultraslow diffusion with a logarithmic time dependence of the MSD. In what follows we consider a stochastic harmonic oscillator driven by a mixture of internal noises, from which we recover the results for a free particle in the limit of vanishing force constant. The purpose of introducing a mixture of different noises is due to some experimental observations, showing that two types of noise are needed to model the motion of a tracked particle in intracellular transport in biological cells [16, 51, 52].

Here we analyse the MSD and DACF for different forms of the friction memory kernel. In section 2 we specify the GLE approach to anomalous diffusion. We analyse the relaxation functions, MSD, and DACF for Dirac-δ, power-law, and combinations friction kernels. The corresponding overdamped limits are analysed and the force-free limit recovered. Distributed-order friction memory kernels are considered in section 3. It is shown that such kernels yield ultraslow diffusion, strong anomaly, and other complex behaviours. The overdamped motion of a harmonic oscillator driven by distributed-order noises is investigated in detail. A summary is presented in section 5.

2. GLE in presence of white and power-law noises

We study the GLE for both white and power-law noises and their combinations. At the end of this section, we also derive the overdamped limit.

2.1. Additive white noises

The simplest case of the GLE is obtained for a test particle connected to a thermal bath of temperature T effecting N additive internal white Gaussian and zero-mean noises, equation (20), in which each component fulfils $\zeta_i(t) = \delta(t)$ (i.e. $\hat{\zeta}_i(s) = 1$). Physically such a joint noise may stem from an environment with different components. As we assume that the noise is internal, from the second FDT we conclude that the friction memory kernel is given by

$$\hat{\gamma}(s) = \sum_{i=1}^{N} \frac{\alpha_i^2}{k_{\mathrm{B}}T} \tag{24}$$

in Laplace space. From relation (11), we obtain the relaxation function $G(t)$ in the form

$$\begin{aligned} G(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^2+\kappa s+\omega^2};t\right\} \\ &= \frac{2}{\sqrt{\kappa^2-4\omega^2}} \exp\left(-\frac{\kappa t}{2}\right) \sinh\left(\frac{t}{2}\sqrt{\kappa^2-4\omega^2}\right), \end{aligned} \tag{25}$$

where $\kappa = \sum_{i=1}^{N} \alpha_i^2/(k_{\mathrm{B}}T) > 2\omega$. The limit $\omega = 0$ leads to the known result $G(t) = [1-\exp(-\kappa t)]/\kappa$ [29], i.e. a constant diffusion coefficient in the linear time dependence of the MSD on time in the long time limit ($t \to \infty$),

$$\langle x^2(t)\rangle \sim 2\mathcal{D}t, \quad \text{where} \quad \mathcal{D} = \frac{(k_{\mathrm{B}}T)^2}{\sum_{i=1}^{N} \alpha_i^2}. \tag{26}$$

We note that in the short time limit $G(t) \sim t$, leading to $\langle x^2(t)\rangle \sim t^2$, which means ballistic motion. Writing $\kappa = 2\omega_c$ in terms of the compound frequency $\omega_c = \sum_{i=1}^{N} \alpha_i^2/(2k_{\mathrm{B}}T)$, then $\mathcal{D}(t) = t\exp(-\omega_c t)$, and the MSD is given by

$$\langle x^2(t)\rangle = \omega_c^{-2}\left[1-(1+\omega_c t)\,\mathrm{e}^{-\omega_c t}\right]. \tag{27}$$

Therefore, the relaxation function $I(t)$ from (14) with $I(t) = \omega^{-2}[1 - C_X(t)]$ is defined in terms of

$$C_X(t) = \mathrm{e}^{-\kappa t/2}\left[\cosh\left(\frac{t}{2}\sqrt{\kappa^2-4\omega^2}\right) + \frac{\kappa}{\sqrt{\kappa^2-4\omega^2}}\sinh\left(\frac{t}{2}\sqrt{\kappa^2-4\omega^2}\right)\right]. \tag{28}$$

With our notation $\kappa = 2\omega_c$ it is given by $C_X(t) = (1+\omega_c t)\mathrm{e}^{-\omega_c t}$. The case $\kappa < 2\omega$, i.e. the case of underdamped motion, yields in the form

$$C_X(t) = \mathrm{e}^{-\kappa t/2}\left[\cos\left(\frac{t}{2}\sqrt{4\omega^2-\kappa^2}\right) + \frac{\kappa}{\sqrt{4\omega^2-\kappa^2}}\sin\left(\frac{t}{2}\sqrt{4\omega^2-\kappa^2}\right)\right]. \tag{29}$$

Thus, the MSD in the long time limit approaches the equilibrium value $\langle x^2(t)\rangle_{\mathrm{eq}} = 2k_{\mathrm{B}}T/\omega^2$. A graphical representation of the NDCF $C_X(t)$ is given in figure 1. From panel (a) we conclude that in the overdamped case, the NDCF shows a monotonic decay to zero, without zero crossings. The underdamped case shows oscillatory behaviour of $C_X(t)$ with zero crossings (panels (b) and (c)), where oscillations become more pronounced for increasing oscillator frequency.

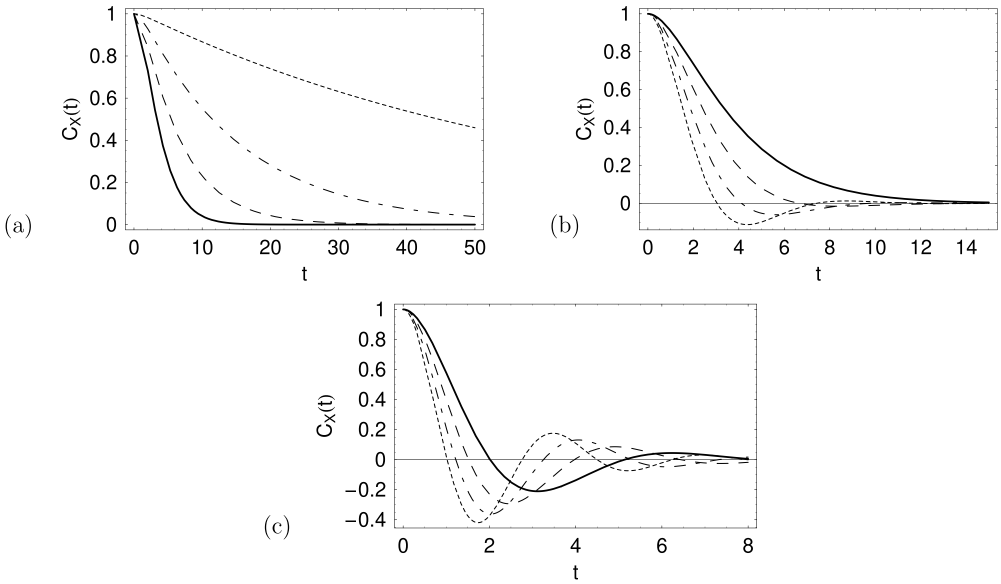

Figure 1. Normalised displacement correlation function for $\kappa = 1$: (a) overdamped motion $\kappa > 2\omega$ (28), $\omega = \omega_c = 1/2$ (solid line), $\omega = 3/8$ (dashed line); $\omega = 1/4$ (dot-dashed line); $\omega = 1/8$ (dotted line); (b) underdamped motion (29), $\omega = \omega_c = 1/2$ (solid line), $\omega = 5/8$ (dashed line); $\omega = 3/4$ (dot-dashed line); $\omega = 7/8$ (dotted line); (c) underdamped motion (29), $\omega = \omega_c = 9/8$ (solid line), $\omega = 11/8$ (dashed line); $\omega = 13/8$ (dot-dashed line); $\omega = 15/8$ (dotted line).

Remark 1. Let us analyse the diffusion coefficient in this case somewhat further. We see that if we consider a single internal white noise the diffusion coefficient is $\mathcal{D}_i = (k_\mathrm{B}T)^2/\alpha_i^2$, for $i = 1, 2, \ldots, N$. From relation (26), by using the relation between harmonic and arithmetic mean, we obtain

$$\mathcal{D} = \frac{1}{\sum_{i=1}^{N} \frac{1}{\mathcal{D}_i}} \leqslant \frac{\sum_{i=1}^{N} \mathcal{D}_i}{N^2}. \tag{30}$$

So we see that if the particular diffusion coefficients $\mathcal{D}_i$ are identical and equal to $\bar{\mathcal{D}}$, the diffusion coefficient for N independent internal white Gaussian noise terms scales inversely to N, $\mathcal{D} = \bar{\mathcal{D}}/N$.

If we write $\mathcal{D}_i = a^2/(2\tau_i)$ in a random walk-like notation, where a^2 represents the squared lattice constant or the variance of the jump length PDF [39, 40, 53], we rewrite the compound diffusion coefficient as

$$\mathcal{D} = \frac{1}{\sum_{i=1}^{N} \frac{2\tau_i}{a^2}} = \frac{a^2}{2\sum_{i=1}^{N} \tau_i} = \frac{a^2}{2N\langle\tau\rangle} = \frac{\overline{\mathcal{D}}}{N}. \tag{31}$$

In this sense the equality in (30) always holds. We note that the mean time $\langle\tau\rangle$ may be a misrepresentation of the largest time scale $\tau_m \max_i\{\tau_i\}$, depending on the underlying set $\{\tau_i\}$.

2.2. Different power-law noises

We now turn to generalising the results of [29], where the authors studied the GLE for free particles and N independent internal noises, by using a harmonic potential, corresponding to the Hookean force $F(x) = -m\omega^2 x$, in the GLE (6). To this end we first recall the considerations in [28], where the authors study the GLE for a harmonic potential and one internal noise, and then generalise these results. We mention that in [28] complementary polynomials are used to analyse the overdamped, underdamped, and critical behaviours of the oscillator. Here, our investigation is based on multinomial Prabhakar-type functions, $\mathcal{E}_{(\vec{\mu}),\beta}(s;\vec{a})$ (see [54] and appendix A). Despite the somewhat complicated notation, their definition and use in analytic calculations are in fact quite straightforward.

2.2.1. Power-law noises. We first consider the single-term internal power-law noise with correlation function $\zeta_1(t) = t^{-\lambda_1}/\Gamma(1-\lambda_1)$ (i.e. $\hat{\zeta}_1(s) = s^{\lambda_1-1}$) for $\lambda_1 \in (0,1)$. Thus, the friction memory kernel due to the second FDT (23) is given by $\gamma_1(t) = \alpha_1^2\zeta_1(t)/(k_\mathrm{B}T)$. Then the GLE (6) reads (using unit mass)

$$\ddot{x}(t) + \frac{\alpha_1^2}{k_\mathrm{B}T}\,{}^{C}D_t^{\lambda_1}x(t) + \omega^2 x(t) = \xi(t), \qquad \dot{x}(t) = v(t). \tag{32}$$

Due to the power-law friction memory, the friction terms can be compactly written in terms of the fractional Caputo derivative (${}^{C}D_t^{\lambda_1}$, and the GLE (32) is often referred to as the fractional Langevin equation [27, 40]. The Caputo operator for $\lambda_1 \in (0,1)$ is defined via ${}^{C}D_t^{\lambda_1}x(t) = \int_0^t \dot{x}(t')(t-t')^{-\lambda_1}\mathrm{d}t'/\Gamma(1-\lambda_1)$, see for example [50, 55]. The relaxation functions $G_1(t)$, $I_1(t)$, and $g_1(t)$ are obtained by inverting the Laplace transform (11). They are given in terms of the multinomial Prabhakar-type functions $\mathcal{E}_{(\vec{\mu}),\beta}(s;\vec{a}) = t^{\beta-1}E_{(\vec{\mu}),\beta}(-a_1t^{\mu_1},\ldots,-a_Nt_N^{\mu})$ where $\vec{\mu} = \mu_1,\ldots\mu_N$ and $\vec{a} = a_1,\ldots,a_N$ (see [54] and appendix A) as

$$G_1(t) = \mathscr{L}^{-1}\left\{\left(s^2 + A_1 s^{\lambda_1} + \omega^2\right)^{-1};t\right\} = \mathcal{E}_{(2,2-\lambda_1),2}\left(t;\omega^2,A_1\right), \tag{33}$$

and

$$I_1(t) = \mathcal{E}_{(2,2-\lambda_1),3}\left(t;\omega^2,A_1\right), \quad g_1(t) = \mathcal{E}_{(2,2-\lambda_1),1}\left(t;\omega^2,A_1\right), \tag{34}$$

where $A_1 = \alpha_1^2/(k_\mathrm{B}T)$. For $t \gg 1$ these functions are estimated by the two-parameter ML function, see (A.8), whose asymptotic behaviours follow from equation (A.10), such that

$$G_1(t) \sim_{t\to\infty} A_1^{-1}t^{\lambda_1-1}E_{\lambda_1,\lambda_1}\left(-\omega^2t^{\lambda_1}/A_1\right) \sim_{t\to\infty} -\frac{A_1}{\omega^4}\frac{t^{-\lambda_1-1}}{\Gamma(-\lambda_1)} = \frac{A_1\lambda_1}{\omega^4}\frac{t^{-\lambda_1-1}}{\Gamma(1-\lambda_1)}, \tag{35}$$

$$I_1(t) \sim_{t\to\infty} A_1^{-1}t^{\lambda_1}E_{\lambda_1,\lambda_1+1}\left(-\omega^2t^{\lambda_1}/A_1\right) \sim_{t\to\infty} \frac{1}{\omega^2}\left[1 - \frac{A_1}{\omega^2}\frac{t^{-\lambda_1}}{\Gamma(1-\lambda_1)}\right], \tag{36}$$

and

$$g_1(t) \sim_{t\to\infty} A_1^{-1}t^{\lambda_1-2}E_{\lambda_1,\lambda_1-1}\left(-\omega^2t^{\lambda_1}/A_1\right) \sim_{t\to\infty} -\frac{A_1}{\omega^4}\frac{t^{-\lambda_1-2}}{\Gamma(-1-\lambda_1)}. \tag{37}$$

The same results can be obtained by calculating the appropriate overdamped relaxation functions such that we have $\lim_{t\to\infty} G_1(t) = G_{o;1}(t)$, $\lim_{t\to\infty} I_1(t) = I_{o;1}(t)$, and $\lim_{t\to\infty} g_1(t) = g_{o;1}(t)$. From the result for the relaxation function $I_1(t)$ we conclude that the MSD in the long time limit approaches the equilibrium (thermal) value $\langle x^2(t)\rangle_\mathrm{eq} = 2k_\mathrm{B}T/\omega^2$ via a power-law decay, i.e.

$$\langle x^2(t)\rangle \sim_{t\to\infty} \langle x^2(t)\rangle_\mathrm{eq}\left[1 - \frac{A_1}{\omega^2}\frac{t^{-\lambda_1}}{\Gamma(1-\lambda_1)}\right]. \tag{38}$$

By asymptotic analysis in the short time limit ballistic motion is also observed since the MSD behaves as $\langle x^2(t)\rangle \sim \frac{t^2}{2} - A_1 \frac{t^{4-\lambda_1}}{\Gamma(5-\lambda_1)}$.

For the force-free case ($\omega = 0$) the MSD becomes

$$\begin{aligned}\langle x^2(t)\rangle &= 2k_{\mathrm{B}}T\mathcal{L}^{-1}\left[\frac{s^{-1-\lambda_1}}{s^{2-\lambda_1}+A_1}\right]\\ &= 2k_{\mathrm{B}}Tt^2E_{2-\lambda_1,3}\left(-A_1t^{2-\lambda_1}\right) \sim \begin{cases} t^2 - A_1\frac{t^{4-\lambda_1}}{\Gamma(5-\lambda_1)}, & t\to 0,\\ t^{\lambda_1}, & t\to\infty,\end{cases}\end{aligned} \tag{39}$$

from which we conclude that from ballistic motion in the short time limit, the particle turns to subdiffusion in the long time limit [29].

2.2.2. N power-law noises. The compound fractional Langevin equation, i.e. the GLE in which we use N independent internal noises, for $m=1$ reads

$$\ddot{x}(t) + \sum_{r=1}^{N} A_r\, {}^{C}D_t^{\lambda_r}x(t) + \omega^2 x(t) = \xi(t). \tag{40}$$

The friction memory kernel $\gamma_N(t)$ becomes $\sum_{r=1}^{N}\gamma_r(t)$, where $\gamma_r(t)$ is defined analogously to $\gamma_1(t)$ above, but instead of a single internal noise we now have N. Here, we assume that $0 < \lambda_1 < \cdots < \lambda_N < 1$. Using short-hand notation we denote the relaxation functions $g_N(t)$, $G_N(t)$, and $I_N(t)$ by the function $F_{N;j}(t)$ indexed by $j=1,2,3$ such that $F_{N;1}(t) = g_N(t)$ is obtained for $j=1$, $F_{N;2}(t) = G_N(t)$ for $j=2$, and $F_{N;3}(t) = I_N(t)$ for $j=3$. The auxiliary functions $F_{N;j}(t)$ read

$$F_{N;j}(t) = \mathcal{E}_{(2,2-\lambda_1,\dots,2-\lambda_N),j}\left(t;\omega^2,A_1,\dots,A_N\right) \tag{41}$$

in terms of the Prabakhar-type function. Following [54] we check the behaviour of $F_{N;j}(t)$ in (41) at long t. In this case we obtain that $F_{N;j}(t)$ tends to the relaxation function for $N=1$, i.e.

$$F_{N;j}(t) \sim_{t\to\infty} A_1^{-1}t^{\lambda_1+j-3}E_{\lambda_1,\lambda_1+j-2}\left(-\omega^2t^{\lambda_1}/A_1\right). \tag{42}$$

Their further asymptotics in the long-time limit $t\to\infty$ depend on the value of j, such that for $j=1,2,3$ we get equations (35)–(37), respectively. Making use of equation (42) we calculate the quantities (i) to (iv) defined in the introductory section is necessary to describe the time evolution of stochastic systems coupled to a thermal bath.

Based on these results the MSD can be shown to be given by the expression

$$\langle x^2(t)\rangle_N = 2k_{\mathrm{B}}T\mathcal{E}_{(2,2-\lambda_1,\dots,2-\lambda_N),3}\left(t;\omega^2,A_1,\dots,A_N\right). \tag{43}$$

Thus, by asymptotic analysis, in the long time limit, we find the behaviour

$$\langle x^2(t)\rangle_N \sim_{t\to\infty} \langle x^2(t)\rangle_{\mathrm{eq}}\left[1 - \frac{A_1}{\omega^2}\frac{t^{-\lambda_1}}{\Gamma(1-\lambda_1)}\right]. \tag{44}$$

The MSD approaches the equilibrium (thermal) value $\langle x^2(t)\rangle_{\mathrm{eq}} = 2k_{\mathrm{B}}T/\omega^2$ in power-law fashion, instead of the exponentially fast relaxation for the normal Ornstein-Uhlenbeck process. Note that similar power-law relaxations are known from subdiffusive continuous time random walks [32, 56] and from the time-averaged MSD of the fractional Langevin equation [17, 57] in an external harmonic potential. From equation (44) we see that the noise with the smaller exponent λ_1 has the dominant contribution to the oscillator behaviour in the long time limit. We also note that the same result for the MSD in the long time limit can be obtained by Tauberian theorems (see appendix B) if we analyse the behaviour of $\hat{I}_N(s)$ in the limit $s \to 0$ [58]. The short time limit $t \to 0$ yields $I_N(t) \simeq t^2/2 - A_N t^{4-\lambda_N}/\Gamma(5-\lambda_N)$, so we conclude that the noise with the largest exponent λ_N dominates the dynamic in the short time limit.

For the force-free case ($\omega = 0$), we recover the result for the MSD obtained in [29], which in the short time limit behaves as $\langle x^2(t)\rangle_N \sim t^2 - A_1 \frac{t^{4-\lambda_N}}{\Gamma(5-\lambda_N)}$, while in the long time limit we find $\langle x^2(t)\rangle_N \sim t^{\lambda_1}$. Thus, the highest exponent is the dominant contribution in the short time limit, while the lowest exponent has the analogous role in the long time limit.

The DACF (14) in the case of thermal initial conditions $\langle x_0^2\rangle = k_B T/\omega^2$, $\langle x_0 v_0\rangle = 0$, and $\langle \xi(t) x_0\rangle = 0$, from equations (14) and (41), we obtain

$$C_X(t) = 1 - \omega^2 \mathcal{E}_{(2,2-\lambda_1,\ldots,2-\lambda_N),3}\left(t;\omega^2, A_1,\ldots,A_N\right). \tag{45}$$

Its asymptotic behaviour in the long time limit via equation (36) reads

$$C_X(t) \sim_{t\to\infty} 1 - \frac{\omega^2}{A_1} t^{\lambda_1} E_{\lambda_1,\lambda_1+1}\left(-\frac{\omega^2}{A_1} t^{\lambda_1}\right) = E_{\lambda_1}\left(-\frac{\omega^2}{A_1} t^{\lambda_1}\right) \sim_{t\to\infty} \frac{A_1}{\omega^2}\frac{t^{-\lambda_1}}{\Gamma(1-\lambda_1)}. \tag{46}$$

Thus we obtain a power-law decay, approaching the zero line from positive values for $\lambda_1 \in (0,1)$, and from negative values for $\lambda_1 \in (1,2)$, if we consider a memory kernel defined in Laplace space as $\hat{\gamma}(s) = \sum_{i=1}^N s^{\lambda_i - 1}$, $1 < \lambda_i < 2$. Since $E_\alpha(-x)$ is a completely monotone function for $\alpha \in (0,1)$ [59–62], we conclude that the normalised displacement correlation function is completely monotone in the long time limit for $\lambda_1 \in (0,1)$. A more detailed analysis of $C_X(t)$ is provided in section 2.4 for the case of high damping, which has more practical application in the theory of anomalous dynamics in single particle tracking and protein dynamics. We mention that the equality in equation (46) was proven in [63] and [64, remark 3].

2.3. Combinations of white and power-law noises

In the same way, as for the power-law noises, we can analyse the relaxation functions for a mixture of δ- and power-law distributed noises. The approach given in [29] can be applied in the case of a harmonic oscillator driven by P white noises and Q power-law noises ($P + Q = N$) as well. Here we consider the special case $\gamma(t) = B_1\delta(t) + B_2 t^{-\lambda}/\Gamma(1-\lambda)$ where $B_1 = \alpha^2/(k_B T)$, $B_2 = \beta^2/(k_B T)$, and $0 < \lambda < 1$ [29, 65].

With the help of equation (A.2) we see that

$$G(t) = \mathcal{E}_{(2,2-\lambda,1),2}\left(t;\omega^2, B_2, B_1\right). \tag{47}$$

From equations (A.3) and (A.4) we can calculate the associated integral and derivative which, respectively, lead to $I(t)$ and $g(t)$. Then, the MSD obtained from equation (12) reads

$$\langle x^2(t)\rangle = 2k_B T \mathcal{E}_{(2,2-\lambda,1),3}\left(t;\omega^2, B_2, B_1\right). \tag{48}$$

Notice that equation (48) in the force-free case $\omega = 0$, after applying equation (A.15), can be recognised as [29, equation (27) for $\lambda_1 = \lambda$, $\lambda_2 = 1$], namely,

$$\langle x^2(t)\rangle = 2k_B T \sum_{n=0}^{\infty} (-B_2)^n t^{(2-\lambda)n+2} E^{n+1}_{1,(2-\lambda)n+3}(-B_1 t). \tag{49}$$

The asymptotic of the MSD at short and long times are calculated from equations (A.5) and (A.6), yielding

$$\langle x^2(t)\rangle \sim_{t\to 0} \frac{t^2}{2} - \frac{t^3}{3!} - \frac{t^4}{4!} - \frac{t^{4-\lambda}}{\Gamma(5-\lambda)} \tag{50}$$

for $t\to 0$, and for $t\gg 1$ we have

$$\langle x^2(t)\rangle \sim_{t\to\infty} \frac{2k_{\mathrm{B}}T}{B_2} t^{\lambda} E_{\lambda,1+\lambda}\left(-\frac{\omega^2}{B_2}t^{\lambda}\right) \sim_{t\to\infty} \frac{2k_{\mathrm{B}}T}{\omega^2}\left[1 - \frac{B_2}{\omega^2}\frac{t^{-\lambda}}{\Gamma(1-\lambda)}\right], \tag{51}$$

which means that the MSD has a power-law decay to the equilibrium value $\langle x^2(t)\rangle_{\mathrm{eq}} = 2k_{\mathrm{B}}T/\omega^2$. We conclude that the power-law noise is dominant in the long time limit, and the white noise in the short time limit, as naively expected. We note that in the force-free case ($\omega = 0$), in the long time limit the particle shows anomalous diffusion of the form $\langle x^2(t)\rangle \sim t^{\lambda}$, while in the short time limit ballistic motion is observed. This means that the fractional exponent has a dominant contribution in the long time limit [29].

For the DACF $C_X(t)$, we obtain from relation (14) that

$$C_X(t) = 1 - \omega^2 \mathcal{E}_{(2,2-\lambda,1),3}\left(t;\omega^2,B_2,B_1\right), \tag{52}$$

from where the long time limit follows,

$$C_X(t) \sim_{t\to\infty} 1 - \frac{\omega^2}{B_2} t^{\lambda} E_{\lambda,1+\lambda}\left(-\frac{\omega^2}{B_2}t^{\lambda}\right) \sim_{t\to\infty} \frac{B_2}{\omega^2}\frac{t^{-\lambda}}{\Gamma(1-\lambda)}. \tag{53}$$

We conclude that $C_X(t)$ in the long time limit is a completely monotone function since $0 < \lambda < 1$, showing a power-law decay to zero.

Following the same procedure one may consider mixtures of white noises, power-law noises, and ML type noises. The calculation of the relaxation function can be represented in terms of multinomial Prabhakar functions [29, 54].

2.4. Overdamped limit

Next, we analyse the high-damping limit, in which the inertial term $m\ddot{x}(t)$ can be neglected. Then, the relaxation functions are defined by equation (18).

The case of a mixture of N internal white noises considered in section 2.1, for the MSD $\langle x^2(t)\rangle_{\mathrm{o}} = 2k_{\mathrm{B}}TI_{\mathrm{o}}(t)$ and the DACF $C_{X,\mathrm{o}}(t)$ yields

$$\langle x^2(t)\rangle_{\mathrm{o}} = 2k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{\frac{s^{-1}}{\kappa s+\omega^2};t\right\} = \frac{2k_{\mathrm{B}}T}{\omega^2}\left[1-\exp\left(-\frac{\omega^2}{\kappa}t\right)\right], \tag{54}$$

$$C_{X,\mathrm{o}}(t) = \exp\left(-\frac{\omega^2}{\kappa}t\right) = \exp\left(-\frac{\mathcal{D}\omega^2}{k_{\mathrm{B}}T}t\right), \tag{55}$$

where κ is given below equation (25) and $\mathcal{D}$ is defined in equation (26). We find that $C_{X,o}(t)$ has a monotonic exponential decay, as expected. In the short time limit the MSD has a linear dependence on time, $\langle x^2(t)\rangle_{\mathrm{o}} \sim t$, as it is expected.

For the case of N power-law noises, it follows from relation (12) in the overdamped case that

$$\langle x^2(t)\rangle_{\mathrm{o}} = \frac{2k_{\mathrm{B}}T}{A_N}\mathcal{E}_{(\lambda_N,\lambda_N-\lambda_1,\ldots,\lambda_N-\lambda_{N-1});\lambda_N+1}\left(t,\frac{\omega^2}{A_N},\frac{A_1}{A_N},\ldots,\frac{A_{N-1}}{A_N}\right), \tag{56}$$

$$C_{X,\mathrm{o}}(t) = 1 - \frac{\omega^2}{A_N}\mathcal{E}_{(\lambda_N,\lambda_N-\lambda_1,\ldots,\lambda_N-\lambda_{N-1});\lambda_N+1}\left(t, \frac{\omega^2}{A_N}, \frac{A_1}{A_N}, \ldots, \frac{A_{N-1}}{A_N}\right). \quad (57)$$

In the long time limit the particle shows a power-law decay of the MSD to the equilibrium value of the form (44), which means that the lowest fractional exponent has a dominant contribution, while in the short time limit anomalous diffusion of the form $\langle x^2(t)\rangle_{\mathrm{o}} \sim t^{\lambda_N}$ is observed, i.e. the highest fractional exponent has the dominant contribution at short times. For the force-free case ($\omega = 0$) the MSD shows characteristic crossover from $\langle x^2(t)\rangle_{\mathrm{o}} \sim t^{\lambda_N}$ for short times to $\langle x^2(t)\rangle_{\mathrm{o}} \sim t^{\lambda_1}$ for long times, i.e. decelerating subdiffusion.

The DACF shows an asymptotic power-law decay in the long time limit, i.e.

$$C_{X,\mathrm{o}}(t) \sim_{t\to\infty} E_{\lambda_1}\left(-\frac{\omega^2}{A_1}t^{\lambda_1}\right) \sim_{t\to\infty} \frac{A_1}{\omega^2}\frac{t^{-\lambda_1}}{\Gamma(1-\lambda_1)}. \quad (58)$$

Thus, we can use equation (16) in the overdamped limit case, which is simpler instead of the GLE (6), to analyse the asymptotic behaviour of the harmonic oscillator in the long time limit.

A graphical representation of the DACF is presented in figure 2. From panel (a) we see that by changing the values of the parameters λ_1 and λ_2 for fixed frequency ω there appears a non-monotonic decay of $C_{X,o}(t)$ without zero crossings, approaching the zero line at infinity (see solid line), a monotonic decay without zero crossings to zero at infinity (dashed line), a non-monotonic decay without zero crossings approaching zero at a finite time instant and at infinity (dot-dashed line), or a non-monotonic decay with zero crossings approaching zero at infinity (dotted line). All long-time decays of the DACF are of power-law form to zero, as it can be anticipated from relation (58). The behaviour of $C_X(t)$ for different values of the frequency ω and fixed values of λ_1 and λ_2 is shown in figures 2(b) and (c). We see that there are different critical frequencies, i.e. the frequency at which $C_{X,o}(t)$ changes its behaviour, for instance, from non-monotonic to monotonic decay without zero crossings, or the frequency at which $C_{X,o}(t)$ crosses the zero line. Such different types of critical frequencies were discussed in [28].

Figure 3 depicts the MSD for unconfined and confined motion. For free motion the MSD may have monotonic, non-monotonic, and oscillatory behaviour, turning into a power-law form in the long time limit. For the confined case the MSD has different behaviours at intermediate times, and in the long-time limit, the MSD has a power-law approach to the equilibrium value $\langle x^2(t)\rangle_{\mathrm{eq}} = 2k_{\mathrm{B}}T/\omega^2$.

In the same way as above, we obtain the following DACF for the case of a mixture of a white noise a and power-law noise (see results 2.3),

$$C_{X,\mathrm{o}}(t) = \sum_{n=0}^{\infty}\left(-\frac{\omega^2}{B_1}\right)^n t^n E^n_{1-\lambda,n+1}\left(-\frac{B_2}{B_1}t^{1-\lambda}\right), \quad (59)$$

for $0 < \lambda < 1$, and

$$C_{X,\mathrm{o}}(t) = \sum_{n=0}^{\infty}\left(-\frac{\omega^2}{B_2}\right)^n t^{\lambda n} E^n_{\lambda-1,\lambda n+1}\left(-\frac{B_1}{B_2}t^{\lambda-1}\right), \quad (60)$$

for $1 < \lambda < 2$, where B_1 and B_2 are given at the beginning of section 2.3. These results are equivalent to those obtained in section 2.3 in the long time limit for the GLE when the inertial term is not neglected, as it should be.

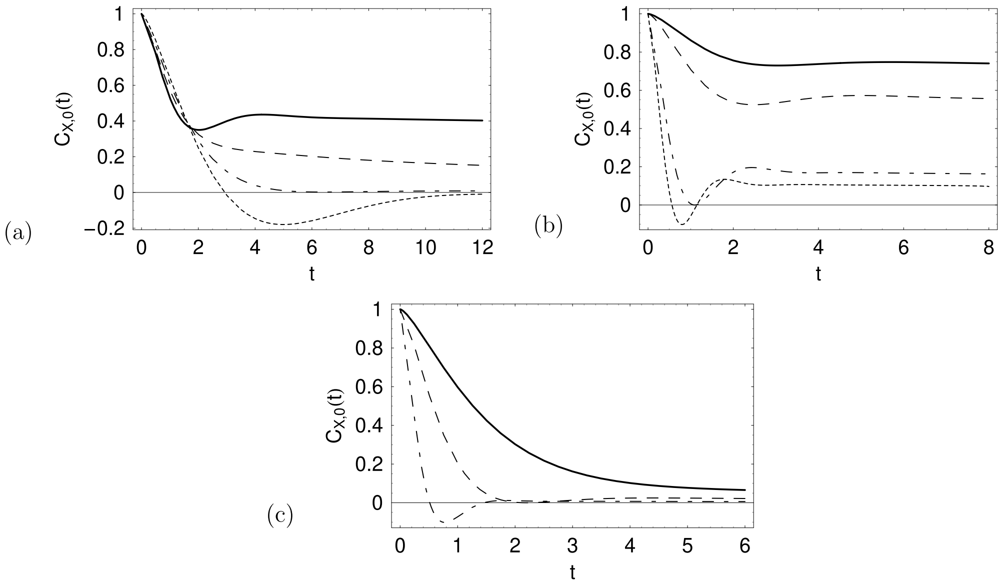

Figure 2. Normalised displacement correlation function (57) for the following cases for $N = 2$: (a) $\omega = 1$, $\lambda_2 = 3/2$, $\lambda_1 = 1/8$, (solid line), $\lambda_1 = 1/2$ (dashed line), $\lambda_1 = 7/8$ (dot-dashed line), $\lambda_1 = 5/4$ (dotted line); (b) $\lambda_1 = 1/8$, $\lambda_2 = 3/4$, $\omega = 0.5$ (solid line), $\omega = 0.75$ (dashed line), $\omega = 1.8959706$ (dot-dashed line), $\omega = 2.5$ (dotted line); (c) $\lambda_1 = 3/4$, $\lambda_2 = 3/2$, $\omega = 1$ (solid line), $\omega = 1.617$ (dashed line), $\omega = 3$ (dot-dashed line).

3. Distributed-order Langevin equations

It has been shown that the distributed-order differential equations are suitable tools for modelling ultraslow relaxation and diffusion processes [29, 55, 66–70]. In the case of distributed order differential equations, one uses the following memory kernel (see for example [70])

$$\gamma(t) = (k_B T)^{-1} \int_0^1 p(\lambda) \frac{t^{-\lambda}}{\Gamma(1-\lambda)} \mathrm{d}\lambda, \tag{61}$$

where $p(\lambda)$ is a dimensionless, non-negative weight function with $\int_0^1 p(\lambda)\mathrm{d}\lambda = c$, where c is a constant. When $c = 1$, $p(\lambda)$ is normalised. The distributed order memory kernel mixes fractional exponents from 0 to 1, and is obtained when the summation in the memory kernel in case of N fractional power-law functions turns to integration. If we substitute the distributed-order memory kernel in the GLE (6) we transform it to the following distributed-order Langevin equation

$$\ddot{x}(t) + \frac{1}{k_B T} \int_0^1 p(\lambda)\, {}^{C}D_t^{\lambda} x(t)\, \mathrm{d}\lambda + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = \xi(t), \quad \dot{x}(t) = v(t). \tag{62}$$

Here we note that assumption (8) is satisfied for distributed-order Langevin equations since $\lim_{s\to 0} s\hat{\gamma}(s) \sim \lim_{s\to 0} \int_0^1 p(\lambda) s^{\lambda} \mathrm{d}\lambda = 0$. Thus we can use the representations of MSD, VACF, time-dependent diffusion coefficient and DACFs in terms of the relaxation functions.

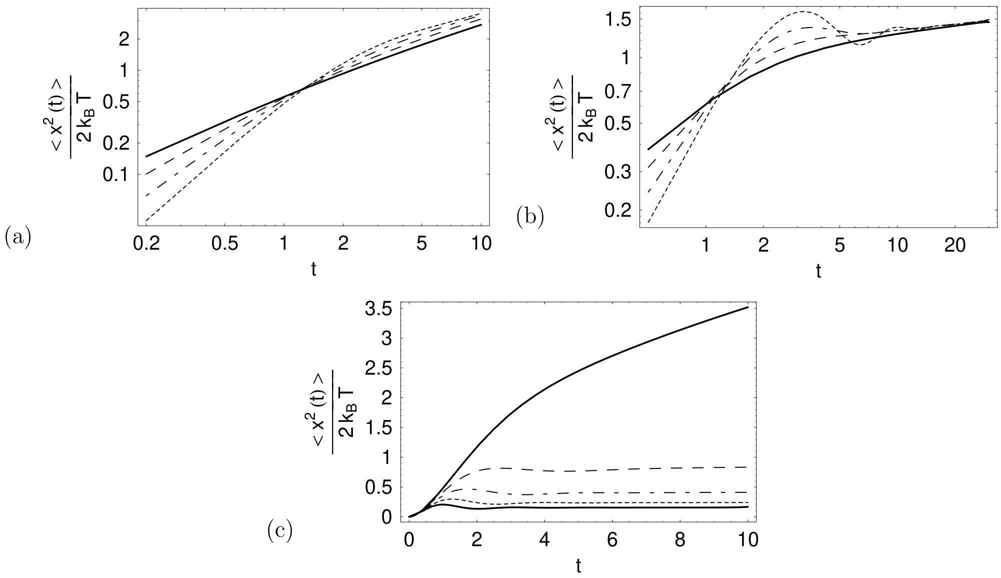

Figure 3. MSD (56) for the following cases for $N = 2$: (a) $\omega = 0$, $\lambda_1 = 1/2$, $\lambda_2 = 1$, (solid line), $\lambda_2 = 5/4$ (dashed line), $\lambda_2 = 3/2$ (dot-dashed line), $\lambda_2 = 7/4$ (dotted line); (b) $\omega = 0$, $\lambda_1 = 0.1$, $\lambda_2 = 1$, (solid line), $\lambda_2 = 5/4$ (dashed line), $\lambda_2 = 3/2$ (dot-dashed line), $\lambda_2 = 7/4$ (dotted line); (c) $\lambda_1 = 1/2$, $\lambda_2 = 7/4$, $\omega = 0$ (upper solid line), $\omega = 1$ (dashed line), $\omega = 1.5$ (dot-dashed line), $\omega = 2$ (dotted line), $\omega = 2.5$ (lower solid line).

We now consider the distributed-order Langevin equation (62) in the presence of the constant external force F. Then we have

$$\ddot{x}(t) + \frac{1}{k_{\mathrm{B}}T}\int_0^1 p(\lambda)\,{}^C D_t^{\lambda} x(t)\,\mathrm{d}\lambda - F = \xi(t), \quad \dot{x}(t) = v(t), \tag{63}$$

from which, by the Laplace transform method, we obtain

$$x(t) = \langle x(t)\rangle_F + \int_0^t G(t-t')\,\xi(t')\,\mathrm{d}t', \tag{64}$$

where $\langle x(t)\rangle_F = FI(t)$ and $\hat{\gamma}(s)$ is given by equation (61), and where

$$G(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2 + s\hat{\gamma}(s)};t\right\} \quad \text{and} \quad I(t) = \mathscr{L}^{-1}\left\{\frac{s^{-1}}{s^2 + s\hat{\gamma}(s)};t\right\}. \tag{65}$$

The latter expression corresponds to (11) for a free particle ($\omega = 0$). Thus, we conclude that the generalised Einstein relation $\langle x^2(t)\rangle_F = [F/(2k_{\mathrm{B}}T)]\langle x^2(t)\rangle_{F=0}$ is satisfied for the distributed-order Langevin equation (62), where $\langle x^2(t)\rangle_{F=0} = 2k_{\mathrm{B}}T\mathscr{L}^{-1}\{s^{-1}[s^2 + s\hat{\gamma}(s)]^{-1};t\}$ is the MSD for the case of a free particle.

3.1. Force-free case

We first consider the distributed-order Langevin equation for a free particle with $\omega = 0$. Note that a weight function of the form $p(\lambda) = \sum_{i=1}^{N}\alpha_i^2\,\delta(\lambda - \lambda_i)$ yields the fractional Langevin equation and we have the compound white Gaussian noise, see section 2.2.2. For $p(\lambda) = \alpha^2$ we obtain the uniformly distributed noise [69]

$$k_B T\gamma(t) = \alpha^2 \int_0^1 \frac{t^{-\lambda}}{\Gamma(1-\lambda)} \mathrm{d}\lambda, \tag{66}$$

which was used by Kochubei in the theory of evolution equations. In [29] we analysed the GLE for a free particle with friction memory kernel of distributed order (66) and showed that the MSD in the long time limit is given by

$$\langle x^2(t)\rangle \sim_{t\to\infty} \frac{2(k_B T)^2}{\alpha^2}\left[C + \log t + \mathrm{e}^t \mathrm{E}_1(t)\right], \tag{67}$$

where $C = 0.577\,216$ is the Euler-Mascheroni (or Euler's) constant, $\mathrm{Ei}(-t) = -\int_t^\infty (\mathrm{e}^{-x}/x)\mathrm{d}x$ is the exponential integral [71], and $\mathrm{E}_1(t) = -\mathrm{Ei}(-t)$. From the asymptotic expansion $\mathrm{E}_1(t) \sim_{t\to\infty} t^{-1}\mathrm{e}^{-t}\sum_{k=0}^{n-1}(-1)^k k! t^{-k}$ [55], which has an error of order $\mathcal{O}(n!t^{-n})$, we obtained that the particle shows ultraslow diffusion, i.e. $\langle x^2(t)\rangle \sim_{t\to\infty} [2(k_B T)^2/\alpha^2](C + \log t)$ [29]. In the short time limit, again ballistic motion is observed.

Consider now the power-law case $p(\lambda) = \alpha^2\beta\lambda^{\beta-1}$, where $\beta > 0$. Then the friction memory kernel becomes

$$k_B T\gamma(t) = \alpha^2 \int_0^1 \beta\lambda^{\beta-1} \frac{t^{-\lambda}}{\Gamma(1-\lambda)} \mathrm{d}\lambda. \tag{68}$$

For the MSD in the long time limit, we obtain (see (65) based on Tauberian theorems)

$$\langle x^2(t)\rangle \sim_{t\to\infty} \frac{2(k_B T)^2}{\alpha^2} \frac{\log^\beta t}{\Gamma(1+\beta)}. \tag{69}$$

From this result, we conclude that this power-law model leads to ultraslow diffusion (for $\beta = 1$) or a strong anomaly. In [72] a strong anomaly means the behaviour of form $\langle x^2(t)\rangle \sim \log^\nu t$, which for $\nu = 4$ has the same form as the MSD of the Sinai diffusion model in a random force field [73, 74], compare also to the discussion in [75]. We also note that the Sinai diffusion here occurs due to the friction memory kernel, while in a quenched disorder landscape the Sinai diffusion occurs due to the thermal random motion of a particle in a random potential [76, 77]. Thus the physics of the logarithmic anomalous diffusion is different in both cases. However, as demonstrated in [73], the statistical behaviour in terms of the MSD and PDF are strikingly similar. Result (69) for the MSD is equivalent to the one obtained by Eab and Lim [68], since the long time limit considered here corresponds to the case of the overdamped limit (the inertial term $\ddot{x}(t)$ is neglected) studied in [68]. The same result for the MSD can be obtained from the distributed-order diffusion equation [67] in which the weight function corresponds to the one considered in the memory kernel (68).

Next we consider the weight function $p(\lambda) = \alpha^2/(\lambda_2 - \lambda_1)$ for $0 \leqslant \lambda_1 < \lambda < \lambda_2 \leqslant 1$, and $p(\lambda) = 0$ otherwise [66]. With the help of relation (65) the MSD yields in the form

$$\begin{aligned}\langle x^2(t)\rangle = 2k_B T\Bigg[\frac{t^2}{2} + \sum_{n=1}^{\infty}\left(-\frac{\alpha^2}{k_B T}\right)^n \frac{1}{(\lambda_2-\lambda_1)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \\ \times \mu(t, n-1, (2-\lambda_2)n + (\lambda_2-\lambda_1)k + 2)\Bigg],\end{aligned} \tag{70}$$

which in the long time limit becomes (use the Tauberian theorem in equation (65))

$$\begin{aligned}\langle x^2(t)\rangle \sim_{t\to\infty} \frac{2(k_B T)^2}{\alpha^2}\sum_{n=1}^{\infty}\frac{t^{\lambda_2-(\lambda_2-\lambda_1)n}}{\Gamma(\lambda_2-(\lambda_2-\lambda_1)n+1)} \\ \times\left[\ln t^{\lambda_2-\lambda_1} - (\lambda_2-\lambda_1)\psi(\lambda_2-(\lambda_2-\lambda_1)n+1)\right],\end{aligned} \tag{71}$$

where we used the Volterra function $\mu(t,\beta,\alpha)$ defined in appendix C, and $\psi = \frac{\Gamma'}{\Gamma}$ is the digamma function. Thus, we conclude that in the long time limit the MSD has the ultraslow form

$$\langle x^2(t)\rangle \sim_{t\to\infty} \frac{2(k_BT)^2}{\alpha^2}\frac{\lambda_2-\lambda_1}{\Gamma(1+\lambda_1)}t^{\lambda_1}\ln t.$$

For $\lambda_1 = 0$ and $\lambda_2 = 1$ we arrive at the result (66) obtained for uniformly distributed noise with a logarithmic MSD dependence on time. In the short time limit the particle performs ballistic motion, as well.

Such relaxation patterns of logarithmic form and ultraslow diffusion have been observed in the analysis of distributed-order relaxation and diffusion equations by Kochubei [69, 78–80] as well as by Mainardi *et al* [55, 70].

3.2. *Harmonic oscillator: overdamped limit*

We now move on to consider the distributed-order GLE for a harmonic oscillator with different weight functions in the overdamped limit. For the uniformly distributed noise (66) we obtain the following form of the MSD

$$\langle x^2(t)\rangle_{\mathrm{o}} = 2k_BT\mathscr{L}^{-1}\left\{\frac{s^{-1}}{\frac{\alpha^2}{k_BT}\frac{s-1}{\log s}+\omega^2};t\right\} = 2k_BT\mathscr{L}^{-1}\left\{\frac{s^{-1}\log s}{\omega^2\log s + As - A};t\right\}, \tag{72}$$

where $A = \alpha^2/(k_BT)$. From Tauberian theorems (see appendix B) we analyse the asymptotic behaviour in the long and short time limits. At long times ($t\to\infty$, equivalent to $s\to 0$) it follows that

$$\begin{aligned}\langle x^2(t)\rangle_{\mathrm{o}} &\sim_{t\to\infty} \frac{2k_BT}{\omega^2}\mathscr{L}^{-1}\left\{\frac{s^{-1}}{1-\frac{A}{\omega^2\log s}};t\right\} \sim_{t\to\infty} \frac{2k_BT}{\omega^2}\mathscr{L}^{-1}\left\{\frac{1}{s}+\frac{\frac{A}{\omega^2}}{s\log s};t\right\}\\ &= \frac{2k_BT}{\omega^2}\left[1+\frac{A}{\omega^2}\nu(t)\right],\end{aligned} \tag{73}$$

where $\nu(t)$ is the Volterra function, see appendix C. Thus, the DACF in the long time limit becomes

$$C_{X,\mathrm{o}}(t) \sim_{t\to\infty} -\frac{A}{\omega^2}\nu(t). \tag{74}$$

At short times ($t\to 0$, or $s\to\infty$), we find for the MSD and DACF that

$$\langle x^2(t)\rangle_{\mathrm{o}} \sim_{t\to 0} \frac{2k_BT}{A}\mathscr{L}^{-1}\left\{\frac{\log s}{s^2};t\right\} = \frac{2k_BT}{A}t\left(\log\frac{1}{t}+1-\gamma\right), \tag{75}$$

and

$$C_{X,\mathrm{o}}(t) \sim_{t\to 0} 1-\frac{\omega^2}{A}t\left(\log\frac{1}{t}+1-\gamma\right), \tag{76}$$

respectively.

Now consider the distributed-order noise (68). From the relaxation function $I_o(t)$ the MSD becomes

$$\langle x^2(t)\rangle_{\mathrm{o}} = \frac{2k_BT}{\omega^2}\mathscr{L}^{-1}\left\{\frac{s^{-1}}{\frac{A}{\omega^2}\nu\left(\log\frac{1}{s}\right)^{-\nu}\gamma(\nu,-\log s)+1};t\right\}, \tag{77}$$

where $\gamma(a,\sigma)=\int_0^\sigma t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$ is the lower incomplete Gamma function. Taking the long time limit yields

$$\begin{aligned}\langle x^2(t)\rangle_{\mathrm{o}} &\sim_{t\to\infty} \frac{2k_{\mathrm{B}}T}{\omega^2}\left\{1-\frac{A}{\omega^2}\Gamma(1+\nu)\mathscr{L}^{-1}\left\{\frac{1}{s}\left(\log\frac{1}{s}\right)^{-\nu};t\right]\right\}\\ &\sim_{t\to\infty} \frac{2k_{\mathrm{B}}T}{\omega^2}\left[1-\frac{A}{\omega^2}\frac{\Gamma(1+\nu)}{\log^\nu t};t\right],\end{aligned} \tag{78}$$

from which we obtain the asymptotic behaviour of the DACF,

$$C_{X,\mathrm{o}}(t)\sim_{t\to\infty}\frac{2k_{\mathrm{B}}T}{\omega^2}\frac{A}{\omega^2}\frac{\Gamma(1+\nu)}{\log^\nu t}. \tag{79}$$

In the short time limit, we find

$$\langle x^2(t)\rangle_{\mathrm{o}}\sim_{t\to 0}\frac{2k_{\mathrm{B}}T}{A}\frac{\log^\nu\frac{1}{t}}{\Gamma(1+\nu)} \tag{80}$$

and

$$C_{X,\mathrm{o}}(t)\sim_{t\to 0}1-\frac{\omega^2}{A}\frac{\log^\nu\frac{1}{t}}{\Gamma(1+\nu)}. \tag{81}$$

Similarly, we obtain for the distributed-order noise (61) with weight function $p(\lambda)=\alpha^2/(\lambda_2-\lambda_1)$ $(0\leqslant\lambda_1<\lambda<\lambda_2\leqslant 1$, and $p(\lambda)=0$ otherwise) the asymptotic forms of the MSD:

$$\langle x^2(t)\rangle_{\mathrm{o}}\sim_{t\to\infty}\frac{2k_{\mathrm{B}}T}{\omega^2}\left\{1-\frac{A/\omega^2}{(\lambda_2-\lambda_1)}\left[\nu(t,-\lambda_2)-\nu(t,-\lambda_1)\right]\right\}, \tag{82}$$

in the long time limit, and

$$\begin{aligned}\langle x^2(t)\rangle_{\mathrm{o}}\sim_{t\to 0}\frac{2k_{\mathrm{B}}T}{A}\sum_{n=0}^{\infty}&\frac{t^{(\lambda_2-\lambda_1)n+\lambda_2}}{\Gamma[(\lambda_2-\lambda_1)n+\lambda_2+1]}\\ &\times\left[\log t^{-(\lambda_2-\lambda_1)}-(\lambda_2-\lambda_1)\psi((\lambda_2-\lambda_1)n+\lambda_2+1)\right]\end{aligned} \tag{83}$$

in the short time limit. From here we can easily find the DACF.

From these results, we conclude that the distributed-order GLE may be used to model various anomalous diffusive behaviours, such as ultraslow diffusion, strong anomaly, and other complex diffusive regimes.

4. Further generalisations

4.1. LE with power-logarithmic distributed order noises

Let us consider the Langevin equation (6) with the logarithmically distributed-order friction kernel

$$\gamma(t)=(k_{\mathrm{B}}T)^{-1}\int_0^1\Gamma(3/2-\lambda)\frac{\log^{\lambda-1}t}{\sqrt{t}}\mathrm{d}\lambda\quad\text{with}\quad\hat{\gamma}(s)=\frac{\pi}{k_{\mathrm{B}}T}\frac{s-1}{\sqrt{s}\log s}, \tag{84}$$

which satisfies the condition (8). The relaxation function $I(t)$ defined by equation (11) assumes the form

$$I(t) = \mathscr{L}^{-1}\left\{\frac{s^{-1}}{s^2 + \frac{\pi}{k_{\mathrm{B}}T}\frac{s(s-1)}{\sqrt{s}\log s} + \omega^2}; t\right\}. \tag{85}$$

As it is challenging to calculate the exact form of equation (85) for general $t > 0$ we concentrate on the asymptotic behaviour for small $s \ll 1$, yielding

$$\hat{I}(s) \sim_{s\to 0} \frac{s^{-1}}{\omega^2}\frac{1}{1 - \frac{\pi}{k_{\mathrm{B}}T\omega^2}\frac{s(1-s)}{\sqrt{s}\log s}} \simeq \frac{1}{\omega^2}\left[s^{-1} + \frac{\pi}{k_B T\omega^2}\frac{1}{\sqrt{s}\log s} - \frac{\pi}{k_{\mathrm{B}}T\omega^2}\frac{\sqrt{s}}{\log s}\right],$$

where we take the zeroth and first-order terms of the series expansion. If s tends zero then the term $\sqrt{s}/\log s$ can be neglected. Thus, we have

$$\hat{I}(s) \sim_{s\to 0} \frac{1}{\omega^2}\left[s^{-1} + \frac{\pi}{k_{\mathrm{B}}T\omega^2}\frac{1}{\sqrt{s}\log s}\right] \quad \text{and} \quad I(t) \sim_{t\to\infty} \frac{1}{\omega^2}\left[1 + \frac{\pi}{k_{\mathrm{B}}T\omega^2}\nu(t,-1/2)\right]. \tag{86}$$

This allows us to find the MSD and DACF for $t \to \infty$,

$$\langle x^2(t)\rangle \sim_{t\to\infty} \frac{2k_{\mathrm{B}}T}{\omega^2} + \frac{2\pi}{\omega^4}\nu(t,-1/2) \quad \text{and} \quad C_X(t) \sim_{t\to\infty} -\frac{\pi}{k_{\mathrm{B}}T\omega^2}\nu(t,-1/2).$$

Note that for $\omega = 0$ we get

$$\begin{aligned}\langle x^2(t)\rangle &= 2k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{\frac{s^{-1}}{s^2 + \frac{\pi}{k_{\mathrm{B}}T}\frac{s(s-1)}{\sqrt{s}\log s}}; t\right\} \sim_{t\to\infty} 2k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{\frac{s^{-1}}{\frac{\pi}{k_{\mathrm{B}}T}\frac{s(s-1)}{\sqrt{s}\log s}}; t\right\} \\ &\sim_{t\to\infty} \frac{2(k_{\mathrm{B}}T)^2}{\pi}\mathscr{L}^{-1}\left\{\frac{\log s}{s^{3/2}(s-1)}; t\right\} \sim_{t\to\infty} \frac{2(k_{\mathrm{B}}T)^2}{\pi}\mathscr{L}^{-1}\left\{\frac{\log\frac{1}{s}}{s^{3/2}}; t\right\} \\ &\sim_{t\to\infty} \frac{2(k_{\mathrm{B}}T)^2}{\pi}\frac{1}{\Gamma(3/2)}t^{1/2}\log t = \frac{4(k_{\mathrm{B}}T)^2}{\pi^{3/2}}t^{1/2}\log t.\end{aligned} \tag{87}$$

We note that in the short time limit ballistic motion is observed, as well, due to the effect of the inertial term.

4.2. Langevin equation with distributed-order Mittag–Leffler noise

In the next example, we study the GLE for a harmonic oscillator with the ML friction memory kernel

$$\gamma(t) = \frac{1}{k_{\mathrm{B}}T}\int_0^1 E_\lambda\left(-t^\lambda\right)\mathrm{d}\lambda \quad \left(\hat{\gamma}(s) = \frac{1}{k_{\mathrm{B}}T}\frac{\log\frac{s+1}{2}}{s\log s}\right), \tag{88}$$

which satisfies the condition (8). The long-time asymptotic of the relaxation function $I(t)$ involved in the MSD and the DACF can be calculated by use of the Tauberian theorem (appendix B) in which we consider the limit $s \to 0$ of $\hat{I}(s)$. Thus, we have

$$I(t) = \mathscr{L}^{-1}\left\{\frac{s^{-1}}{s^2 + \frac{1}{k_\mathrm{B}T}\frac{\log\frac{s+1}{2}}{\log s} + \omega^2}; t\right\} \sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{k_\mathrm{B}T\omega^2}\frac{\log 2}{s\log s}; t\right\}$$
$$= \frac{1}{\omega^2}\left[1 - \frac{\log 2}{k_\mathrm{B}T\omega^2}\nu(t)\right]. \tag{89}$$

In the force-free case with $\omega = 0$ we observe ultraslow diffusion, since the relaxation function $I(t)$ has a logarithmic time dependence,

$$I(t) \sim_{t\to\infty} \frac{k_\mathrm{B}T}{\log 2}\mathscr{L}^{-1}\left\{\frac{\log\frac{1}{s}}{s}; t\right\} = \frac{k_\mathrm{B}T}{\log 2}\log t = k_\mathrm{B}T\log_2 t. \tag{90}$$

In the short time limit the relaxation function behaves as $I(t) \sim t^2$.

4.3. Distributed-order Langevin equation with Caputo-Prabhakar derivative

In this part, we now consider the GLE (6) for a harmonic oscillator with the choice of the distributed-order Prabhakar friction memory kernel

$$\gamma(t) = \frac{1}{k_\mathrm{B}T}\int_0^1 t^{-\lambda}E^{\delta}_{\rho,1-\lambda}(-t^\rho)\,\mathrm{d}\lambda \quad \text{with} \quad \hat{\gamma}(s) = \frac{1}{k_\mathrm{B}T}\frac{s-1}{s\log s}\left(1+s^{-\rho}\right)^{-\delta}. \tag{91}$$

We consider two cases, the first one for $0<\rho,\delta<1$ and the second one for $-1<\rho,\delta<0$. In both cases the condition (8) is satisfied. If we substitute the distributed-order Prabhakar friction memory kernel in the GLE (6) we arrive to the following generalised Langevin equation

$$\ddot{x}(t) + \frac{1}{k_\mathrm{B}T}\int_0^1\left({}_\mathrm{C}\mathcal{D}^{\delta,\lambda}_{\rho,1,0+}x\right)(t)\,\mathrm{d}\lambda + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = \xi(t), \quad \dot{x}(t) = v(t), \tag{92}$$

with regularised fractional derivative

$$\left({}_\mathrm{C}\mathcal{D}^{\delta,\lambda}_{\rho,\omega,0+}f\right)(t) = \int_0^t (t-t')^{-\lambda}E^{\delta}_{\rho,1-\lambda}\left(-\omega[t-t']^\rho\right)\frac{\mathrm{d}}{\mathrm{d}t'}f(t')\,\mathrm{d}t'.$$

Using the Tauberian theorem we calculate the asymptotics of the relaxation function $I(t)$. **(i)** In the case $0<\rho,\delta<1$ and for $t\to\infty$ we have

$$I(t) \sim_{t\to\infty} \mathscr{L}^{-1}\left\{\frac{s^{-1}}{\omega^2 + \frac{1}{k_\mathrm{B}T}\frac{s-1}{\log s}\left(1+s^{-\rho}\right)^{-\delta}}; t\right\} \sim_{t\to\infty} \mathscr{L}^{-1}\left\{\frac{s^{-1}}{\omega^2 + \frac{1}{k_\mathrm{B}T}\frac{s-1}{\log s}s^{\rho\delta}}; t\right\}$$
$$\sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{s^{-1}}{1 + \frac{1}{k_\mathrm{B}T\omega^2}\frac{s^{\rho\delta}}{\log\frac{1}{s}}}; t\right\} \sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{k_\mathrm{B}T\omega^2}\frac{s^{-1+\rho\delta}}{\log\frac{1}{s}}; t\right\}$$
$$\sim_{t\to\infty} \frac{1}{\omega^2}\left[1 - \frac{1}{k_\mathrm{B}T\omega^2\Gamma(1-\rho\delta)}\frac{t^{-\rho\delta}}{\log t}\right]. \tag{93}$$

From the asymptotic form of $I(t)$ we obtain the long time behaviours of the MSD and DACF,

$$\langle x^2(t)\rangle \sim_{t\to\infty} \frac{2k_\mathrm{B}T}{\omega^2} - \frac{1}{\omega^4\Gamma(1-\rho\delta)}\frac{t^{-\rho\delta}}{\log t} \tag{94}$$

and

$$C_X(t) \sim_{t\to\infty} \frac{1}{k_\mathrm{B}T\omega^2\Gamma(1-\rho\delta)}\frac{t^{-\rho\delta}}{\log t}. \tag{95}$$

In addition, note that in the free particle case ($\omega=0$) the long-time limit (or for the over-damped case when we neglect the term with s^2) becomes

$$I(t) \sim_{t\to\infty} k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{\frac{s^{-1}\left(1+s^{-\rho}\right)^{\delta}\log s}{s-1};t\right\} \sim_{t\to\infty} k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{\frac{\log s}{s^{1+\rho\delta}\left(s-1\right)};t\right\}$$
$$\sim_{t\to\infty} k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{s^{-1-\rho\delta}\log\frac{1}{s};t\right\} \sim_{t\to\infty} k_{\mathrm{B}}T\frac{t^{\rho\delta}}{\Gamma\left(1+\rho\delta\right)}\log t. \tag{96}$$

(ii) Replacing ρ by $-\rho$ and δ by $-\delta$ in equations (91) and (92) we find

$$\gamma\left(t\right)=\frac{1}{k_{\mathrm{B}}T}\int_0^1 t^{-\lambda}E_{-\rho,1-\lambda}^{-\delta}\left(-t^{-\rho}\right)\mathrm{d}\lambda \quad \text{and} \quad \hat{\gamma}\left(s\right)=\frac{1}{k_{\mathrm{B}}T}\frac{s-1}{s\log s}\left(1+s^{\rho}\right)^{\delta}, \tag{97}$$

where $0<\rho,\delta<1$. The relaxation function $I(t)$ for $\omega\neq 0$ reads

$$I(t) \sim_{t\to\infty} \mathscr{L}^{-1}\left\{\frac{s^{-1}}{\frac{1}{k_{\mathrm{B}}T}\frac{s-1}{\log s}\left(1+s^{\rho}\right)^{\delta}+\omega^2};t\right\} \sim_{t\to\infty} \mathscr{L}^{-1}\left\{\frac{s^{-1}}{\frac{1}{k_{\mathrm{B}}T}\frac{s-1}{\log s}+\omega^2};t\right\}$$
$$=\frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{s^{-1}}{1-\frac{1}{k_{\mathrm{B}}T\omega^2}\frac{1-s}{\log s}};t\right\} \sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{1}{s}+\frac{1}{k_{\mathrm{B}}T\omega^2}\frac{1}{s\log s};t\right\}$$
$$=\frac{1}{\omega^2}\left[1+\frac{\nu\left(t\right)}{k_{\mathrm{B}}T\omega^2}\right]. \tag{98}$$

For $\omega=0$ the long time asymptotic of $I(t)$ is equal to

$$I(t)=k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{\frac{s^{-1}}{\frac{s-1}{\log s}\left(1+s^{\rho}\right)^{\delta}};t\right\} \sim_{t\to\infty} k_{\mathrm{B}}T\mathscr{L}^{-1}\left\{\frac{\log s}{s\left(s-1\right)};t\right\}$$
$$=k_{\mathrm{B}}T\left[C+\log t+\mathrm{e}^{t}E_1\left(t\right)\right]. \tag{99}$$

Note that the relevant MSD is the same as obtained for the force-free case by using the distributed-order fractional derivative, i.e. equation (67) for $\alpha=1$.

4.4. Langevin equation with distributed-order Volterra function

We consider the GLE for which the friction kernel is based on the Volterra functions $\nu(t,-\alpha)$ and $\mu(t,-\beta)$ for $0<\alpha,\beta<1$, see appendix C. In the first case the friction memory kernel in t and s reads

$$\gamma_1\left(t\right)=\frac{1}{k_{\mathrm{B}}T}\int_0^1 \nu\left(t,-\alpha\right)\mathrm{d}\alpha \quad \text{and} \quad \hat{\gamma}_1\left(s\right)=\frac{1}{k_{\mathrm{B}}T}\frac{s-1}{s\log^2 s}, \tag{100}$$

thus satisfying condition (8). For the GLE for the stochastic harmonic oscillator ($\omega\neq 0$) the long time limit ($t\to\infty$) of the relaxation function $I_1(t)$ yields in the form

$$I_1\left(t\right)=\mathscr{L}^{-1}\left\{\frac{s^{-1}}{s^2+\frac{1}{k_{\mathrm{B}}T}\frac{s-1}{\log^2 s}+\omega^2};t\right\} \sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{1}{s}+\frac{1}{k_{\mathrm{B}}T\omega^2}\frac{1}{s\log^2 s};t\right\}$$
$$=\frac{1}{\omega^2}\left[1+\frac{\mu\left(t,1\right)}{k_{\mathrm{B}}T\omega^2}\right]. \tag{101}$$

In the force-free case ($\omega = 0$) we obtain [60, equation (45)]

$$
\begin{aligned}
I_1(t) &= \mathscr{L}^{-1}\left\{\frac{s^{-1}}{s^2+\frac{1}{k_\mathrm{B}T}\frac{s-1}{\log^2 s}};t\right\} \sim_{t\to\infty} k_\mathrm{B}T\mathscr{L}^{-1}\left\{\frac{\log^2 s}{s(s-1)};t\right\} \\
&= k_\mathrm{B}T\Bigg[-\left(C^2+\pi^2/6\right)\mathrm{e}^t + 2t\mathrm{e}^t\Phi_{1;1}^{\star,(1,1)}(-t,3,1) - 2(C+\log t)\mathrm{e}^t E_1(t) \\
&\quad - (2C+\log t)(\mathrm{e}^t+1)(\log t) - C^2 + \pi^2/6\Bigg]. \qquad (102)
\end{aligned}
$$

Here $\Phi_{1;1}^{\star,(1,1)}$ is the Hurwitz–Lerch function, see [81].

In the second case, the friction in t and s spaces turns out to be

$$\gamma_2(t) = \frac{1}{k_\mathrm{B}T}\int_0^1 \mu(t,-\beta)\,\mathrm{d}\beta \quad \text{and} \quad \hat{\gamma}_2(s) = \frac{1}{k_\mathrm{B}T}\frac{\log s - 1}{s(\log s)(\log\log s)}. \qquad (103)$$

Here we consider only the force-free case, i.e. $\omega = 0$. In that case we find the asymptotic of $I_2(t)$ at long times. Using that $\log\log s$ is a slowly varying function, by Tauberian theorems we find that

$$
\begin{aligned}
I_2(t) &\sim_{t\to\infty} k_\mathrm{B}T\mathscr{L}^{-1}\left\{\frac{(\log s)(\log\log s)}{s(\log s - 1)};t\right\} = k_\mathrm{B}T\mathscr{L}^{-1}\left\{\frac{\log\log s}{s\left(1-\frac{1}{\log s}\right)};t\right\} \\
&\sim_{t\to\infty} k_\mathrm{B}T\mathscr{L}^{-1}\left\{\frac{1}{s}\log\log\frac{1}{s^{-1}};t\right\} = k_\mathrm{B}T\log\log\frac{1}{t}. \qquad (104)
\end{aligned}
$$

Finally, we consider a memory kernel of distributed order in respect to both parameters in the Volterra μ function:

$$\gamma_3(t) = \frac{1}{k_\mathrm{B}T}\int_0^1\int_0^1 \mu(t,\beta,\alpha-1)\,\mathrm{d}\beta\mathrm{d}\alpha \qquad (105)$$

and

$$\hat{\gamma}_3(s) = \frac{1}{k_\mathrm{B}T}\int_0^1\frac{\mathrm{d}\alpha}{s^\alpha}\int_0^1\frac{\mathrm{d}\beta}{\log^{\beta+1}s} = \frac{1}{k_\mathrm{B}T}\frac{(s-1)(\log s-1)}{s\left(\log^3 s\right)(\log\log s)}. \qquad (106)$$

For $\omega \neq 0$ the long time limit of the relaxation function $I_3(t)$ yields

$$
\begin{aligned}
I_3(t) &= \mathscr{L}^{-1}\left\{\frac{s^{-1}}{s^2+\frac{1}{k_\mathrm{B}T}\frac{(s-1)(\log s-1)}{s\left(\log^3 s\right)(\log\log s)}+\omega^2};t\right\} \\
&\sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{s^{-1}}{1-\frac{1}{k_\mathrm{B}T\omega^2}\frac{(1-s)(\log s-1)}{s\left(\log^3 s\right)(\log\log s)}};t\right\} \\
&\sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{1}{s}+\frac{1}{k_\mathrm{B}T\omega^2}\frac{1}{s^2\left(\log^2 s\right)(\log\log s)};t\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\sim_{t\to\infty} \frac{1}{\omega^2}\mathscr{L}^{-1}\left\{\frac{1}{s}+\frac{1}{k_BT\omega^2}\frac{1}{s^2\left(\log^2\frac{1}{s}\right)\left(\log\log\frac{1}{s^{-1}}\right)};t\right\}\\
&\sim_{t\to\infty} \frac{1}{\omega^2}\left\{1+\frac{1}{k_BT\omega^2}\frac{t}{\left(\log^2 t\right)\left(\log\log\frac{1}{t}\right)}\right\}.
\end{aligned}
\tag{107}
$$

5. Summary

We studied the GLE for an external harmonic potential and the free particle limit for various cases of the friction memory kernel. In particular, we derived the associated MSD and DACF for the different chosen forms for the friction memory kernel of Dirac delta, power-law, and distributed-order forms. Various anomalous diffusive behaviours, such as subdiffusion, superdiffusion, ultraslow diffusion, and strong anomaly, are observed. Special attention was paid to distributed-order GLEs and distributed-order diffusion-type equation. For different forms of the weight functions, we obtained ultraslow diffusion, strong anomaly, and other complex diffusive behaviours.

It will be interesting to compare the results obtained here for the GLE to the case for external noise [10], i.e. when the second FDT is not satisfied. In both cases, superstatistical and stochastic variations of the diffusion coefficient and anomalous scaling exponent (Hurst exponent) have been analysed recently [47, 82–86]. Studying such concepts in the frameworks developed here will significantly enlarge our current range of stochastic models for disordered systems. We also note potential generalisations with respect to subordinated GLE models such as those studied in [87, 88], fractional GLE [14, 31], as well as the presence of stochastic resetting [89] in the system, including resetting in the memory kernel [90, 91].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Multinomial Prabhakar type functions $\mathcal{E}_{(\vec{\mu}),\beta}(t;\vec{a})$

The multinomial Prabhakar function $\mathcal{E}_{(\vec{\mu}),\beta}(t;\vec{a})$ with $\mu_1>\mu_2>\cdots>\mu_m>0$, see [54], is related to the multinomial ML function $E_{(\vec{\mu}),\beta}(-a_1t^{\mu_1},\ldots,-a_mt^{\mu_m})$ as follows

$$
\mathcal{E}_{(\vec{\mu}),\beta}(t;\vec{a})=t^{\beta-1}E_{(\vec{\mu}),\beta}\left(-a_1t^{\mu_1},\ldots,-a_mt^{\mu_m}\right).
\tag{A.1}
$$

Its Laplace transform reads

$$\mathscr{L}\left\{\mathcal{E}_{(\vec{\mu}),\beta}\left(t;\vec{a}\right);s\right\} = \frac{s^{\mu_1-\beta}}{s^{\mu_1}+a_m s^{\mu_1-\mu_m}+\ldots+a_2 s^{\mu_1-\mu_2}+a_1}. \tag{A.2}$$

Theorem 2.3 from [54] implies that the following identities hold true:

$$\int_0^t \mathcal{E}_{(\vec{\mu}),\beta}\left(\xi;\vec{a}\right)\mathrm{d}\xi = \mathcal{E}_{(\vec{\mu}),\beta+1}\left(t;\vec{a}\right), \tag{A.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{(\vec{\mu}),\beta}\left(t;\vec{a}\right) = \mathcal{E}_{(\vec{\mu}),\beta-1}\left(t;\vec{a}\right), \qquad \beta > 1. \tag{A.4}$$

The asymptotic behaviours of the multinomial Prabhakar function for $t \ll 1$ and $t \gg 1$ are equal to

$$\mathcal{E}_{(\vec{\mu}),\beta}\left(t;\vec{a}\right) \sim_{t\to 0} \frac{t^{\beta-1}}{\Gamma\left(\beta\right)} - \sum_{j=1}^{m}\frac{t^{\beta-1+\mu_j}}{\Gamma\left(\beta+\mu_j\right)}, \tag{A.5}$$

$$\mathcal{E}_{(\vec{\mu}),\beta}\left(t;\vec{a}\right) \sim_{t\to\infty} a_2^{-1}t^{\beta-\mu_2-1}E_{\mu_1-\mu_2,\beta-\mu_2}\left(-a_1 a_2^{-1}t^{\mu_1-\mu_2}\right), \tag{A.6}$$

respectively. Equation (A.1) implies the series representation of $\mathcal{E}_{(\vec{\mu}),\beta}(s;\vec{a})$, that is

$$\mathcal{E}_{(\vec{\mu}),\beta}\left(t;\vec{a}\right) = \sum_{k=0}^{\infty}\sum_{\substack{k_1+\ldots+k_m=k\\ k_1,\ldots,k_m\geqslant 0}}\frac{\left(-1\right)^k k!}{k_1!\ldots k_m!}\frac{\prod_{j=1}^{m}a_j^{k_j}t^{\beta-1+\sum_{j=0}^{m}\mu_j k_j}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_j k_j\right)} \tag{A.7}$$

from which it appears that we can represent the multinomial Prabhakar function by the sums of the three-parameter ML function [92], i.e.

$$E_{\mu,\beta}^{\delta}\left(-\lambda t^{\mu}\right) = \frac{1}{\Gamma\left(\delta\right)}\sum_{r=0}^{\infty}\frac{\Gamma\left(\delta+r\right)\left(-\lambda t^{\mu}\right)^r}{r!\Gamma\left(\beta+\mu r\right)}, \tag{A.8}$$

whose the Laplace transform reads

$$\mathscr{L}\left\{t^{\beta-1}E_{\mu,\beta}^{\delta}\left(-\lambda t^{\mu}\right);s\right\} = \frac{s^{\mu\delta-\beta}}{\left(\lambda+s^{\mu}\right)^{\delta}}. \tag{A.9}$$

For $\delta = 1$ the three-parameter ML function becomes a two-parameter ML function, while for $\beta = \delta = 1$ it becomes one parameter ML function. The asymptotic expansion of the three-parameter ML function follows from the expression [93, 94]

$$E_{\rho,\beta}^{\delta}\left(-z\right) = \frac{z^{-\delta}}{\Gamma\left(\delta\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\delta+n\right)}{\Gamma\left(\beta-\rho\left(\delta+n\right)\right)}\frac{\left(-z\right)^{-n}}{n!}, \tag{A.10}$$

with $z > 1$, and $0 < \rho < 2$.

For example, for $m = 2$, we have

$$\mathcal{E}_{(\mu_1,\mu_2),\beta}\left(t;a_1,a_2\right) = \sum_{j=0}^{\infty}\left(-a_2\right)^j t^{\beta+\mu_2 j-1}E_{\mu_1,\beta+\mu_2 j}^{1+j}\left(-a_1 t^{\mu_1}\right) \tag{A.11}$$

$$= \sum_{j=0}^{\infty}\left(-a_1\right)^j t^{\beta+\mu_1 j-1}E_{\mu_2,\beta+\mu_1 j}^{1+j}\left(-a_2 t^{\mu_2}\right). \tag{A.12}$$

Moreover, equation (A.2) for $m = 2$ under condition $|a_1 s^{-\mu_1} + a_2 s^{-\mu_2}| > 1$ gives

$$\mathcal{E}_{(\mu_1,\mu_2),\beta}(t;a_1,a_2) = \sum_{r=0}^{\infty} \frac{(-1)^r}{a_1^{r+1}} t^{\beta-\mu_2(r+1)-1} E^{r+1}_{\mu_1-\mu_2,\beta-\mu_2(r+1)}\left(-\frac{a_2}{a_1} t^{\mu_1-\mu_2}\right) \tag{A.13}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{a_2^{r+1}} t^{\beta-\mu_1(r+1)-1} E^{r+1}_{\mu_2-\mu_1,\beta-\mu_1(r+1)}\left(-\frac{a_1}{a_2} t^{\mu_2-\mu_1}\right). \tag{A.14}$$

From equation (A.7) for $m = 3$ after some laborious calculation can be derived, e.g. following formula which is used in our manuscript

$$\mathcal{E}_{(\mu_1,\mu_2,\mu_3),\beta}(t;a_1,a_2,a_3) = \sum_{j=0}^{\infty} \frac{(-a_3)^j}{j!} \sum_{k=0}^{\infty} \frac{(-a_1)^k}{k!} (k+j)! \times t^{\beta-1+\mu_1 k+\mu_3 j} E^{j+k+1}_{\mu_2,\beta+\mu_1 k+\mu_3 j}\left(-a_2 t^{\mu_2}\right). \tag{A.15}$$

Appendix B. Tauberian theorems [95]

If the Laplace transform pair $\hat{r}(s)$ of the function $r(t)$ behaves like

$$\hat{r}(s) \sim s^{-\rho} L\left(s^{-1}\right), \quad s \to 0, \quad \rho > 0, \tag{B.1}$$

where $L(t)$ is a slowly varying function at infinity, then $r(t)$ has the following asymptotic behaviour [95]

$$r(t) \sim \frac{1}{\Gamma(\rho)} t^{\rho-1} L(t), \quad t \to \infty. \tag{B.2}$$

A slowly varying function at infinity means that

$$\lim_{t\to\infty} \frac{L(at)}{L(t)} = 1, \quad a > 0. \tag{B.3}$$

The Tauberian theorem works also for the opposite asymptotic, i.e. for $t \to 0$.

Appendix C. The Volterra family functions

Volterra's function is defined as follows [96–98]

$$\mu(t,\beta,\alpha) = \frac{1}{\Gamma(1+\beta)} \int_0^{\infty} \frac{t^{u+\alpha} u^{\beta}}{\Gamma(u+\alpha+1)} \mathrm{d}u, \quad \Re(\beta) > -1 \quad \text{and} \quad t > 0, \tag{C.1}$$

whose particular cases are

$$\begin{aligned} \alpha = \beta = 0: &\qquad \nu(t) = \mu(t,0,0), \\ \alpha \neq 0, \beta = 0: &\qquad \nu(t,\alpha) = \mu(t,0,\alpha), \\ \alpha = 0, \beta \neq 0: &\qquad \mu(t,\beta) = \mu(t,\beta,0). \end{aligned}$$

The Laplace transform of the Volterra's function $\mu(t,\beta,\alpha)$ is given by [98]

$$\mathscr{L}\{\mu(t,\beta,\alpha);s\} = \frac{1}{s^{\alpha+1}\log^{\beta+1} s}. \tag{C.2}$$

Appendix D. Derivation of equation (84)

Since,

$$\int_0^\infty \mathrm{e}^{-st}t^a\left(\log^{\lambda-1}t\right)\mathrm{d}t = \frac{\partial^{\lambda-1}}{\partial a^{\lambda-1}}\int_0^\infty \mathrm{e}^{-st}t^a\mathrm{d}t = \frac{\partial^{\lambda-1}}{\partial a^{\lambda-1}}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\} = I_0^{1-\lambda}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\}$$
$$= \frac{\Gamma(a+1)}{\Gamma(1-\lambda)}\int_0^s (s-t)^{-\lambda}t^{-a-1}\mathrm{d}t = \frac{\Gamma(a+1)\Gamma(-a)}{s^{\lambda+a}\Gamma(1-\lambda-a)}, \tag{D.1}$$

for $a=-1/2$, we get

$$\int_0^\infty \mathrm{e}^{-st}t^{-1/2}\left(\log^{\lambda-1}t\right)\mathrm{d}t = \frac{\pi}{s^{\lambda-1/2}\Gamma(3/2-\lambda)}, \quad 0<\lambda<1. \tag{D.2}$$

Here $I_0^{1-\lambda}$ denotes the Riemann–Liouville (fractional) integral, defined by [50, 55]

$$\left(I_a^\mu f\right)(t) = \frac{1}{\Gamma(\mu)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\mu}}\mathrm{d}\tau, \quad t>a, \quad \Re(\mu)>0. \tag{D.3}$$

Then,

$$k_\mathrm{B}T\hat{\gamma}(s) = \int_0^1 \Gamma(3/2-\lambda)\,\mathrm{d}\lambda\left[\int_0^\infty \mathrm{e}^{-st}t^{-1/2}\left(\log^{\lambda-1}t\right)\mathrm{d}t\right]$$
$$= \pi\sqrt{s}\int_0^1\frac{\mathrm{d}\lambda}{s^\lambda} = \pi\frac{s-1}{\sqrt{s}\log s}. \tag{D.4}$$

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References

[1] Langevin P 1908 *C. R.* **146** 530
[2] Zwanzig R 2001 *Nonequilibrium Statistical Mechanics* (Oxford University Press)
[3] Coffey W T and Kalmykov Y P 2012 *The Langevin Equation* (World Scientific)
[4] Kupferman R 2004 *J. Stat. Phys.* **114** 291
[5] Goychuk I 2012 *Adv. Chem. Phys.* **150** 187 (available at: www.physik.uni-augsburg.de/theo1/hanggi/History/Langevin1908.pdf)
[6] Lizana L, Ambjörnsson T, Taloni A, Barkai E and Lomholt M L 2010 *Phys. Rev.* E **81** 051118
[7] Kubo R 1966 *Rep. Prog. Phys.* **29** 255
[8] Mainardi F and Pironi P 1996 *Extr. Math.* **10** 140
[9] Despósito M A and Viñales A D 2008 *Phys. Rev.* E **77** 031123
Despósito M A and Viñales A D 2009 *Phys. Rev.* E **80** 021111
[10] Klimontovich Y L 1991 *Turbulent Motion and the Structure of Chaos* (Kluwer)
[11] Viñales A D and Despósito M A 2007 *Phys. Rev.* E **75** 042102
Viñales A D, Wang K G and Despósito M A 2009 *Phys. Rev.* E **80** 011101
Viñales A D and Despósito M A 2006 *Phys. Rev.* E **73** 016111

[12] Pottier N 2003 *Physica* A **317** 371
Pottier N and Mauger A 2000 *Physica* A **282** 77
[13] Sandev T 2012 *Phys. Macedonica* **61** 59
[14] Sandev T, Metzler R and Tomovski Ž 2014 *J. Math. Phys.* **55** 023301
[15] Franosch T, Grimm M, Belushkin M, Mor F M, Foffi G, Forro L and Jeney S 2011 *Nature* **478** 7367
[16] Jeon J -H, Tejedor V, Burov S, Barkai E, Selhuber-Unkel C, Berg-Sørensen K, Oddershede L and Metzler R 2011 *Phys. Rev. Lett.* **106** 048103
[17] Jeon J -H, Leijnse N, Oddershede L B and Metzler R 2013 *New J. Phys.* **15** 045011
[18] Nørregaard K, Metzler R, Ritter C, Berg-Sørensen K and Oddershede L 2017 *Chem. Rev.* **117** 4342
[19] Selhuber-Unkel C, Weiss M, Scott S, Barooji Y F, Sabri A, Erler J T, Metzler R and Oddershede L B 2023 *Phys. Chem. Chem. Phys.* **25** 1513
[20] Kou S C and Xie X S 2004 *Phys. Rev. Lett.* **93** 180603
[21] Yang H, Luo G, Karnchanaphanurach P, Louie T-M, Reich I, Cova S, Xun L and Xie X S 2003 *Science* **302** 262
[22] Einstein A 1905 *Ann. Phys., Lpz.* **322** 549
[23] von Smoluchowski M 1906 *Ann. Phys., Lpz.* **21** 756
[24] Spiechowicz J, Marchenko I G, Hänggi P and Łczka J 2022 *Entropy* **25** 42
[25] Sutherland W 1905 *Phil. Mag.* **9** 781
[26] Wang K G 1992 *Phys. Rev.* A **45** 833
Wang K G and Tokuyama M 1999 *Physica* A **265** 341
[27] Lutz E 2001 *Phys. Rev.* E **64** 051106
[28] Burov S and Barkai E 2008 *Phys. Rev.* E **78** 031112
Burov S and Barkai E 2008 *Phys. Rev. Lett.* **100** 070601
[29] Sandev T and Tomovski Ž 2014 *Phys. Lett.* A **378** 1
[30] Sandev T, Tomovski Ž and Dubbeldam J L A 2011 *Physica* A **390** 3627
[31] Sandev T, Metzler R and Tomovski Ž 2012 *Fract. Calc. Appl. Anal.* **15** 426
[32] Metzler R and Klafter J 2000 *Phys. Rep.* **339** 1
Metzler R and Klafter J 2004 *J. Phys. A: Math. Gen.* **37** R161
[33] Barkai E, Garini Y and Metzler R 2012 *Phys. Today* **65** 29
[34] Castiglione P, Mazzino A, Muratore-Ginanneschi P and Vulpiani A 1999 *Physica* D **134** 75
[35] Höfling F and Franosch T 2013 *Rep. Progr. Phys.* **76** 046602
[36] Krapf D and Metzler R 2019 *Phys. Today* **72** 48
[37] Klafter J and Sokolov I M 2005 *Phys. World* **18** 29
[38] Vilk O *et al* 2022 *Phys. Rev. Res.* **4** 033055
[39] Burov S, Jeon J -H, Metzler R and Barkai E 2011 *Phys. Chem. Chem. Phys.* **13** 1800
[40] Metzler R, Jeon J -H, Cherstvy A G and Barkai E 2014 *Phys. Chem. Chem. Phys.* **16** 24128
[41] Muñoz-Gil G *et al* 2021 *Nat. Commun.* **12** 6253
[42] Riascos A P and Mateos J L 2014 *Phys. Rev.* E **90** 032809
Riascos A P and Mateos J L 2015 *J. Stat. Mech.* P 07015
Michelitsch T M, Collet B A, Riascos A P, Nowakowski A F and Nicolleau F C G A 2017 *J. Phys. A: Math. Theor.* **50** 055003
Michelitsch T M, Collet B A, Riascos A P, Nowakowski A F and Nicolleau F 2017 *J. Phys. A: Math. Theor.* **50** 505004
Michelitsch T, Riascos A P, Collet B, Nowakowski A and Nicolleau F 2019 *Fractional Dynamics on Networks and Lattices* (Wiley)
[43] Seckler H and Metzler R 2022 *Nat. Commun.* **13** 6717
[44] Sokolov I M 2012 *Soft Matter* **8** 9043
[45] Molina-Garcia D, Sandev T, Safdari H, Pagnini G, Chechkin A and Metzler R 2018 *New J. Phys.* **20** 103027
[46] Jeon J -H, Martinez-Seara Monne H, Javanainen M and Metzler R 2012 *Phys. Rev. Lett.* **109** 188103
[47] Ślęzak J, Metzler R and Magdziarz M 2018 *New J. Phys.* **20** 023026
[48] van der Straeten E and Beck C 2011 *Physica* A **390** 951
[49] Vojta T, Skinner S and Metzler R 2019 *Phys. Rev.* E **100** 042142
[50] Sandev T and Tomovski Ž 2019 *Fractional Equations and Models: Theory and Applications* (Springer Nature)
[51] Tabei S M A, Burov S, Kim H Y, Kuznetsov A, Huynh T, Jureller J, Philipson L H, Dinner A R and Scherer N F 2013 *Proc. Natl Acad. Sci. USA* **110** 4911
[52] Weigel A V, Simon B, Tamkun M M and Krapf D 2011 *Proc. Natl Acad. Sci. USA* **108** 6438

[53] He Y, Burov S, Metzler R and Barkai E 2008 *Phys. Rev. Lett.* **101** 058101
[54] Bazhlekova E 2021 *Fract. Calc. Appl. Anal.* **24** 88
[55] Mainardi F 2010 *Fractional Calculus and Waves in Linear Viscoelesticity: An Introduction to Mathematical Models* (Imperial College Press)
[56] Metzler R, Barkai E and Klafter J 1999 *Phys. Rev. Lett.* **82** 3563
[57] Jeon J -H and Metzler R 2012 *Phys. Rev.* E **85** 021147
[58] Gorenflo R and Mainardi F 2005 *J. Phys.: Conf. Ser.* **7** 1
[59] Capelas de Oliveira E, Mainardi F and Vaz Jr J 2011 *Eur. Phys. J.* **193** 161–71
[60] Górska K, Pietrzak T, Sandev T and Tomovski Ž 2023 *J. Comput. Appl. Math.* **433** 115306
[61] Mainardi F 2014 *Discrete Contin. Dyn. Syst.* B **19** 2267
[62] Pollard H 1948 *Bull. Am. Math. Soc.* **54** 1115
[63] Górska K, Horzela A, Bratek L, Dattoli G and Penson K A 2018 *J. Phys. A: Math. Theor.* **51** 135202
[64] Górska K, Horzela A and Penson K A 2023 *J. Phys. A: Math. Theor.* **56** 313001
[65] Tateishi A A, Lenzi E K, da Silva L R, Ribeiro H V, Picoli S and Mendes R S 2012 *Phys. Rev.* E **85** 011147
[66] Chechkin A V, Gorenflo R and Sokolov I M 2002 *Phys. Rev.* E **66** 046129
[67] Chechkin A V, Klafter J and Sokolov I M 2003 *Europhys. Lett.* **63** 326
[68] Eab C H and Lim S C 2011 *Phys. Rev.* E **83** 031136
[69] Kochubei A N 2011 *Integr. Equ. Oper. Theory* **71** 583
[70] Mainardi F, Mura A, Gorenflo R and Stojanovic M 2007 *J. Vib. Control* **13** 1249
[71] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1955 *Higher Transcedential Functions* vol 3 (McGraw-Hill)
[72] Dräger J and Klafter J 2000 *Phys. Rev. Lett.* **84** 5998
[73] Godec A, Chechkin A V, Barkai E, Kantz H and Metzler R 2014 *J. Phys. A: Math. Theor.* **47** 492002
[74] Sinai Y G 1982 *Theor. Probab. Appl.* **27** 256
[75] Goychuk I, Kharchenko V I and Metzler R 2017 *Phys. Rev.* E **96** 052134
[76] Bouchaud J P and Georges A 1990 *Phys. Rep.* **195** 127
Bouchaud J P, Comtet A, Georges A and Le Doussal P 1990 *Ann. Phys.* **201** 285
[77] Padash A, Aghion E, Schulz A, Barkai E, Chechkin A V, Metzler R and Kantz H 2022 *New J. Phys.* **24** 073026
[78] Kochubei A N 2008 *J. Math Anal. Appl.* **340** 252
[79] Kochubei A N 2008 *Ukranian Math. J.* **60** 551
[80] Kochubei A N 2009 *J. Phys. A: Math. Theor* **42** 315203
[81] Lin S -D and Srivastava H M 2004 *Appl. Math. Comput.* **154** 725
[82] Balcerek M, Wylomańska A, Burnecki K, Metzler R and Krapf D 2023 *New J. Phys.* **25** 103031
[83] Balcerek M, Burnecki K, Thapa S, Wylomańska A and Chechkin A 2022 *Chaos* **32** 093114
[84] Han D, Korabel N, Chen R, Johnston M, Gavrilova A, Allan V J, Fedotov S and Waigh T A 2020 *eLife* **9** e52224
[85] Ślęzak J and Metzler R 2023 *J. Phys. A: Math. Theor.* A **56** 35LT01
[86] Wang W, Balcerek M, Burnecki K, Chechkin A V, Janušonis S, Slezak J, Vojta T, Wylomańska A and Metzler R 2023 *Phys. Rev. Res.* **5** L032025
[87] Fox Z R, Barkai E and Krapf D 2021 *Nat. Commun.* **12** 6162
[88] Liang Y, Wang W and Metzler R 2023 *Phys. Rev.* E **108** 024143
[89] Evans M R and Majumdar S N 2011 *Phys. Rev. Lett.* **106** 160601
Evans M R, Majumdar S N and Schehr G 2020 *J. Phys. A: Math. Theor.* **53** 193001
[90] Petreska I, Pejov L, Sandev T, Kocarev L and Metzler R 2022 *Fractal Fract.* **6** 88
[91] Sandev T and Iomin A 2022 *Special Functions of Fractional Calculus: Applications to Diffusion and Random Search Processes* (World Scientific)
[92] Prabhakar T R 1971 *Yokohama Math. J.* **19** 7
[93] Garra R and Garrappa R 2018 *Commun. Nonlinear Sci. Numer. Sim.* **56** 314
[94] Giusti A, Colombaro I, Garra R, Garrappa R, Polito F, Popolizio M and Mainardi F 2020 *Frac. Calc. Appl. Anal.* **23** 9
[95] Feller W 1968 *An Introduction to Probability Theory and Its Applications* vol II (Wiley)
[96] Apelblat A 2010 *Integral Transforms and Volterra Function* (Nova Sci. Pub., Inc.)
[97] Apelblat A 2013 *Volterra Functions* (Nova Sci. Pub., Inc.)
[98] Bateman H and Erdélyi A 1955 *Higher Transcendental Functions* vol 3 (Robert E. Krieger Pub. Comp)