fractal and fractional

Article

Topological Subordination in Quantum Mechanics

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Abstract: An example of non-Markovian quantum dynamics is considered in the framework of a geometrical (topological) subordination approach. The specific property of the model is that it coincides exactly with the fractional diffusion equation, which describes the geometric Brownian motion on combs. Both classical diffusion and quantum dynamics are described using the dilatation operator $x\frac{d}{dx}$. Two examples of geometrical subordinators are considered. The first one is the Gaussian function, which is due to the comb geometry. For the quantum consideration with a specific choice of the initial conditions, it corresponds to the integral representation of the Mittag–Leffler function by means of the subordination integral. The second subordinator is the Dirac delta function, which results from the memory kernels that define the fractional time derivatives in the fractional diffusion equation.

Keywords: subordination; comb model; anomalous diffusion; non-Markovian quantum dynamics; dilatation operator

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1. Introduction

In this paper, we consider both classical anomalous diffusion and quantum non-Markovian dynamics in the two dimensional comb geometry, both of which are controlled by the operator $x\frac{d}{dx}$. Its Hermitian counterpart is the dilatation (contraction) operator, which has the Hamiltonian form $\hat{\mathcal{H}}_0 = -ix\hbar\frac{d}{dx} - i\hbar/2 \equiv \hat{x}\hat{p}_x - i\hbar/2$. The system attracts much attention in quantum mechanics and number theory in connection with the Riemann Hypothesis [1–3], where $\mathcal{H}_0$ is known as a Berry–Keating–Connes Hamiltonian [4]. This Hamiltonian has been considered in various general contexts, including space–time conformal transformations [5–7], the quantum Mellin transform [8], and a random walk [9]. In the Hamiltonian dynamics, it describes inverted potentials [10] with the dynamics near hyperbolic points [11], including a singular behaviour [12] and exponential spreading of the phase space [13], and the relaxation dynamics in the comb geometry [14].

We have analysed plausible quantum non-Markovian, non-unitary dynamics derived from the Markovian-unitary quantum dynamics [15]. This is achieved via a corresponding subordinator, which is due to the comb geometry. As is known [16], according to a subordination approach, one can derive a variety of (non-Markovian) processes from Markovian ones, such as the Brownian motion with stationary transition probabilities $G_0(x,t)$, by introducing a so-called operational time. In this case, a new process is described using a subordination integral as follows

$$G(x,t) = \int_0^\infty G_0(x,u)h(u,t)\,du, \tag{1}$$

where u is the operational time and $h(u,t)$ is a subordination function (or subordinator) that depends on both physical time t and operational time u, respectively.

Subordination has been introduced by Bochner [17] as a notion of a subordinated semi-group, and its interpretation and systematic presentation in terms of the operational time has been suggested for the fractional calculus [18]. Our interest in the subordination approach in the comb geometry relates to the fact that the latter is responsible for anomalous or fractional diffusion, which is described by a linear fractional Fokker–Planck equation (FFPE). A comb model was originally introduced to understand the anomalous diffusion in percolating clusters [19–22]. Various settings of the comb geometry structures lead to various FFPEs, whose effects include both subdiffusion [23,24], including ultra-slow diffusion [25], and superdiffusion [26–28]. The nontrivial nature of transport along combs is discernible from the fact that the motion along the branches results in a long-range memory for the motion along the backbone, where the corresponding anomalous behaviour of the transport takes place [29]. The implementation of comb models in various natural phenomena is reflected in recent reviews [29–34]. In contemporary studies, the comb model is employed in a variety of applications including explanations of experimental realizations in percolation clusters, energy transfer in dendritic polymers [35], diffusion of drugs in the circulatory system [36], anomalous diffusion in neurons [37,38], random walks of active species in porous media [39] utilized in microelectronics [40], electron transport in disordered nanostructured semiconductors [41], and so on. Another important issue discussed in the framework of the comb model [42] is related to understanding the geometry impact on diffusion with stochastic resetting [43] and first-passage properties in nonequilibrium systems [44].

An elegant way for a heuristic formulation of the comb model in the framework of a Fokker–Planck equation has been suggested in [22], and its description in the framework of the fractional calculus is presented in Appendices A and B. In particular in Appendix B, we introduce the details of how the transport in the comb geometry leads to the fractional transport described by the FFPE. In this sense, the relation between the comb geometry and the subordination approach is straightforward and is established in Section 2.

A general form of this equation for the description of slow and ultra-slow diffusion has been suggested by means of memory kernels $\gamma(t)$ and $\eta(t)$, and the corresponding integro-differential comb equation reads [25]

$$\int \gamma(t-t')\,\frac{\partial}{\partial t'}P(x,y,t')\,dt' = \delta(y)\int \eta(t-t')\,L_{FP}P(x,y,t')\,dt' + D_y\,\frac{\partial^2}{\partial y^2}P(x,y,t). \tag{2}$$

Here $-\infty < x, y < \infty$, and the initial condition is

$$P(x,y,t=0) = \frac{1}{2}\,\delta(|x|-x_0)\delta(y), \tag{3}$$

where an arbitrary $x_0 \in \mathbb{R}$ is specified later in the text. The boundary conditions are set to zero at infinity for both the probability density function (PDF) $P(x,y,t)$ and its first derivatives with respect to x and y. The memory kernels $\gamma(t)$ and $\eta(t)$ are integrable non-negative functions of time, which approach zero in the long time limit. The PDF $P(x,y,t)$ defines the test particle at position (x,y) at time t. Here, x measures the direction along the backbone of the comb, which is controlled by the Fokker–Planck operator $\delta(y)\,L_{FP}$ at $y=0$, while y is the distance along the side branches (also fingers or teeth). The latter are continuously distributed along the backbone. The analytical form of the model, suggested by heuristic arguments with $L_{FP} = D_x\,\partial_x^2$ in [22], supposes an inhomogeneous two-dimensional diffusion, where the non-zero diagonal components of a diffusion tensor, $D_x\,\delta(y)$ and D_y are the diffusion coefficients in the x and y directions, correspondingly. This special case is considered in Appendix B.

According to this formulation, the problem at hand is to develop a subordinated quantum dynamics, which is a quantum counterpart of classical turbulent diffusion in the comb geometry. To this end in Section 2, the FFPE related to the comb model (2) is constructed together with the solution in the subordination form. Its quantum counterpart is analysed in Section 3. A subordination approach for some generalisation of both fractional diffusion and fractional quantum dynamics is suggested in Section 4. In Section 5 we present the conclusion of the obtained results. Additional material necessary for the analysis is considered in the Appendices.

2. Subordinated Comb Diffusion

In the present analysis, we consider $L_{FP} = -v \frac{\partial}{\partial x} x$, which leads to a turbulent-like diffusion analogous to the geometrical Brownian motion, when the mean squared displacement (MSD) grows exponentially in time [14,26,45]. By means of commutation, the quantum counterpart of L_{FP} relates to the Hamiltonian $\hat{\mathcal{H}}_0$ as well.

In what follows, we use dimensionless variables and parameters without loss of generality. In particular, the velocity v and the diffusion coefficient $D_y \equiv D$ form the parameters D/v and D/v^2, which are used for dimensionless space and time variables, respectively.

It is interesting to admit that for the drift term, the corresponding diffusion equation can be obtained from a Langevin equation. Correspondingly, a possible realisation of the turbulent transport can be discussed in the framework of a Langevin equation in a so-called Matheron–de Marsily form [46]

$$\dot{X} = v\,\delta(Y)\,X, \quad \dot{Y} = \xi(t), \tag{4}$$

where $\xi(t)$ is a random Gaussian delta correlated process $\langle \xi(t)\xi(t')\rangle = 2\,\delta(t-t')$, where $D \equiv D_y$ is a diffusion coefficient. That is, white noise affects the velocity of the drift along the x axis. Following a standard procedure, one obtains a Fokker–Planck equation for the PDF.

$$P(x,y,t) = \langle \delta(X(t)-x)\delta(Y(t)-y)\rangle$$

as follows

$$\frac{\partial}{\partial t}P(x,y,t) = -v\,\delta(y)\,\frac{\partial}{\partial x}xP(x,y,t) + D\,\frac{\partial^2}{\partial y^2}P(x,y,t). \tag{5}$$

We use the same boundary and symmetrical initial conditions as in Equation (2). The solution to the equation is a convolution integral. That is, the PDF $P(x,y,t)$ reads

$$P(x,y,t) = P(x,y=0,t) * P_y(y,t) = \int_0^t P(x,y=0,t-t')P_y(y,t')\,dt',$$

where $P(x,y=0,t)$ is a non-conserving PDF on the backbone and $P_y(y,t)$ is the Lévy–Smirnov density $\mathcal{L}[P_y(y,t)](s) = \exp\big(-|y|\sqrt{s/D}\big)$, see Appendix B. In the Laplace space, $\tilde{P}(x,y,s) = \mathcal{L}[P(x,y,t)]$, we have

$$\tilde{P}(x,y,s) = \tilde{P}(x,y=0,s)\,\exp\Big(-|y|\sqrt{s/D}\Big). \tag{6}$$

Our main interest here is related with finding the marginal PDF,

$$P_1(x,t) = \int_{-\infty}^{\infty} P(x,y,t)\,dy. \tag{7}$$

To find the corresponding equation, we integrate Equation (5) with respect to y, which after the Laplace transform reads as

$$s\tilde{P}_1(x,s) - \delta(x) = -v\,\frac{\partial}{\partial x}x\tilde{P}(x,y=0,s). \tag{8}$$

Then, the integration of Equation (6) with respect to y yields $\tilde{P}(x,y=0,s) = \frac{1}{2}\sqrt{s/D}\,\tilde{P}_1(x,s)$, and substituting the result in Equation (8), we obtain

$$\tilde{P}_1(x,s) = \frac{s^{-1/2}}{s^{1/2} + \frac{1}{2}vD^{-1/2}\frac{\partial}{\partial x}x} P_1(x,t=0), \tag{9}$$

where $P_1(x,t=0) = \delta(x)$ is the initial condition. The Laplace inversion of Equation (9) yields the FFPE for the marginal PDF of the backbone anomalous transport, as follows

$$_C D_t^{1/2} P_1(x,t) = -\frac{1}{2}vD^{-1/2}\frac{\partial}{\partial x}xP_1(x,t), \tag{10}$$

where $_C D_t^{\alpha}$ is the Caputo fractional derivative of order α, see Equation (A4). This equation has an exact solution in the form of the log-normal distribution [26,45].

We are, however, interested in presenting the solution in the form of the subordination integral defined in Equation (1). To this end, the denominator in Equation (9) is rewritten in the exponential operator form,

$$\begin{aligned} \tilde{P}_1(x,s) &= \int_0^\infty s^{-1/2}\, e^{-us^{1/2}} \exp\left(-u\frac{1}{2}vD^{-1/2}\frac{\partial}{\partial x}x\right) P_1(x,t=0)\, du \\ &\equiv \int_0^\infty \tilde{h}(s,u)\, \hat{G}_0(x,u)\, P_1(x,u=0)\, du, \end{aligned} \tag{11}$$

where $P_1(x,t=0) = P_1(x,u=0)$ and $\tilde{h}(s,u) = s^{-1/2}\, e^{-us^{1/2}}$ are the subordinator that subordinates the FFPE (10) to a standard drift equation for the evolution operator $\hat{G}_0(x,u) = \exp\left(-u\frac{1}{2}vD^{-1/2}\frac{\partial}{\partial x}x\right)$, in the form

$$\frac{\partial}{\partial u}\hat{G}_0 = -\frac{1}{2}vD^{-1/2}\frac{\partial}{\partial x}x\hat{G}_0. \tag{12}$$

The formal solution to Equation (12) is

$$\hat{G}_0(x,u) = \exp\left(-u\frac{1}{2}vD^{-1/2}\frac{\partial}{\partial x}x\right) P_1(x,u=0).$$

Note that since all parameters and variables are dimensionless, the diffusion coefficient $vD^{-1/2}$ in Equation (12) can be taken to be the same as in Equation (10).

3. Subordinated Quantum Mechanics

It is worth stressing that (remarkably) the FFPE (10) and the drift Equation (12) are quantum Schrödinger equations as well. Indeed, let us multiply both equations by $i\hbar$, where for our dimensionless variables' consideration, $\hbar$ is a dimensionless Planck constant and we can use the same power of the Planck constant for both equations. Then, Equations (10) and (12) become the fractional Schrödinger equation (FSE) and the conventional (standard) Schrödinger equation, respectively,

$$i\hbar\, _C D_t^{1/2}\psi(x,t) = \omega\, \hat{p}\hat{x}\psi(x,t) \tag{13}$$

and

$$i\hbar\frac{\partial}{\partial u}\psi_0(x,u) = \omega\, \hat{p}\hat{x}\psi_0(x,u) \tag{14}$$

where $\omega = \frac{1}{2}vD^{-1/2}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and $\hat{x} = x$ are the quantum momentum operator and the quantum coordinate, respectively with the commutation rule $[x,\hat{p}] = i\hbar$. For convenience of notation and without restriction of generality, we set $\omega = 1$.

An essential difference from the classical diffusion case is that the quantum initial condition is the wave function $\psi(x,t=0) = \psi_0(x,u=0)$. Correspondingly, the PDF is $|\psi(x,t=0)|^2$. These functions can be square integrable functions, such that

$$\int |\psi(x,t=0)|^2\, dx = \int |\psi_0(x,u=0)|^2\, dx = 1.$$

However, it is not mandatory and the normalisation condition to the Dirac δ function is also possible and considered in the present analysis, see Appendix C.

It is reasonable to choose the quantum initial conditions in the form of the eigenfunctions of the Hermitian Hamiltonian $\hat{\mathcal{H}}_0$. In this case of the continuous spectrum, the normalisation condition needs to be calculated with some care, see the discussion in Appendix C. The dilatation operator

$$\hat{\mathcal{H}}_0 = \hbar\left[-ix\frac{\partial}{\partial x} - i/2\right] = \hat{p}x + i\hbar/2$$

determines the complete set of eigenfunctions $\chi_e(x)$ with the eigenvalues e according to the eigenvalue problem $\hat{\mathcal{H}}\chi_e(x) = \hbar\, e\, \chi_e(x)$, where e is the continuous spectrum and the eigenfunctions are [1,4]

$$\chi_e(x) = \frac{1}{\sqrt{N|x|}}\exp(ie\ln|x|), \tag{15}$$

which satisfies the boundary conditions $\chi_e(x = \pm\infty) = 0$ and $N = 4\pi$. Therefore,

$$\psi(x, t = 0) = \psi_0(x, u = 0) = \chi_e(x). \tag{16}$$

In this case, both Equations (13) and (14) describe relaxation processes from the initial energy level e according to the non-Hermitian Hamiltonian

$$\hat{\mathcal{H}} = \omega\,\hat{p}x = \omega\,\hat{\mathcal{H}}_0 - i\hbar\omega/2 = \hat{\mathcal{H}}_0 - i\hbar/2, \tag{17a}$$

$$\hat{\mathcal{H}}\chi_e(x) = (e - i\hbar/2)\chi_e(x) = e_i\,\chi_e(x), \tag{17b}$$

where we set $\omega = 1$. The solution to the FSE (13) subordinated to Equation (14) is defined by the subordination integral in Equation (11). Taking into account the eigenvalue problem (17), the solution in the Laplace space reads as follows,

$$\tilde{\psi}(x, s) = \int_0^\infty s^{-1/2}\, e^{-us^{1/2}}\, e^{-iu(e-i\hbar/2)/\hbar}\chi_e(x)\, du. \tag{18}$$

This leads to the integral representation of the Mittag–Leffler function, namely [47]

$$\begin{aligned}\psi(x, t) =& E_{1/2}\Big(-i(e - i\hbar/2)t^{1/2}/\hbar\Big)\chi_e(x) = \mathcal{L}^{-1}\left[\frac{s^{-1/2}}{s^{1/2} - (e - i\hbar/2)/i\hbar}\right]\chi_e(x) \\ =& \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{u^2}{4t}}\, e^{-iu(e-i\hbar/2)/\hbar}\chi_e(x)\, du,\end{aligned} \tag{19}$$

where $E_\alpha(zt^\alpha)$ is the one parameter Mittag–Leffler function, see Equation (A8) for $\nu \to 1/2$, $\beta \to 1$ and $z \to -i(e - i\hbar/2)/\hbar$.

It should be pointed that from the analysis of Sections 2 and 3, it follows that the one dimensional quantum dynamics is the same as the classical one, and since the latter is well defined on the backbone, the former is well defined in the same area as well. This coincidence results from the fact that these two phenomena are described just by the same Equation (10), which is the FFPE for the marginal PDF of the backbone anomalous transport and at the same time, it is the FSE for the wave function of the quantum dynamics on the same backbone.

Discussion of the Preliminary Results

From this consideration of the toy model, we obtain two important results for the classical and quantum dynamics on the comb. The first one is that the inhomogeneous advection on the comb, which is described by means of the FFPE (10), describes also the fractional dynamics of a quantum particle with the non-Hermitian Hamiltonian $\hat{\mathcal{H}} = \hat{p}x = x\hat{p} - i\hbar/2$ in the framework of the FSE (13). In the latter case, the time evolution of the system follows the Mittag–Leffler function $U(t) = E_\alpha(-e_i t^\alpha)$, where $e_i = e - i\hbar/2$ is the continuous spectrum with the decay term. To some extent, this result follows immediately from the separation of variables and the fact that the Mittag–Leffler function is the eigenfunction of the Caputo fractional derivative. In this case, the last two lines of Equation (19) are both the integral representation of the Mittag–Leffler function and the subordination integral. In this connection, this is a particularly simple case, when the evolution can be easily expressed by the Mittag–Leffler function.

The second result is that in the general case, say of the memory kernel $\gamma(t)$, as in Equation (2), the subordination integral is a convenient form of the solution for the evolution operator $U(t)$, which is subordinated to the Schrödinger Equation (14). This situation is considered in the next section. In contemporary studies, the model is employed in a variety of applications including explanations of experimental realisations in percolation clusters, energy transfer in dendritic polymers [35], diffusion of drugs in the circulatory system [36], anomalous diffusion in neurons [37,38], and the random walk of active species in porous media [39] utilized in microelectronics [40].

4. Generalised Comb Model

Let us now consider the generalised comb Equation (2). Performing the Laplace transform, we obtain

$$\tilde{\gamma}(s)\left[s\tilde{P}(x,y,s) - P_0(x,y)\right] = \delta(y)\,\tilde{\eta}(s)\,L_{FP}\tilde{P}(x,y,s) + D\,\frac{\partial^2}{\partial y^2}\tilde{P}(x,y,s). \tag{20}$$

In this case, the ansatz (6) reads

$$\tilde{P}(x,y,s) = \tilde{P}(x,y=0,s)\,\exp\left(-|y|\sqrt{s\tilde{\gamma}(s)/D}\right), \tag{21}$$

which yields the relation to the marginal PDF $\tilde{P}_1(x,s)$

$$\tilde{P}(x,y=0,s) = \sqrt{s\tilde{\gamma}(s)/4D}\,\tilde{P}_1(x,s). \tag{22}$$

Integrating Equation (20) with respect to y, then accounting for expression (22) and performing a quantisation in complete analogy of Section 3, we arrive at the FSE for the wave function in the Laplace space,

$$i\hbar\,\tilde{\xi}(s)[s\tilde{\psi}(x,s) - \psi_0(x)] = \hat{\mathcal{H}}\tilde{\psi}, \tag{23}$$

where

$$\tilde{\xi}(s) = \frac{1}{\tilde{\eta}(s)}\sqrt{\frac{\tilde{\gamma}(s)}{s}}. \tag{24}$$

The quantum solution presented in the form of the subordination integral is

$$\psi(x,t) = \mathcal{L}^{-1}\left[\frac{\tilde{\xi}(s)}{s\tilde{\xi}(s) + i\hat{\mathcal{H}}/\hbar}\right]\psi_0(x) = \left[\int_0^\infty h(t,u)e^{-iu\frac{\hat{\mathcal{H}}}{\hbar}}\,du\right]\psi_0(x), \tag{25}$$

where the subordinator $h(t,u)$ is determined by the inverse Laplace transform, as follows

$$h(t,u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\xi}(s)\,e^{-us\tilde{\xi}(s)}\,e^{st}\,ds. \tag{26}$$

Note that for $\gamma(t) = \eta(t) = \delta(t)$ we immediately arrive at the result of Equation (19). In more sophisticated cases of the memory kernels, like kernels of distributed order of the form

$$\gamma(t) = \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\,d\alpha, \tag{27}$$

it is not possible to obtain solutions in an analytical form, see also the discussion in [25]. We are finishing this section with one more example with $\tilde{\tilde{\xi}}(s) = 1$.

4.1. Memory Kernels with $\tilde{\tilde{\xi}}(s) = 1$

A variety of realisations of the memory kernel have been considered in [25]. Let us consider the memory kernels $\gamma(t)$ and $\eta(t)$, such that $\tilde{\tilde{\xi}}(s) = 1$ when the memory kernels compensate the comb geometry. In this case, the subordinator is $h(t,u) = \delta(u-t)$ and the quantum dynamics is governed by the standard Schrödinger Equation (14).

For example, for $\gamma(t) = \frac{t^{-1/2}}{\Gamma(1/2)}$ and $\eta(t) = \frac{t^{-1/4}}{\Gamma(3/4)}$, then $\tilde{\tilde{\xi}}(s) = 1$. In the general case of the power law forms for the memory kernels $\gamma(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}$ and $\eta(t) = \frac{t^{\mu-1}}{\Gamma(\mu)}$ with $0 < \mu, \nu < 1$, we have $\tilde{\tilde{\xi}}(s) = 1$ for $\nu + 1 = 2\mu$. In the case when $2\mu < \nu + 1$, the quantum dynamics follows the Mittag–Leffler function $E_\alpha\big(i\hat{\mathcal{H}}t^\alpha/\hbar\big)$, where $\alpha = \frac{1}{2}(\nu+1) - \mu$.

4.1.1. Comb Wave Equation

The second example relates to the wave equation in the comb geometry. Let us consider $\gamma(t) = -\frac{d}{dt}\delta(t)$ and $\eta(t) = \delta(t)$, then $\tilde{\tilde{\xi}} = 1$. In this case, Equation (2) reads

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)F(x,y,t) = 0, \quad \Delta = \delta(y)\, L_{FP} + D\,\frac{\partial^2}{\partial y^2}, \tag{28}$$

where the Laplace operator Δ has the comb structure, which eventually leads to a diffusion-wave-like process on the comb with an additional initial condition $\frac{\partial}{\partial t}F(x,y,t=0) = 0$.

Indeed, let us integrate Equation (28) with respect to y with $f(x,t) = \int F(x,y,t)\,dy$. Then, performing the Laplace transform and substitution analogous to Equation (22),

$$\tilde{F}(x,y,s) = \tilde{P}(x,y=0,s)\,\exp\left(-|y|\sqrt{s\tilde{\gamma}(s)/D}\right), \tag{29a}$$

$$\tilde{F}(x,y=0,s) = \sqrt{s\tilde{\gamma}(s)/4D}\,\tilde{f}(x,s), \tag{29b}$$

we arrive at its "fractional" counterpart that reads

$$\tilde{\tilde{\xi}}(s)\big[s\tilde{f}(x,s) - f_0(x)\big] = L_{FP}\tilde{f}, \tag{30}$$

which is just a diffusion equation for $\tilde{\tilde{\xi}}(s) = 1$.

Again, multiplying Equation (30) by $i\hbar$, we arrive at the conventional Schrödinger equation

$$i\hbar\,\frac{\partial}{\partial t}\psi(x,t) = \hat{\mathcal{H}}\psi(x,t) \tag{31}$$

with the Hamiltonian $\hat{\mathcal{H}}$ defined in Equation (17a). An important feature here is the zero boundary conditions at infinity as the necessary conditions for the considerations of Equations (28), (30) and (31).

5. Conclusions

We have derived both the fractional geometric Brownian motion on combs and the corresponding quantum particle dynamics based on the dilatation operator $x\frac{d}{dx}$, based on the same dynamic equation. However, there is an essential difference between these descriptions. Namely, the classical case is described by the probability density function, which is integrable. In contrast, the quantum case is described by the wave function, which is square integrable. Both cases are considered in the framework of the subordination approach, when both the fractional Fokker–Planck equation and the fractional Schrödinger equation are subordinated to the their conventional counterparts. We considered two cases of geometrical subordinators. The first one is the Gaussian function, which is due to the comb geometry. For the quantum consideration with a specific choice of the initial conditions, it corresponds to the kernel of the integral representation of the Mittag–Leffler function. The second subordinator is the Dirac delta function that results from memory kernels that compensate the geometry effect in the generalised comb model.

In the latter case, there is an interesting effect of reduction of the wave equation to the diffusion equation in the comb geometry. In particular, let us consider the comb wave equation as follows

$$\frac{\partial^2}{\partial t^2}F(x,y,t) = \delta(y)\left(\frac{\partial}{\partial x}x\right)^2 F(x,y,t) + \frac{\partial^2}{\partial y^2}F(x,y,t) \tag{32}$$

with the initial conditions $P(x,y,t=0) = P_0(x,y)$, when the first derivative with respect to time is zero at $t=0$. The boundary conditions are set to zero at infinity. The diffusion operator $\left(\frac{\partial}{\partial x}x\right)^2$ results from the multiplicative noise in the corresponding Langevin equation (in the Matheron–de Marsily form (4)) [45]. Performing the standard procedure of Sections 2 and 4.1.1 (see also Appendix B), we arrive at the Fokker–Planck equation for the marginal function $f(x,t) = \int F(x,y,t)\,dy$, that is

$$\frac{\partial}{\partial t}f(x,s) = \frac{1}{2}\left(\frac{\partial}{\partial x}x\right)^2 f(x,s), \tag{33}$$

where $f(x,t)$ becomes the marginal PDF that describes the geometrical Brownian motion.

It must be noted that in the quantum analysis, we considered specific initial conditions in the form of the eigenfunctions of the dilatation operator $\hat{\mathcal{H}}$, where we only touched some specific aspects of the evolution of the initial wave packet due to the non-Hermitian dilatation operator. This approach eventually corresponds to the variable separation, concentrating our attention to the form of the corresponding subordinators for the evolution functions. However, in a general choice of the initial condition, the latter can be presented in the form of the spectral decomposition over the eigenfunctions, for example the initial wave function can be $\psi_0(x) = \int A(e)\,\chi_e(x)\,de$. As shown above, this fractional Cauchy problem can be considered by either approach in the framework of both FFPE (10) and FSE (13), which are equivalent to the relation $P_1(x,t=0) = |\psi_0(x)|^2$. In this case, the evolution of the MSD is well defined, and grows exponentially with time in both cases of fractional diffusion [45] and fractional quantum mechanics [13,48]. Note that a general subordination approach for time-fractional evolution equations is studied as well [49].

It should be pointed out that the fractional property of the transport (and correspondingly the marginal PDF) is described by the subordinator in the form of the normal distribution in Equations (11), (18) and (19), which is the topological subordinator. It reflects the classical comb geometry and normal diffusion along the fingers. Remarkably, the obtained Equation (10) describes also the one dimensional fractional time quantum dynamics. It follows from the simple multiplication by $i\hbar$, which is the identity operation.

The comb geometry can be more sophisticated [29,34]. For example, by considering a three dimensional (3D) comb or a comb with the power law distribution of the fingers length. In this case, the topological subordinator $h(u,t)$ is described by the Fox H-functions, and in the Laplace space, it has a simple form $\tilde{h}(u,s) = s^{\nu-1}e^{-us^{\nu}}$, where the fractional index is $0 < \nu < 1$, and for the 3D comb we have $\nu = 1/4$.

There is another interesting result that follows from the analysis and its generalisation. Namely, let us define the fractional time quantum evolution according to the subordinator $h(u,t)$ in the standard way, $\Psi(x,t) = \int_0^\infty h(u,t)\psi(x,u)du$, where $\tilde{h}(u,s) = s^{\nu-1}e^{-us^{\nu}}$ in the Laplace space. In this case, if the conventional quantum mechanics with a Hamiltonian $\hat{H}$ is due to the wave function $\psi(x,u) = e^{-i\hat{H}t/\hbar}\psi_0(x)$, then the corresponding FTSE reads

$$i\hbar\,{}_C D_t^{\nu}\Psi = \hat{H}\Psi.$$

In conclusion, in this developing field, which is similar to the concept of quantum walks, the full physical meaning of the model is not yet established. For the topology of the comb, we could say that it started from pure curiosity and then the obtained effects turned out to be very interesting. So, while a clear physical interpretation is missing, the study could be of interest in the development of stochastic features in quantum physics. Moreover, we can view the fingers/teeth of the comb as an environment with some kinds of trapping quasi-states (or confining potentials).

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Abbreviations

The following abbreviations are used in this manuscript:

PDF	Probability density function
MSD	Mean squared displacement
FFPE	Fractional Fokker–Planck equation
FSE	Fractional Schrödinger equation

Appendix A. A Brief Survey on Fractional Integration

Extended reviews of fractional calculus can be found, e.g., in [50–54]. Fractional integration of the order of α is defined by the Riemann–Liouville operator

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}\, dy, \tag{A1}$$

where $\alpha > 0$, $x > a$ and $\Gamma(z)$ are the Gamma function. Fractional derivation was developed as a generalisation of integer order derivatives and is defined as the inverse operation to the fractional integral. Therefore, the fractional derivative is defined as the inverse operator to I_x^α, namely $I_x^\alpha D f(x) = I_x^{-\alpha} f(x)$ and $I_x^\alpha = D_x^{-\alpha}$. Its explicit form is

$$D^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x f(y)(x-y)^{-1-\alpha} dy. \tag{A2}$$

This integral diverges for arbitrary $\alpha > 0$, and a regularisation procedure is introduced with two alternative definitions of D^α. For an integer n defined as $n-1 < \alpha < n$, we obtain the Riemann–Liouville fractional derivative,

$${}_{\mathrm{RL}}D^\alpha f(x) \equiv D_x^\alpha f(x) = \frac{d^n}{dx^n} I_x^{n-\alpha} f(x)\,, \tag{A3}$$

and the Caputo fractional derivative,

$${}_{\mathrm{C}}D_x^\alpha f(x) = I_x^{n-\alpha} \frac{d^n}{dx^n} f(x)\,. \tag{A4}$$

There is no constraint on the lower limit a. For example, when $a = 0$, one has

$$ {}_{\mathrm{RL}}D_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\, x^{\beta-\alpha}. $$

This fractional derivation with the fixed low limit is also called the left fractional derivative. One can introduce the right fractional derivative as well, where the upper limit a is fixed and $a > x$. For example, the right fractional integral is

$$ I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^a (y-x)^{\alpha-1} f(y) dy \,. \tag{A5} $$

Another important property is $D^\alpha I^\beta = I^{\beta-\alpha}$, where subscripts are omitted for brevity. Note that the inverse combination is not valid. In general, $I^\beta D^\alpha \neq I^{\beta-\alpha}$, since it depends on the lower limits of the integration [52]. We also use here a convolution rule for the Laplace transform for $0 < \alpha < 1$

$$ \mathcal{L}[I_x^\alpha f(x)] = s^{-\alpha} \tilde{f}(s). \tag{A6} $$

Note that for arbitrary $\alpha > 1$, the treatment of the Caputo fractional derivative by the Laplace transformation is more convenient than that of the Riemann–Liouville one.

Solutions considered in the paper can be also obtained in the form of the Mittag–Leffler function by means of the Laplace inversion [47,52,55]

$$ E_{\nu,\beta}(zr^\nu) = \frac{r^{1-\beta}}{2\pi i} \int_C \frac{s^{\nu-\beta}\, e^{sr}}{s^\nu - z}\, ds, \tag{A7} $$

where C is a suitable contour of integration, starting and finishing at $-\infty$ and encompassing a circle $|s| \leq |z|^{1/\nu}$ in the positive direction, and $\nu, \beta > 0$.

It is also convenient to present the Mittag–Leffler $E_{\nu,\beta}(-zr^\nu)$ function in the series representation. Expanding the denominator formally, we have

$$ E_{\nu,\beta}(zr^\nu) = \sum_{n=0}^{\infty} \frac{r^{1-\beta}}{2\pi i} \int_C s^{\nu-\beta}\, e^{sr}\, s^{-n\nu}\, z^n\, ds = \sum_{n=0}^{\infty} \frac{(zr^\nu)^n}{\Gamma(n\nu+\beta)}. \tag{A8} $$

From the series representation (A8), the asymptotic behaviour can be easily found as well [47]. In particular, for $E_{\nu,\beta}(-zr^\nu)$ in the limit $|zr^\nu| \ll 1$, we obtain the stretched exponential behaviour of the Mittag–Leffler function,

$$ E_{\nu,\beta}(-zr^\nu) \approx \frac{1}{\Gamma(\beta)}\left[1 - \frac{\Gamma(\beta)zr^\nu}{\Gamma(\nu+\beta)}\right] \approx \frac{1}{\Gamma(\beta)} \exp\left[-\frac{\Gamma(\beta)zr^\nu}{\Gamma(\nu+\beta)}\right]. \tag{A9} $$

In the opposite case of $|zr^\nu| \gg 1$, we have [47],

$$ E_{\nu,\beta}(-zr^\nu) \approx \frac{(zr^\nu)^{-1}}{\Gamma(\beta-\nu)}, \tag{A10} $$

where $|\arg(zr^\nu)| < (1-\nu/2)\pi$.

Appendix B. Fractional Fokker–Planck Equation

Considering Equation (2) with $\gamma(t) = \eta(t) = \delta(t)$ and $L_{FP} = D_x \frac{\partial^2}{\partial x^2}$, we arrive at the well known form of the comb model, suggested in [22], which reads

$$ \frac{\partial}{\partial t} P(x,y,t) = \delta(y)\, D_x \frac{\partial^2}{\partial x^2} P(x,y,t) + D_y \frac{\partial^2}{\partial y^2} P(x,y,t). \tag{A11} $$

The same initial and boundary conditions as in Equation (2) are imposed. Integration with respect to y leads to an equation for the marginal distribution $P_1(x,t) = \int_{-\infty}^{\infty} P(x,y,t)\, dy$:

$$\frac{\partial}{\partial t}P_1(x,t) = D_x \frac{\partial^2}{\partial x^2}P(x,y=0,t), \tag{A12}$$

where $\hat{P}(x,y=0,s) = f(x,s)$. To obtain this equation in a closed form, we perform the Laplace transform for the solution, which is

$$\hat{P}(x,y,s) = \hat{P}(x,y=0,s)\,\exp\Big(-\sqrt{s/D_y}\,|y|\Big), \tag{A13}$$

and integrate it with respect to y. This yields the relation $\hat{P}(x,y=0,s) = \sqrt{s/4D_y}\,\hat{P}_1(x,s)$. Using this relation and the Laplace transforming Equation (A12), we find

$$s^{1/2}\hat{P}_1(x,s) - \delta(x)\,s^{-1/2} = \frac{D_x}{2\sqrt{D_y}}\frac{\partial^2}{\partial x^2}\hat{P}_1(x,s). \tag{A14}$$

Performing the Laplace inversion, we obtain the integro-differential equation

$$_{\mathrm{C}}D_t^{1/2}P_1(x,t) = \frac{D_x}{2\sqrt{D_y}}\frac{\partial^2}{\partial x^2}P_1(x,t). \tag{A15}$$

The integro-differential operator $_{\mathrm{C}}D_t^{1/2}$ is the so-called Caputo fractional derivative, defined in Equation (A4), which reads

$$_{\mathrm{C}}D_t^{\alpha}P_1(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-t')^{\alpha}}\frac{dP_1(t')}{d\,t'}\,dt', \quad 0<\alpha<1, \tag{A16}$$

while Equation (A15) is called the fractional Fokker–Planck equation (FFPE). Instead of the Caputo derivative, it is possible to employ the Riemann–Liouville fractional derivative, defined in Equation (A3), which reads

$$\mathcal{D}_t^{\alpha}P_1(t) \equiv {}_{\mathrm{RL}}D^{\alpha}P_1(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\,t}\int_0^t \frac{P_1(t')}{(t-t')^{\alpha}}\,dt', \quad 0<\alpha<1. \tag{A17}$$

This operator leads to the different form of the FFPE [56], as follows

$$\frac{\partial}{\partial t}P_1(x,t) = \frac{D_x}{2\sqrt{D_y}}\,{}_{\mathrm{RL}}D_t^{1-\alpha}\frac{\partial^2}{\partial x^2}P_1(x,t), \tag{A18}$$

where $\alpha = 1/2$. For arbitrary $0<\alpha<1$, this equation is a general form of the FFPE with the solution (A28)

$$P_1(x,t) = \frac{1}{\sqrt{D_\alpha t^\alpha}}\,H_{1,1}^{1,0}\left[\frac{x^2}{D_\alpha t^\alpha}\middle|\begin{array}{c}(1-\alpha/2,\alpha)\\(0,2)\end{array}\right]. \tag{A19}$$

In that case, the diffusion coefficient must be generalised as well, $D_x/(2\sqrt{D_y}) = D_{1/2} \to D_\alpha$.

Solution in the Form of the Fox H-Function

The Fox H-function is defined in terms of the Mellin–Barnes integral [55,57],

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\middle|\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array}\right] = \frac{1}{2\pi i}\int_\Omega \Theta(s)\,z^{-s}\,ds, \tag{A20}$$

where

$$\Theta(s) = \frac{\Big\{\prod_{j=1}^{m}\Gamma(b_j+sB_j)\Big\}\Big\{\prod_{j=1}^{n}\Gamma(1-a_j-sA_j)\Big\}}{\Big\{\prod_{j=m+1}^{q}\Gamma(1-b_j-sB_j)\Big\}\Big\{\prod_{j=n+1}^{p}\Gamma(a_j+sA_j)\Big\}}, \tag{A21}$$

with $0 \le n \le p$, $1 \le m \le q$ and $a_i, b_j \in C$, while $A_i, B_j \in R+$, for $i = 1, \ldots, p$, and $j = 1, \ldots, q$. The contour Ω starting at $\sigma - i\infty$ and ending at $\sigma + i\infty$, separates the poles of the functions $\Gamma(b_j + sB_j)$, $j = 1, \ldots, m$ from those of the function $\Gamma(1 - a_i - sA_i)$, $i = 1, \ldots, n$.

Now, let us solve FFPEs (A15) and (A18) in terms of the Fox H-functions. Performing the Fourier and Laplace transformations, we obtain the Montroll–Weiss equation, as follows

$$\bar{\tilde{P}}_1(k,s) \equiv P(k,s) = \frac{s^{\alpha-1}}{s^\alpha + D_\alpha k^2}, \tag{A22}$$

where we take $\bar{\tilde{P}}_0(k,s) = 1$. Then employing formula (A7) for the Mittag–Leffler function [47,52,55] one obtains

$$P(k,t) \equiv \bar{P}_1(k,t) = E_{\alpha,1}\Big(-D_\alpha k^2 t^\alpha\Big). \tag{A23}$$

The two parameter Mittag–Leffler function (A7) is a special case of the Fox H-function, which can be represented by means of the Mellin–Barnes integral (A20)

$$\begin{aligned} E_{\alpha,\beta}(-z) &= \frac{1}{2\pi i}\int_\Omega \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)} z^{-s}\,ds = H^{1,1}_{1,2}\left[z\,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\beta,\alpha) \end{matrix}\right] \\ &= \frac{1}{\delta} H^{1,1}_{1,2}\left[z^{1/\delta}\,\middle|\, \begin{matrix} (0,1/\delta) \\ (0,1/\delta),(1-\beta,\alpha/\delta) \end{matrix}\right]. \end{aligned} \tag{A24}$$

The Fourier-cosine transformation of Equations (A23) and (A24) yields [29,55]

$$\begin{aligned} P_\rho(x,t) &= \frac{1}{2\pi}\int_0^\infty k^{\rho-1}\cos(kx)\, H^{1,1}_{1,2}\left[\sqrt{D_\alpha t^\alpha}|k|\,\middle|\, \begin{matrix} (0,1/2) \\ (0,1),(0,\alpha/2) \end{matrix}\right] dk \\ &= \frac{1}{|x|^\rho} H^{2,1}_{3,3}\left[\frac{x^2}{D_\alpha t^\alpha}\,\middle|\, \begin{matrix} (1,1),(1,\alpha),(\frac{1+\rho}{2},1) \\ (1,2),(1,1),(\frac{1+\rho}{2},1) \end{matrix}\right]. \end{aligned} \tag{A25}$$

For $\rho = 1$, we obtain the solution of Equation (A22). Taking into account the properties of the Fox H-function, we obtain

$$\frac{1}{|x|} H^{2,1}_{3,3}\left[\frac{x^2}{D_\alpha t^\alpha}\,\middle|\, \begin{matrix} (1,1),(1,\alpha),(1,1) \\ (1,2),(1,1),(1,1) \end{matrix}\right] = \frac{1}{|x|} H^{2,0}_{2,2}\left[\frac{x^2}{D_\alpha t^\alpha}\,\middle|\, \begin{matrix} (1,\alpha),(1,1) \\ (1,2),(1,1) \end{matrix}\right]. \tag{A26}$$

Use of property

$$x^\delta\, H^{m,n}_{p,q}\left[x\,\middle|\, \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix}\right] = H^{m,n}_{p,q}\left[x\,\middle|\, \begin{matrix} (a_p+\delta A_p,A_p) \\ (b_q+\delta B_q,B_q) \end{matrix}\right]$$

reduces Equation (A25) to

$$P_1(x,t) \equiv P(x,t) = \frac{1}{\sqrt{D_\alpha t^\alpha}} H^{2,0}_{2,2}\left[\frac{x^2}{D_\alpha t^\alpha}\,\middle|\, \begin{matrix} (1-\alpha/2,\alpha),(1/2,1) \\ (0,2),(1/2,1) \end{matrix}\right]. \tag{A27}$$

Using the property of Equation (A26), we obtain

$$P_1(x,t) = \frac{1}{\sqrt{D_\alpha t^\alpha}} H^{1,0}_{1,1}\left[\frac{x^2}{D_\alpha t^\alpha}\,\middle|\, \begin{matrix} (1-\alpha/2,\alpha) \\ (0,2) \end{matrix}\right]. \tag{A28}$$

The straightforward generalisation to another toy model is also possible. Namely, the FFPE of the form

$${}_C D_t^\alpha P(x,t) = x^\alpha\, {}_C D_x^\alpha P(x,t), \quad x \in \mathbb{R}_+, \quad 0 < \alpha < 2, \tag{A29}$$

describes the fractional diffusion of a tracer with the PDF $P(x,t)$ with corresponding initial and boundary conditions. By means of the multiplication by $(i\hbar)^\alpha$, one arrives at the fractional quantum dynamics.

Appendix C. Eigenfunctions of the Dilatation Operator

The dilatation operator

$$\hat{\mathcal{H}}_0 = \hbar\left[-ix\frac{\partial}{\partial x} - i/2\right] = \hat{p}_x x + i\hbar/2$$

determines the complete set of eigenfunctions $\chi_\omega(x)$ with the eigenvalues ω according to the eigenvalue problem $\hat{\mathcal{H}}\chi_\omega(x) = \hbar\,\omega\,\chi_\omega(x)$, where ω is the continuous spectrum and the eigenfunctions are [1,4]

$$\chi_\omega(x) = \frac{1}{\sqrt{N|x|}}\exp(i\omega\ln|x|), \tag{A30}$$

which satisfies the boundary conditions $\chi_\omega(x = \pm\infty) = 0$ and $N = 4\pi$. For the continuous spectrum, the normalisation condition is

$$\int_{-\infty}^{\infty}\chi^*_{\omega'}(x)\,\chi_\omega(x)\,dx = \delta(\omega - \omega'), \tag{A31}$$

while the completeness relation is $\int\chi^*_\omega(x')\,\chi_\omega(x)\,d\omega = \delta(x - x')$ (see e.g., [58]). Note also that for $x > 0$, the normalisation constant is half as large, $N = 2\pi$ [4].

It is worth mentioning that a mathematically rigorous calculation of the normalisation constant for the wave function $\chi_\omega(x)$ can be presented as well by following the presentation of [59]. Since the operator $x\hat{p}$ has the continuous spectrum ω, the eigenfunctions $\chi(\omega, x) \equiv \chi_\omega(x)$ are not square integrable. Therefore, the normalisation condition exists not for the eigenfunction but for the "eigendifferential" [59] $\Delta\chi(\omega, x)$, which reads

$$\Delta\chi(\omega, x) = \int_\omega^{\omega+\Delta\omega}\chi(\omega', x)\,d\omega'.$$

Substituting here Equation (A30), one obtains

$$\Delta\chi(\omega, x) = \frac{2}{\sqrt{N|x|}\ln|x|}\exp(i(\omega + \Delta\omega)\ln|x|)\,\sin\frac{\Delta\omega\ln|x|}{2}.$$

This solution is already square integrable and has the following normalisation form

$$\lim_{\Delta\omega\to 0}\frac{1}{\Delta\omega}\int_{-\infty}^{\infty}|\Delta\chi(\omega, x)|^2\,dx = 1.$$

To take the limit, the integrand can be presented as follows [14]

$$|\Delta\chi(\omega, x)|^2 = \frac{4}{N|x|\ln^2|x|}\sin\frac{\Delta\omega\ln|x|}{2}\,\sin\frac{\Delta\Omega\ln|x|}{2},$$

where $\Delta\Omega = \Delta\omega + \Delta\omega_1$. We can perform this trick, since the "eigendifferentials" for not overlapping spectral regions $\Delta\omega$ and $\Delta\omega_1$ are orthogonal [59]. Then, taking the limit $\Delta\omega = 0$ and carrying out the variable change $z = (\Delta\omega_1/2)\ln|x|$ and taking into account that $\frac{4}{N}\int_{-\infty}^{\infty}\frac{\sin z}{z}\,dz = \frac{4\pi}{N}$, we obtain $N = 4\pi$, which coincides exactly with the dimensionless normalisation constant N in Equation (A30).

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