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SURVEY PAPER

VELOCITY AND DISPLACEMENT CORRELATION FUNCTIONS FOR FRACTIONAL GENERALIZED LANGEVIN EQUATIONS

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Abstract

We study analytically a generalized fractional Langevin equation. General formulas for calculation of variances and the mean square displacement are derived. Cases with a three parameter Mittag-Leffler frictional memory kernel are considered. Exact results in terms of the Mittag-Leffler type functions for the relaxation functions, average velocity and average particle displacement are obtained. The mean square displacement and variances are investigated analytically. Asymptotic behaviors of the particle in the short and long time limit are found. The model considered in this paper may be used for modeling anomalous diffusive processes in complex media including phenomena similar to single file diffusion or possible generalizations thereof. We show the importance of the initial conditions on the anomalous diffusive behavior of the particle.

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1. Introduction

Anomalous diffusion has been found in various physical and biological systems [22, 23, 26, 50, 17, 57, 25]. The mean square displacement (MSD) of the particle shows a power law dependence on time $\langle x^2(t)\rangle \sim t^\alpha$, becoming subdiffusion in case $0 < \alpha < 1$ and superdiffusion for $1 < \alpha$ [35], and normal classical diffusion for $\alpha = 1$. Several approaches to anomalous diffusion exist. Starting from the generalized Langevin equation (GLE) [1, 26, 29, 39, 20, 2], introduced by Kubo [27], the fractional diffusion equation [34] (see also Refs. [41] and [52]), fractional Fokker-Planck equation [34, 35, 37], generalized Chapman-Kolmogorov equation [33], fractional generalized Langevin equation (FGLE) [12, 28, 15]. To analyze these equations the properties of different Mittag-Leffler (M-L) type functions [40, 48, 44, 22, 38, 7, 51, 19] are of great importance. Thus, Mainardi and Pironi [30] introduced a fractional Langevin equation as a particular case of a GLE, and for the first time represented the velocity and displacement correlation functions in terms of the M-L functions (see also Ref. [31]).

The continuous time random walk (CTRW) [46], which has a finite variance $\langle \delta x^2\rangle$ of jumps lengths and broad distribution of waiting times τ of the form $\psi(\tau) \simeq (\tau^*)^\alpha/\tau^{1+\alpha}$, with $0 < \alpha < 1$ also represents another pathway to anomalous diffusion. It can be shown that the CTRW in the diffusion limit is equivalent to the fractional diffusion equation [35], and that there appears an inequivalence of time versus ensemble averages [21, 3, 4]. There have been proposed different CTRW models that generates interesting behaviors in short, intermediate and long times (see Ref. [16] and references therein). CTRW also describes superdiffusion in the spatiotemporally coupled Lévy walk case, and Lévy flights [35, 37, 18].

Fractional Brownian motion (FBM), introduced by Kolmogorov, can be used to model anomalous diffusive processes. It represents a random process driven by a Gaussian noise ξ with correlations $\langle \xi(0)\xi(t)\rangle \simeq \alpha(\alpha-1)t^{\alpha-2}$, and is of great interest due to its wide application [11, 32, 29, 36]. In contrast to the GLE, FBM is not subject to the fluctuation-dissipation theorem, see below.

The anomalous diffusion of a particle of mass $m = 1$ driven by a stationary random force $\xi(t)$ can be explained by analyzing the GLE [27, 30]:

$$\dot{v}(t) + \int_0^t \gamma(t-t')v(t')\mathrm{d}t' = \xi(t),$$
$$\dot{x}(t) = v(t), \tag{1.1}$$

where $v(t)$ is the velocity of the particle at time $t > 0$, $x(t)$ is the particle displacement and $\gamma(t)$ is the frictional memory kernel. The internal noise $\xi(t)$ is of a zero-mean ($\langle \xi(t)\rangle = 0$), and with an arbitrary correlation function

$$\langle\xi(t)\xi(t')\rangle = C(t'-t). \tag{1.2}$$

The correlation function $C(t)$ may be dependent on the frictional memory kernel via the second fluctuation-dissipation theorem [27, 30] in the following way:

$$C(t) = k_B T\gamma(t), \tag{1.3}$$

where k_B is the Boltzmann constant and T is the absolute temperature of the environment. This is a case of internal noise, when the fluctuation and dissipation come from same source, and the system will reach the equilibrium state. From the other side, the fluctuation and dissipation may come from different sources, so the fluctuation-dissipation theorem (1.2) does not hold, and the system will not reach a unique equilibrium state [56, 53, 47]. Note that in case of internal white Gaussian noise $\xi(t)$, the GLE (1.1) would correspond to the classical Langevin equation [30]. The FBM and GLE motion are ergodic (time and ensemble averages are same [3, 9, 24]).

In this paper the anomalous diffusion is investigated by analyzing FGLE with a three parameter Mittag-Leffler frictional memory kernel. It is a generalization of the GLE (1.1) in which the integer order derivatives are substituted by fractional order derivatives, for example, of the Caputo form [8]:

$${}_C D_{0+}^{\gamma} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\gamma)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\gamma+1-m}} \mathrm{d}\tau & \text{if } m-1<\gamma<m, \\ \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m} & \text{if } \gamma = m, \end{cases} \tag{1.4}$$

where $m \in N$. In the work of Fa [15] a FGLE with nonlocal dissipative force is investigated. Such FGLEs are recently used by Lim and Teo [28], Eab and Lim [12] to model a single file diffusion, which means that in the short time limit it behaves as normal diffusion ($\langle x^2(t)\rangle \sim O(t)$) and as anomalous diffusion ($\langle x^2(t)\rangle \sim O\left(t^{1/2}\right)$) in the long time limit.

It has been mentioned that in the case of internal white Gaussian noise, the GLE (1.1) corresponds to the classical Langevin equation. In many papers the GLE with an internal noise with a power law correlation functions of form $C(t) = C_\lambda \frac{t^{-\lambda}}{\Gamma(1-\lambda)}$, where $\Gamma(\cdot)$ is the Euler-gamma function, C_λ is a proportionality coefficient independent of time and which can depends on the exponent λ ($0 < \lambda < 1$ or $1 < \lambda < 2$), has been used for modeling anomalous diffusion [29, 2, 56, 53, 47, 10, 30, 31]. One parameter M-L correlation function $C(t) = \frac{C_\lambda}{\tau^\lambda} E_\lambda(-(t/\tau)^\lambda)$ of an internal noise also is used [54, 55, 5], where τ is the characteristic memory time,

C_λ is a proportionality coefficient independent of time ($0 < \lambda < 2$) and $E_\lambda(\cdot)$ is the one parameter M-L function (4.1). Camargo et al. [6] introduced a fractional GLE with a two parameter M-L correlation function $C(t) = \frac{C_\lambda}{\tau^\lambda} t^{\nu-1} E_{\lambda,\nu}(-(t/\tau)^\lambda)$, where $E_{\lambda,\nu}(\cdot)$ is the two parameter M-L function (4.2). In Ref. [42] we have introduced a three parameter M-L correlation function $C(t) = \frac{C_{\alpha,\beta,\delta}}{\tau^{\alpha\delta}} E^\delta_{\alpha,\beta}\left(-\frac{t^\alpha}{\tau^\alpha}\right)$, where $E^\delta_{\alpha,\beta}(\cdot)$ is the three parameter M-L function (4.4), $C_{\alpha,\beta,\delta}$ is a proportionality coefficient independent of time ($\alpha > 0$, $\beta > 0$, $\delta > 0$, $0 < \alpha\delta < 2$), and we have studied the asymptotic behavior of a harmonic oscillator and a free particle. In our recent paper [43] we have studied the GLE with a three parameter M-L correlation function

$$C(t) = \frac{C_{\alpha,\beta,\delta}}{\tau^{\alpha\delta}} t^{\beta-1} E^\delta_{\alpha,\beta}\left(-\frac{t^\alpha}{\tau^\alpha}\right), \tag{1.5}$$

where τ is the characteristic memory time, $C_{\alpha,\beta,\delta}$ is a proportionality coefficient independent of time ($\alpha > 0$, $\beta > 0$, $\delta > 0$). Note that, by using relations (4.7) and (4.8), the noise term (1.5) satisfies the assumption $\lim_{t\to\infty} \gamma(t) = \lim_{s\to 0} s\hat{\gamma}(s) = 0$ [43], where $\hat{\gamma}(s) = \mathcal{L}[\gamma(t)](s)$ is the Laplace transform of $\gamma(t)$, for $\beta < 1 + \alpha\delta$. We have shown that for different values of α, β and δ the anomalous diffusion (subdiffusion or superdiffusion) occurs. In this paper we consider FGLE with an internal noise with three parameter M-L correlation function of form (1.5).

The three parameter M-L noise (1.5) for $\delta = 1$ yields the two parameter M-L noise introduced in Ref. [6]. For $\beta = \delta = 1$ it corresponds to the one-parameter M-L noise [54, 55, 5]. In the limit $\tau \to 0$ for $\beta = \delta = 1$ and $\alpha \neq 1$ the power law correlation function is obtained. For $\alpha = \beta = \delta = 1$ it is obtained a correlation function of form $C(t) = \frac{C_{1,1,1}}{\tau} e^{-t/\tau}$, which in the limit $\tau \to 0$ it corresponds to a white Gaussian noise (a standard Brownian motion).

This paper is organized as follows. In Section **2** general expressions for the relaxation functions, average velocity and average particle displacement, variances and MSD are derived. The case with an internal noise with a three parameter M-L correlation function is investigated. The asymptotic behaviors in the short and long time limits of the MSD are analyzed. The appearance of anomalous diffusion (subdiffusion and superdiffusion) is found. Cases for modeling single file-type diffusion are discussed. A Summary of the paper is provided in Section **3**. In Appendices (Section **4**), some properties and formulas for the M-L functions, and for fractional derivatives and integrals are presented.

2. Extended FGLE

Here we consider the following extended FGLE for a particle of mass $m = 1$ with a three parameter M-L memory kernel (1.5) and the Caputo time fractional derivatives:

$$\begin{aligned} {}_C D_{0+}^{\mu} v(t) + \int_0^t \gamma(t - t') v(t') \mathrm{d}t' &= \xi(t), \\ {}_C D_{0+}^{\nu} x(t) &= v(t), \end{aligned} \tag{2.1}$$

where ${}_C D_{0+}^{\mu}$ and ${}_C D_{0+}^{\nu}$ are the Caputo time fractional derivatives (4.13), $0 < \mu \leq 1$, $0 < \nu \leq 1$, $\mu + \nu > 1$, and the memory kernel is of form (1.5). If we substitute the second equation of (2.1) into first equation of (2.1) it is obtained a term of form ${}_C D_{0+}^{\mu+\nu} x(t)$. So, in order equation (2.1) to be a fractional generalization of equation (1.1), it is obtained that $\mu + \nu > 1$. From the other side, since $0 < \mu \leq 1$ and $0 < \nu \leq 1$, it follows that $\mu + \nu \leq 2$.

Equation of form (2.1) is introduced by Lim and Teo [28] to model a single file diffusion. They considered cases of internal and external noises with correlations of white Gaussian and power law forms. The case $\nu = 1$ with Dirac delta, exponential and power law correlation functions, and their combination is investigated in Refs.[12, 28, 15] in detail. The case $\mu = \nu = 1$ with an internal noise with a three parameter M-L correlation function of form (1.5) is considered in Ref.[43]. Here we note that, Eab and Lim [13] recently introduced fractional Langevin equation of distributed order and discussed its possible application for modeling single file diffusion and ultraslow diffusion.

2.1. Relaxation functions, variances and MSD

We use the Laplace transform method to analyze the FGLE. Thus, relations (2.1) and (4.17) yield

$$\hat{V}(s) = v_0 \frac{s^{\mu-1}}{s^{\mu} + \hat{\gamma}(s)} + \frac{1}{s^{\mu} + \hat{\gamma}(s)} \hat{F}(s), \tag{2.2}$$

$$\hat{X}(s) = x_0 \frac{1}{s} + v_0 \frac{s^{\mu-\nu-1}}{s^{\mu} + \hat{\gamma}(s)} + \frac{s^{-\nu}}{s^{\mu} + \hat{\gamma}(s)} \hat{F}(s), \tag{2.3}$$

where $\hat{V}(s) = \mathcal{L}[v(t)](s)$, $\hat{X}(s) = \mathcal{L}[x(t)](s)$, $\hat{\gamma}(s) = \mathcal{L}[\gamma(t)](s)$, $\hat{F}(s) = \mathcal{L}[\xi(t)](s)$. Here we introduce the following functions

$$\hat{g}(s) = \frac{1}{s^{\mu} + \hat{\gamma}(s)}, \tag{2.4}$$

$$\hat{G}(s) = \frac{s^{-\nu}}{s^{\mu} + \hat{\gamma}(s)}, \tag{2.5}$$

$$\hat{I}(s) = \frac{s^{-2\nu}}{s^{\mu} + \hat{\gamma}(s)}. \tag{2.6}$$

By applying inverse Laplace transform to relations (2.2) and (2.3), for the displacement $x(t)$ and velocity $v(t) = {}_C D_{0+}^{\nu} x(t)$ it follows:

$$v(t) = \langle v(t) \rangle + \int_0^t g(t-t')\xi(t')\mathrm{d}t', \tag{2.7}$$

$$x(t) = \langle x(t) \rangle + \int_0^t G(t-t')\xi(t')\mathrm{d}t', \tag{2.8}$$

with $G(0) = 0$, where

$$\langle v(t) \rangle = v_0 \cdot {}_C D_{0+}^{\mu+\nu-1} G(t), \tag{2.9}$$

$$\langle x(t) \rangle = x_0 + v_0 \cdot {}_C D_{0+}^{\mu+\nu-1} I(t), \tag{2.10}$$

and $g(t) = \mathcal{L}^{-1}\left[\hat{g}(s)\right](t)$, $G(t) = \mathcal{L}^{-1}\left[\hat{G}(s)\right](t)$, $I(t) = \mathcal{L}^{-1}\left[\hat{I}(s)\right](t)$ are known as relaxation functions. From relations (2.4), (2.5), (2.6) and (4.17) it follows that ${}_C D_{0+}^{\nu} G(t) = g(t)$ and ${}_C D_{0+}^{\nu} I(t) = G(t)$.

From relations (2.7), (2.8) and (1.3), in case of an internal noise, follow the following general expressions of variances

$$\begin{aligned} \sigma_{xx} &= \langle x^2(t) \rangle - \langle x(t) \rangle^2 = 2\int_0^t \mathrm{d}t_1 G(t_1) \int_0^{t_1} \mathrm{d}t_2 G(t_2) C(t_1-t_2) \\ &= 2k_B T \left[\frac{1}{\Gamma(\nu)} \int_0^t \mathrm{d}\xi G(\xi)\xi^{\nu-1} - \int_0^t \mathrm{d}\xi G(\xi) {}_C D_{0+}^{\mu} G(\xi)\right], \end{aligned} \tag{2.11}$$

$$\begin{aligned} \sigma_{xv} &= \langle \left(v(t) - \langle v(t) \rangle\right)\left(x(t) - \langle x(t) \rangle\right) \rangle \\ &= \int_0^t \mathrm{d}t_1 g(t_1) \int_0^t \mathrm{d}t_2 G(t_2) C(t_1-t_2) \\ &= k_B T \left[\frac{1}{\Gamma(\nu)} \int_0^t \mathrm{d}\xi g(\xi)\xi^{\nu-1} - \int_0^t \mathrm{d}\xi g(\xi) {}_C D_{0+}^{\mu} G(\xi) \right. \\ &\quad \left. - \int_0^t \mathrm{d}\xi G(\xi) {}_{RL} D_{0+}^{\mu} g(\xi)\right], \end{aligned} \tag{2.12}$$

$$\begin{aligned} \sigma_{vv} &= \langle v^2(t) \rangle - \langle v(t) \rangle^2 = 2\int_0^t \mathrm{d}t_1 g(t_1) \int_0^{t_1} \mathrm{d}t_2 g(t_2) C(t_1-t_2) \\ &= -2k_B T \int_0^t \mathrm{d}\xi g(\xi) {}_{RL} D_{0+}^{\mu} g(\xi), \end{aligned} \tag{2.13}$$

where it is used the symmetry of the correlation function $C(t_1 - t_2)$. For $\mu = 1$ and $0 < \nu < 1$ it is obtained

$$\sigma_{xx} = k_B T \left[\frac{2}{\Gamma(\nu)} \int_0^t \mathrm{d}\xi G(\xi)\xi^{\nu-1} - G^2(t) \right], \tag{2.14}$$

$$\sigma_{xv} = k_B T \left[\frac{1}{\Gamma(\nu)} \int_0^t \mathrm{d}\xi g(\xi)\xi^{\nu-1} - g(t)G(t) \right], \tag{2.15}$$

$$\sigma_{vv} = k_B T \left[1 - g^2(t) \right]. \tag{2.16}$$

The case $\nu = 1$ and $0 < \mu < 1$ yields the following results [15, 12]

$$\sigma_{xx} = 2k_B T \left[I(t) - \int_0^t \mathrm{d}\xi G(\xi) {}_C D_{0+}^{\mu} G(\xi) \right], \tag{2.17}$$

$$\sigma_{xv} = \frac{1}{2} \frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}t} = k_B T G(t) \left[1 - {}_C D_{0+}^{\mu} G(t) \right], \tag{2.18}$$

$$\sigma_{vv} = -2k_B T \int_0^t \mathrm{d}\xi g(\xi) {}_{RL} D_{0+}^{\mu} g(\xi). \tag{2.19}$$

Furthermore, for $\mu = \nu = 1$ we obtained the well known expressions [56, 53]

$$\sigma_{xx} = k_B T \left[2I(t) - G^2(t) \right], \tag{2.20}$$

$$\sigma_{xv} = k_B T G(t) \left[1 - g(t) \right], \tag{2.21}$$

$$\sigma_{vv} = k_B T \left[1 - g^2(t) \right]. \tag{2.22}$$

From relation (2.11) for the MSD we obtain

$$\begin{aligned} \langle x^2(t) \rangle = \; & x_0^2 + 2x_0 v_0 {}_C D_{0+}^{\mu+\nu-1} I(t) + v_0^2 \left[{}_C D_{0+}^{\mu+\nu-1} I(t) \right]^2 \\ & + 2k_B T \int_0^t \mathrm{d}\xi G(\xi) \frac{\xi^{\nu-1}}{\Gamma(\nu)} - 2k_B T \int_0^t \mathrm{d}\xi G(\xi) {}_C D_{0+}^{\mu} G(\xi), \end{aligned} \tag{2.23}$$

from where it follows

$$\begin{aligned} D(t) \; & = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle x^2(t) \rangle = x_0 v_0 {}_C D_{0+}^{\mu+\nu} I(t) + v_0^2 {}_C D_{0+}^{\mu+\nu-1} I(t) {}_C D_{0+}^{\mu+\nu} I(t) \\ & + k_B T G(t) \frac{t^{\nu-1}}{\Gamma(\nu)} - k_B T G(t) {}_C D_{0+}^{\mu} G(t). \end{aligned} \tag{2.24}$$

For $\mu = 1$ and $0 < \nu < 1$ it follows

$$D(t) = x_0 v_0 G'(t) + \left(v_0^2 - k_B T \right) G(t) G'(t) + k_B T G(t) \frac{t^{\nu-1}}{\Gamma(\nu)}. \tag{2.25}$$

The case $\nu = 1$ and $0 < \mu < 1$ yields the time-dependent diffusion coefficient [39, 30, 31]

$$\begin{aligned} D(t) &= x_0 v_{0C} D_{0+}^{\mu} G(t) + v_{0C}^2 D_{0+}^{\mu} I(t)_C D_{0+}^{\mu} G(t) + k_B T G(t) \\ &\quad - k_B T G(t)_C D_{0+}^{\mu} G(t), \end{aligned} \tag{2.26}$$

and for $\mu = \nu = 1$ we obtain [39, 30, 31, 43]

$$D(t) = x_0 v_0 g(t) + \left(v_0^2 - k_B T\right) G(t) g(t) + k_B T G(t), \tag{2.27}$$

which in case of thermal initial conditions ($x_0 = 0$, $v_0^2 = k_B T$) turns to the well known results [10]

$$D(t) = k_B T G(t), \tag{2.28}$$

and

$$\langle x^2(t) \rangle = 2 k_B T I(t). \tag{2.29}$$

2.2. Explicit forms

In case of a frictional memory kernel of the M-L type (1.5), by using relation (4.5) it follows [43]

$$\hat{\gamma}(s) = \frac{C_{\alpha,\beta,\delta}}{k_B T \tau^{\alpha\delta}} \cdot \frac{s^{\alpha\delta-\beta}}{\left(s^{\alpha} + \tau^{-\alpha}\right)^{\delta}}. \tag{2.30}$$

From relations (2.4), (2.5), (2.6) and (4.6) for the relaxation functions we obtain

$$g(t) = \sum_{k=0}^{\infty} (-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k+\mu-1} E_{\alpha,(\mu+\beta)k+\mu}^{\delta k} \left(-(t/\tau)^{\alpha}\right), \tag{2.31}$$

$$G(t) = \sum_{k=0}^{\infty} (-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k+\mu+\nu-1} E_{\alpha,(\mu+\beta)k+\mu+\nu}^{\delta k} \left(-(t/\tau)^{\alpha}\right), \tag{2.32}$$

$$I(t) = \sum_{k=0}^{\infty} (-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k+\mu+2\nu-1} E_{\alpha,(\mu+\beta)k+\mu+2\nu}^{\delta k} \left(-(t/\tau)^{\alpha}\right), \tag{2.33}$$

where $\gamma_{\alpha,\beta,\delta} = \frac{C_{\alpha,\beta,\delta}}{k_B T \tau^{\alpha\delta}}$, and $G(0) = 0$ since $\mu + \nu > 1$. The convergence of series (2.31), (2.32) and (2.33) may be proved following the procedure in Ref. [43].

Employing relation (4.15) to (2.9) and (2.10), the average velocity and average particle displacement become, respectively,

$$\langle v(t) \rangle = v_0 \sum_{k=0}^{\infty} (-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k} E_{\alpha,(\mu+\beta)k+1}^{\delta k} \left(-(t/\tau)^{\alpha}\right), \tag{2.34}$$

$$\langle x(t)\rangle = x_0 + v_0 \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k+\nu} E_{\alpha,(\mu+\beta)k+\nu+1}^{\delta k}\left(-(t/\tau)^{\alpha}\right). \quad (2.35)$$

Note that if $\mu = \nu = 1$ the results in Ref.[43] are recovered. Furthermore, if $\tau \to 0$, by using (4.7), relations (2.31), (2.32) and (2.33) yield

$$g(t) = t^{\mu-1} E_{\mu+\beta-\alpha\delta,\mu}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{\mu+\beta-\alpha\delta}\right), \quad (2.36)$$

$$G(t) = t^{\mu+\nu-1} E_{\mu+\beta-\alpha\delta,\mu+\nu}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{\mu+\beta-\alpha\delta}\right), \quad (2.37)$$

$$I(t) = t^{\mu+2\nu-1} E_{\mu+\beta-\alpha\delta,\mu+2\nu}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{\mu+\beta-\alpha\delta}\right), \quad (2.38)$$

where $\mu + \beta - \alpha\delta > 0$. Note that for $\beta = \delta = 1$ the results in Ref. [28] are obtained. Substitution of $\nu = \beta = \delta = 1$ yields the results from Refs. [15, 12, 28]. The case $\beta = \delta = \mu = \nu = 1$, $\tau \to 0$, $0 < \alpha < 2$ (i.e. in case of a power law correlation function; see for example Refs. [29, 43]) yields the known result

$$g(t) = E_{2-\alpha}\left(-\frac{C_{\alpha,1,1}}{k_B T} t^{2-\alpha}\right), \quad (2.39)$$

$$G(t) = t E_{2-\alpha,2}\left(-\frac{C_{\alpha,1,1}}{k_B T} t^{2-\alpha}\right), \quad (2.40)$$

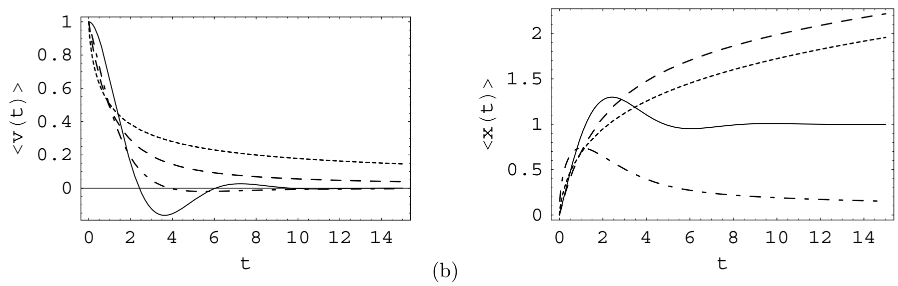
$$I(t) = t^2 E_{2-\alpha,3}\left(-\frac{C_{\alpha,1,1}}{k_B T} t^{2-\alpha}\right). \quad (2.41)$$

From relations (2.34) and (2.35), for the average velocity and average particle displacement in case when $\tau \to 0$ we obtain

$$\langle v(t)\rangle = v_0 E_{\mu+\beta-\alpha\delta}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{\mu+\beta-\alpha\delta}\right), \quad (2.42)$$

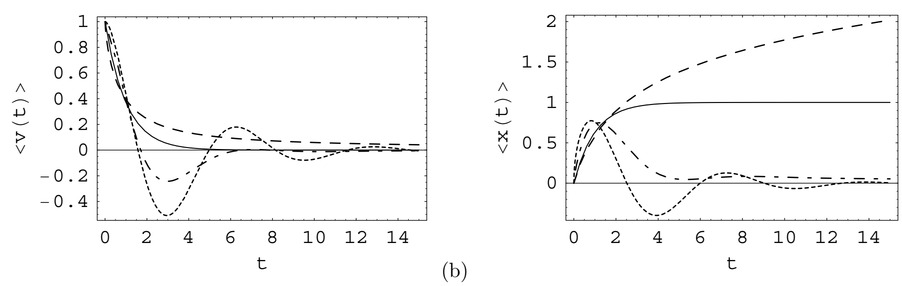
$$\langle x(t)\rangle = x_0 + v_0 t^{\nu} E_{\mu+\beta-\alpha\delta,\nu+1}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{\mu+\beta-\alpha\delta}\right), \quad (2.43)$$

which are generalizations of the mean velocity and mean particle displacement for the fractional Langevin equation, considered in Ref. [29] ($0 < \alpha < 2$, $\beta = \delta = \mu = \nu = 1$). Graphical representation of the mean velocity and mean particle displacement is given in Figures 1, 2 and 3.



(a) (b)

FIGURE 1. Graphical representation in case when $\tau = 1$, $C_{\alpha,\beta,\delta} = 1$, $k_BT = 1$, $x_0 = 0$, $v_0 = 1$ of: (a) Mean particle velocity (The following parameters are used: $\alpha = \beta = \delta = \mu = 1$ (solid line); $\alpha = \beta = \delta = \mu = 1/2$ (dashed line); $\alpha = \beta = \delta = 1/2$, $\mu = 3/4$ (dot-dashed line); $\alpha = \delta = \mu = 1/2$, $\beta = 1/4$ (dotted line)); (b) Mean particle displacement (The following parameters are used:$\alpha = \beta = \delta = \mu = \nu = 1$ (solid line); $\alpha = \beta = \delta = \mu = 1/2$, $\nu = 1$ (dashed line); $\alpha = \beta = \delta = \nu = 1/2$, $\mu = 3/4$ (dot-dashed line); $\alpha = \delta = \mu = 1/2$, $\beta = 1/4$, $\nu = 3/4$ (dotted line)).



(a) (b)

FIGURE 2. Graphical representation in case when $\tau \to 0$, $C_{\alpha,\beta,\delta} = 1$, $k_BT = 1$, $x_0 = 0$, $v_0 = 1$ of: (a) Mean particle velocity (The following parameters are used: $\alpha = \beta = \delta = \mu = 1$ (solid line); $\alpha = \beta = \delta = \mu = 1/2$ (dashed line); $\alpha = \delta = 3/4$, $\beta = 3/2$, $\mu = 1/2$ (dot-dashed line); $\alpha = \delta = 3/4$, $\beta = 3/2$, $\mu = 3/4$ (dotted line)); (b) Mean particle displacement (The following parameters are used: $\alpha = \beta = \delta = \mu = \nu = 1$ (solid line); $\alpha = \beta = \delta = \mu = 1/2$, $\nu = 1$ (dashed line); $\alpha = \delta = \nu = 3/4$, $\beta = 3/2$, $\mu = 1/2$ (dot-dashed line); $\alpha = \delta = \mu = 3/4$, $\beta = 3/2$, $\nu = 1/2$ (dotted line)).

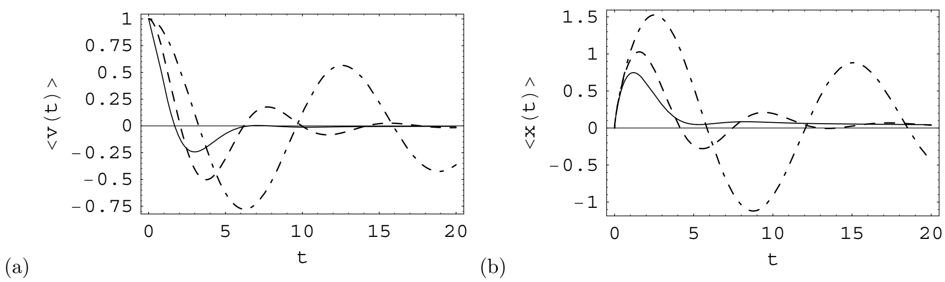


FIGURE 3. Graphical representation in case when $C_{\alpha,\beta,\delta} = 1$, $k_BT = 1$, $x_0 = 0$, $v_0 = 1$ of: (a) Mean particle velocity; (b) Mean particle displacement. The following parameters are used: $\alpha = \delta = \mu = 1/2$, $\beta = 3/2$, $\nu = 3/4$; $\tau = 0$ (solid line); $\tau = 1$ (dashed line); $\tau = 10$ (dot-dashed line).

Relation (2.24) in the case of the three parameter M-L frictional memory kernel (1.5) becomes

$$\begin{aligned} D(t) \quad &= v_0 \sum_{k=0}^{\infty} (-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k+\nu-1} E_{\alpha,(\mu+\beta)k+\nu}^{\delta k} \left(-(t/\tau)^\alpha\right) \\ &\times \left[x_0 + v_0 \sum_{k=0}^{\infty} (-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k+\nu} E_{\alpha,(\mu+\beta)k+\nu+1}^{\delta k} \left(-(t/\tau)^\alpha\right) \right] \\ &- k_BT \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{k+l} \gamma_{\alpha,\beta,\delta}^{k+l} t^{(\mu+\beta)(k+l)+\mu+2\nu-2} \\ &\times E_{\alpha,(\mu+\beta)k+\mu+\nu}^{\delta k} \left(-(t/\tau)^\alpha\right) E_{\alpha,(\mu+\beta)l+\nu}^{\delta l} \left(-(t/\tau)^\alpha\right) \\ &+ k_BT \frac{1}{\Gamma(\nu)} \sum_{k=0}^{\infty} (-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(\mu+\beta)k+\mu+2\nu-2} E_{\alpha,(\mu+\beta)k+\mu+\nu}^{\delta k} \left(-(t/\tau)^\alpha\right). \end{aligned} \tag{2.44}$$

In case of thermal initial conditions ($x_0 = 0$, $v_0^2 = k_BT$) it follows

$$D(t) = k_BT \left[S_1(t) - S_2(t) + S_3(t)\right], \tag{2.45}$$

where

$$\begin{aligned} S_1(t) \quad &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{k+l} \gamma_{\alpha,\beta,\delta}^{k+l} t^{(\mu+\beta)(k+l)+2\nu-1} \\ &E_{\alpha,(\mu+\beta)k+\nu}^{\delta k} \left(-(t/\tau)^\alpha\right) E_{\alpha,(\mu+\beta)l+\nu+1}^{\delta l} \left(-(t/\tau)^\alpha\right), \end{aligned} \tag{2.46}$$

$$S_2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\gamma_{\alpha,\beta,\delta}^{k+l}t^{(\mu+\beta)(k+l)+\mu+2\nu-2} E_{\alpha,(\mu+\beta)k+\mu+\nu}^{\delta k}\left(-(t/\tau)^{\alpha}\right)E_{\alpha,(\mu+\beta)l+\nu}^{\delta l}\left(-(t/\tau)^{\alpha}\right), \quad (2.47)$$

$$S_3(t) = \frac{1}{\Gamma(\nu)}\sum_{k=0}^{\infty}(-1)^{k}\gamma_{\alpha,\beta,\delta}^{k}t^{(\mu+\beta)k+\mu+2\nu-2}E_{\alpha,(\mu+\beta)k+\mu+\nu}^{\delta k}\left(-(t/\tau)^{\alpha}\right), \quad (2.48)$$

where $\tau \neq 0$. From relation (2.25), note that if $\mu = 1$, $S_1(t)$ and $S_2(t)$ vanishes, so $D(t) = k_B T S_3(t)$. We note that the analysis for the short and long time behavior of the MSD given bellow may be done in a same way also for the limit $\tau \to 0$.

By using relation (4.7) in the long time limit ($t \to \infty$) we obtain the following asymptotic behaviors:

$$\begin{aligned}\tilde{S}_1(t) &= t^{2\nu-1}\sum_{k=0}^{\infty}\frac{\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)^k}{\Gamma\left((\mu+\beta-\alpha\delta)k+\nu\right)}\sum_{l=0}^{\infty}\frac{\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)^l}{\Gamma\left((\mu+\beta-\alpha\delta)l+\nu+1\right)}\\ &= t^{2\nu-1}E_{\mu+\beta-\alpha\delta,\nu}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)E_{\mu+\beta-\alpha\delta,\nu+1}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)\\ &= O\left(t^{2\nu-1-2(\mu+\beta-\alpha\delta)}\right), \quad t\to\infty, \end{aligned} \quad (2.49)$$

$$\begin{aligned}\tilde{S}_2(t) &= t^{\mu+2\nu-2}\sum_{k=0}^{\infty}\frac{\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)^k}{\Gamma\left((\mu+\beta-\alpha\delta)k+\nu\right)}\sum_{l=0}^{\infty}\frac{\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)^l}{\Gamma\left((\mu+\beta-\alpha\delta)l+\mu+\nu\right)}\\ &= t^{\mu+2\nu-2}E_{\mu+\beta-\alpha\delta,\nu}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)E_{\mu+\beta-\alpha\delta,\mu+\nu}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)\\ &= O\left(t^{\mu+2\nu-2-2(\mu+\beta-\alpha\delta)}\right), \quad t\to\infty, \end{aligned} \quad (2.50)$$

$$\begin{aligned}\tilde{S}_3(t) &= \frac{1}{\Gamma(\nu)}t^{\mu+2\nu-2}\sum_{k=0}^{\infty}\frac{\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)^k}{\Gamma\left((\mu+\beta-\alpha\delta)k+\mu+\nu\right)}\\ &= \frac{t^{\mu+2\nu-2}}{\Gamma(\nu)}E_{\mu+\beta-\alpha\delta,\mu+\nu}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT}t^{\mu+\beta-\alpha\delta}\right)\\ &= O\left(t^{2\nu-2+\alpha\delta-\beta}\right), \quad t\to\infty. \end{aligned} \quad (2.51)$$

Thus, the MSD in the long time limit ($t \to \infty$) becomes

$$\langle x^2(t)\rangle \sim \begin{cases} O\left(t^{2(\alpha\delta-\beta+\nu-\mu)}\right) & \text{if } 2\mu \le \alpha\delta-\beta+1, \\ O\left(t^{\alpha\delta-\beta+2\nu-1}\right) & \text{if } 2\mu > \alpha\delta-\beta+1 \text{ or } \mu=1, \end{cases} \quad (2.52)$$

from where we conclude appearance of anomalous diffusive behavior. Depending on the power of t in the MSD we distinguish cases of subdiffusion, superdiffusion or normal diffusion. Note that this behavior may change if for some combination of parameters α, β, δ, μ and ν the first term from some of the asymptotic expansions of $S_1(t)$, $S_2(t)$ and $S_3(t)$ vanish. In that cases, the second terms from the asymptotic expansion formulas (4.9) and (4.7) should be used.

From relations (2.46), (2.47) and (2.48) in the short time limit ($t \to 0$) we obtain

$$\langle x^2(t)\rangle \sim \begin{cases} O\left(t^{2\nu}\right) & \text{if } 1 \le 2\mu+\beta, \\ O\left(t^{2\mu+2\nu+\beta-1}\right) & \text{if } 1 > 2\mu+\beta, \end{cases} \quad (2.53)$$

From relations (2.53) and (2.54) can be seen that the anomalous diffusion exponent in the short and long time limit is different. We can conclude that this model may be used for describing single file-type diffusion or possible generalizations thereof, such as accelerating and retarding anomalous diffusion [14]. For example, in the short time limit the anomalous diffusion exponent may be greater than the one in the long time limit. Note that for different initial conditions ($x_0 \neq 0$) different anomalous behavior may be obtained, for example, $\langle x^2(t)\rangle \sim O\left(t^{\nu}\right)$, for $t \to 0$.

For $\nu = 1$ in the long time limit ($t \to \infty$) the MSD is given by

$$\langle x^2(t)\rangle \sim \begin{cases} O\left(t^{2(\alpha\delta-\beta+1-\mu)}\right) & \text{if } 2\mu \le \alpha\delta-\beta+1, \\ O\left(t^{\alpha\delta-\beta+1}\right) & \text{if } 2\mu > \alpha\delta-\beta+1 \text{ or } \mu=1. \end{cases} \quad (2.54)$$

Note that for $\mu = 1$ we can obtain the results from Ref.[43] ($\beta - 1 < \alpha\delta < \beta$ - subdiffusion; $\beta < \alpha\delta < 1+\beta$ - superdiffusion). The case $\beta = \delta = \mu = 1$ corresponds to the well known result $\langle x^2(t)\rangle \sim t^{\alpha}$ ($0 < \alpha < 1$ - subdiffusion; $1 < \alpha < 2$ - superdiffusion) [29]. In the short time limit ($t \to 0$) for the MSD follows

$$\langle x^2(t)\rangle \sim \begin{cases} O\left(t^{2}\right) & \text{if } 1 \le 2\mu+\beta, \\ O\left(t^{2\mu+\beta+1}\right) & \text{if } 1 > 2\mu+\beta. \end{cases} \quad (2.55)$$

We note that for $\nu = 1$ and $\beta > 0$ in the short time limit the MSD has a power law dependence on time with an exponent greater than 1. The particle, in the short time limit, shows a super diffusive behavior (when the exponent is greater than 1) which turns to simple ballistic motion when the exponent is equal to 2.

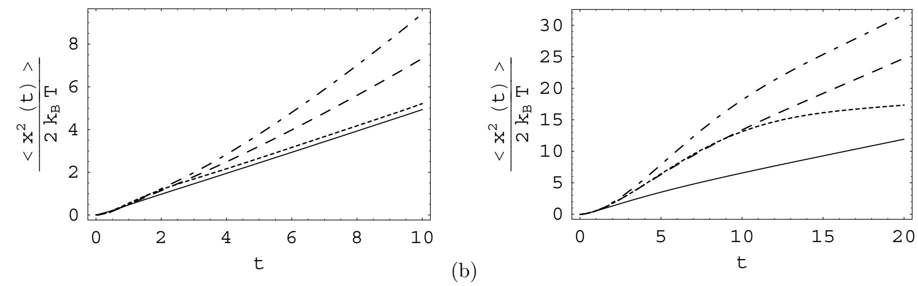


FIGURE 4. Graphical representation of MSD for (a) $\tau = 1$, (b) $\tau = 10$. The following parameters are used: $C_{\alpha,\beta,\delta} = 1$, $k_B T = 1$; $\alpha = 1$, $\beta = \delta = 1/2$, $\mu = 3/10$, $\nu = 4/5$ (solid line); $\alpha = \beta = \delta = 1$, $\mu = 1/4$, $\nu = 7/8$ (dashed line); $\alpha = 5/4$, $\beta = \delta = 1$, $\mu = 1/2$, $\nu = 9/10$ (dot-dashed line); $\alpha = 1$, $\beta = \delta = 3/2$, $\mu = 3/10$, $\nu = 4/5$ (dotted line).

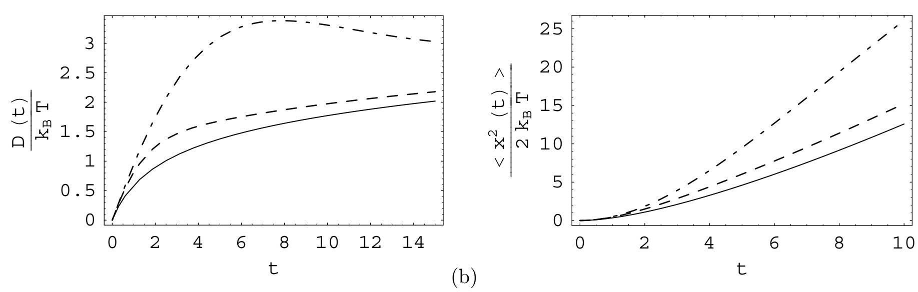


FIGURE 5. Graphical representation of (a) $D(t)$ and (b) MSD, in case of thermal initial conditions ($v_0^2 = k_B T = 1$, $x_0 = 0$) and $\tau \to 0$ (solid line); $\tau = 1$ (dashed line); $\tau = 10$ (dot-dashed line). The following parameters are used: $C_{\alpha,\beta,\delta} = 1$, $k_B T = 1$; $\alpha = 3/4$, $\beta = 1/2$, $\delta = \mu = \nu = 1$.

In Figures 4, 5 and 6 graphical representation of $D(t)$ and MSD is given. In Figures 7 and 8 we represent MSD in case when $\mu = 1$ and $\nu = 1$, respectively. As an addition, in Figure 9 the MSD (2.23) in case of Dirac delta and power law frictional memory kernels for different values of parameters is presented.

REMARK **2.1**. We note that the MSD depends on the initial conditions. As it was shown (see relation (2.55)), in case of thermal initial conditions, for $\nu = 1$ and $\beta > 0$, the anomalous diffusion exponent is greater than 1.

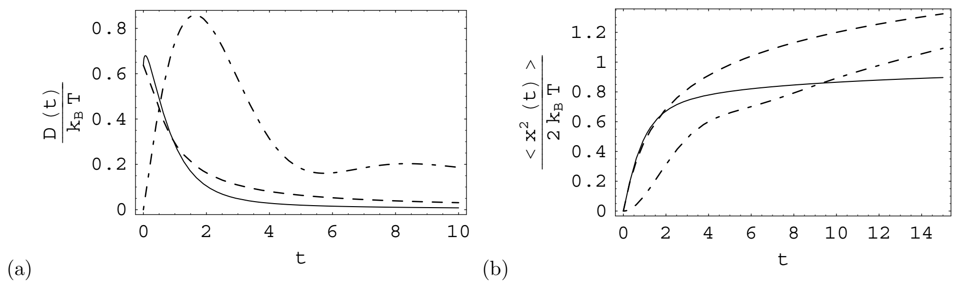


FIGURE 6. Graphical representation of (a) $D(t)$, (b) MSD, in case of thermal initial conditions ($v_0^2 = k_BT = 1$, $x_0 = 0$) and $\tau \to 0$. The following parameters are used: $C_{\alpha,\beta,\delta} = 1$, $k_BT = 1$; $\alpha = \beta = \delta = \nu = 1/2$, $\mu = 3/4$ (solid line); $\alpha = \delta = \nu = 1/2$, $\beta = 1/4$, $\mu = 3/4$ (dashed line); $\alpha = \delta = \mu = \nu = 1$, $\beta = 3/2$ (dot-dashed line). The dot-dashed line in (b) is obtained when MSD is divided by 4.

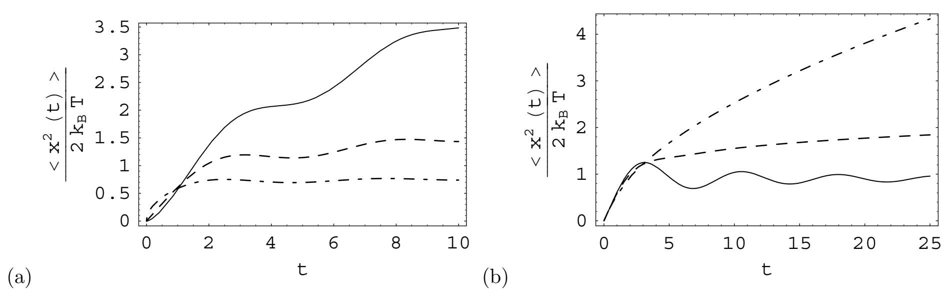


FIGURE 7. Graphical representation of MSD (2.23) in case of thermal initial conditions ($v_0^2 = k_BT = 1$, $x_0 = 0$), $\mu = 1$ and frictional memory kernel of form (1.5); $C_{\alpha,\beta,\delta} = 1$; $k_BT = 1$; $\tau = 1$; (a) $\alpha = \beta = 3/2$, $\delta = 1$, $\nu = 3/4$ (solid line), $\nu = 1/2$ (dashed line), $\nu = 1/4$, (dot-dashed line); (b) $\nu = 1/2$; $\alpha = \delta = 1$; $\beta = 3/2$ (solid line); $\beta = 1$ (dashed line); $\beta = 1/2$ (dot-dashed line).

This means that there is no appearance of normal diffusion in the short time limit. But, if $x_0 \neq 0$ for $\nu = 1$ one can show that $\langle x^2(t)\rangle \sim O(t)$, $t \to 0$, i.e. the particle has normal diffusive behavior. It can be shown that in the long time limit the particle may have anomalous diffusive behavior of form (2.54). Thus, for example, in the case $\beta - \alpha\delta = \frac{1}{2}$, $\mu > 1/4$ the anomalous diffusion exponent is equal to 1/2. Same anomalous diffusion exponent appears in the case $\alpha\delta - \beta + 3/4 = \mu \leq 1/4$.

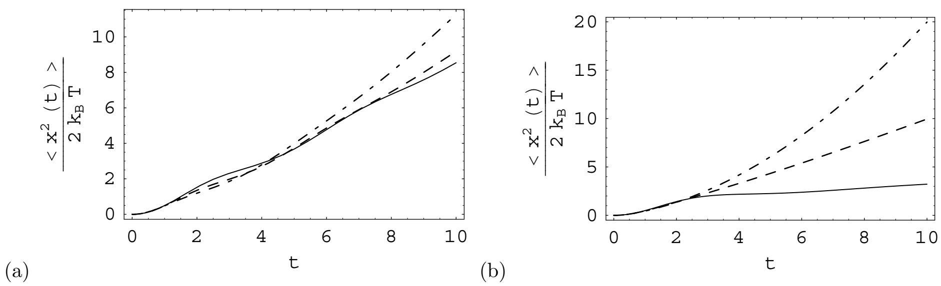


FIGURE 8. Graphical representation of MSD (2.23) in case of thermal initial conditions ($v_0^2 = k_B T = 1$, $x_0 = 0$), $\nu = 1$ and frictional memory kernel of form (1.5); $C_{\alpha,\beta,\delta} = 1$; $k_B T = 1$; $\tau = 1$; (a) $\alpha = \beta = 3/2$, $\delta = 1$, $\mu = 3/4$ (solid line), $\mu = 1/2$ (dashed line), $\mu = 1/4$, (dot-dashed line); (b) $\mu = 1/2$; $\alpha = \delta = 1$; $\beta = 3/2$ (solid line); $\beta = 1$ (dashed line); $\beta = 1/2$ (dot-dashed line).

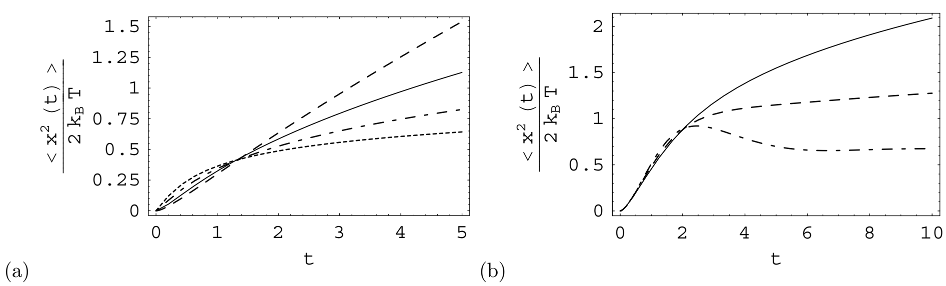


FIGURE 9. Graphical representation of MSD (2.23) in case of thermal initial conditions ($v_0^2 = k_B T = 1$, $x_0 = 0$) and frictional memory kernel of form: (a) $\gamma(t) = 2\lambda\delta(t)$; $\lambda = 1$; $\mu = \nu = 3/4$ (solid line), $\mu = 5/8$, $\nu = 7/8$ (dashed line), $\mu = 3/4$, $\nu = 5/8$ (dot-dashed line), $\mu = 7/8$, $\nu = 1/2$ (dotted line); (b) $\gamma(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$; $\mu = \nu = 3/4$; $\alpha = 3/4$ (solid line); $\alpha = 1/2$ (dashed line); $\alpha = 1/4$ (dot-dashed line).

REMARK **2.2**. Following same procedure as previous, one can investigate anomalous diffusion for the following recently introduced FGLE by Camargo et al. [5]:

$$
\begin{aligned}
&{}_C D_{0+}^{\mu} x(t) + \int_0^t \gamma(t-t')\, {}_C D_{0+}^{\nu} x(t')\mathrm{d}t' = \xi(t), \qquad (2.56)\\
&\dot{x}(t) = v(t),
\end{aligned}
$$

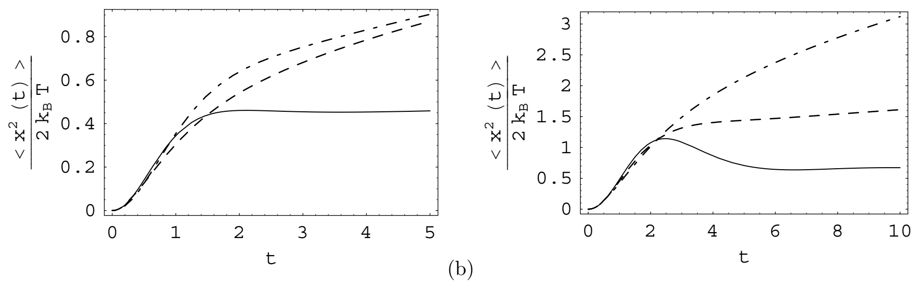


FIGURE 10. Graphical representation of MSD (2.58) in case of thermal initial conditions ($v_0^2 = k_BT = 1$, $x_0 = 0$) and frictional memory kernel of form: (a) $\gamma(t) = 2\lambda\delta(t)$; $\lambda = 1$; $\mu = 3/2$, $\nu = 1/4$ (solid line); $\mu = 3/2$, $\nu = 1/2$ (dashed line); $\mu = 7/4$, $\nu = 1/2$ (dot-dashed line); (b) $\gamma(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$; $\mu = 3/2$, $\nu = 3/4$; $\alpha = 1/4$ (solid line); $\alpha = 1/2$ (dashed line); $\alpha = 3/4$ (dot-dashed line).

where $1 < \mu \leq 2$ and $0 < \nu \leq 1$. For the variance one can obtain

$$\sigma_{xx} = 2k_BT \left[\frac{1}{\Gamma(\nu)} \int_0^t \mathrm{d}\xi G(\xi)\xi^{\nu-1} - \int_0^t \mathrm{d}\xi G(\xi) {}_C D_{0+}^{\mu-\nu} G(\xi)\right]. \quad (2.57)$$

Thus, the MSD is given by

$$\begin{aligned} \langle x^2(t)\rangle \quad &= x_0^2 + 2x_0 v_{0C} D_{0+}^{\mu-1} I(t) + v_0^2 \left[{}_C D_{0+}^{\mu-1} I(t)\right]^2 \\ &+2k_BT \left[\frac{1}{\Gamma(\nu)} \int_0^t \mathrm{d}\xi G(\xi)\xi^{\nu-1} - \int_0^t \mathrm{d}\xi G(\xi) {}_C D_{0+}^{\mu-\nu} G(\xi)\right], \end{aligned} \quad (2.58)$$

with $G(0) = 0$, where

$$\hat{g}(s) = \frac{s}{s^\mu + s^\nu \hat{\gamma}(s)}, \quad (2.59)$$

$$\hat{G}(s) = \frac{1}{s^\mu + s^\nu \hat{\gamma}(s)}, \quad (2.60)$$

$$\hat{I}(s) = \frac{s^{-1}}{s^\mu + s^\nu \hat{\gamma}(s)}. \quad (2.61)$$

From relations (2.59), (2.60) and (2.61) follows that $g(t) = G'(t)$ and $G(t) = I'(t)$.

In Figure 10 the MSD (2.58) in case of Dirac delta and power law frictional memory kernels for different values of parameters is presented.

3. Conclusions

In this paper we derive general formulas for calculation of variances and MSD for a FGLE with two time fractional derivatives. Exact expressions for the relaxation functions, mean velocity and mean particle displacement, variances and MSD in case of a three parameter M-L frictional memory kernel are obtained. Many previously obtained results are recovered. Asymptotic behaviors of the MSD in the short and long time limit are analyzed. It is shown that anomalous diffusion occurs. Cases for modeling single file-type diffusion or possible generalizations thereof are discussed.

The presented FGLE approach is based on a direct generalization of the GLE and provides a very flexible model to describe stochastic processes in complex systems with only few parameters.

4. Appendices

4.1. Appendix I: The Mittag-Leffler functions

The standard M-L function, introduced by Mittag-Leffler, is defined by [38]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{4.1}$$

where ($z \in C; \Re(\alpha) > 0$). The two-parameter M-L function, which is introduced and investigated later, is given by [38]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{4.2}$$

where ($z, \beta \in C; \Re(\alpha) > 0$). The M-L functions (4.1) and (4.2) are entire functions of order $\rho = 1/\Re(\alpha)$ and type 1. Note that $E_{\alpha,1}(z) = E_\alpha(z)$. These functions are generalization of the exponential, hyperbolic and trigonometric functions since $E_{1,1}(z) = e^z$, $E_{2,1}(z^2) = \cosh(z)$, $E_{2,1}(-z^2) = \cos(z)$ and $E_{2,2}(-z^2) = \sin(z)/z$.

The Laplace transform of the two-parameter M-L function is [38]:

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\pm a t^\alpha)\right](s) = \int_0^\infty e^{-st}t^{\beta-1}E_{\alpha,\beta}(\pm a t^\alpha)\mathrm{d}t = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}, \tag{4.3}$$

where $\Re(s) > |a|^{1/\alpha}$.

Prabhakar [40] introduced the following three-parameter M-L function:

$$E_{\alpha,\beta}^{\delta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \tag{4.4}$$

where $(\beta, \delta, z \in C; \Re(\alpha) > 0)$, $(\delta)_k$ is the Pochhammer symbol ($(\delta)_0 = 1, (\delta)_k = \Gamma(\delta + k)/\Gamma(\delta)$). It is an entire function of order $\rho = 1/\Re(\alpha)$ and type 1. Note that $E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$ (see also [48, 7]). Capelas et al [7] discussed the complete monotonicity of these M-L functions and have showed that they are suitable models for non-Debye relaxation phenomena in dielectrics. Moreover, Stanislavsky and Weron [49] recently gave numerical approximation of the three parameter M-L function, based on the Dirichlet average of the two parameter M-L function.

The Laplace transform of the three-parameter M-L type function is given by [48, 40, 7]:

$$\mathcal{L}\left[t^{\beta-1}E^{\delta}_{\alpha,\beta}(\omega t^{\alpha})\right](s) = \frac{s^{\alpha\delta-\beta}}{(s^{\alpha}-\omega)^{\delta}}, \tag{4.5}$$

where $|\omega/s^{\alpha}| < 1$. Also, the following Laplace transform formula is true for the three parameter M-L function (4.4) [51]:

$$\frac{s^{\zeta(\varrho-1)}}{s^{\varrho} - \lambda\left[\frac{s^{\rho\gamma-\varrho}}{(s^{\rho}+\nu)^{\gamma}}\right]} = \mathcal{L}\left[\sum_{k=0}^{\infty}\lambda^{k}x^{2\varrho k+\varrho+\zeta-\zeta\varrho-1}E^{\gamma k}_{\rho,2\varrho k+\varrho+\zeta-\zeta\varrho}(-\nu x^{\rho})\right](s). \tag{4.6}$$

The asymptotic behavior of the three parameter M-L function can be obtained from [44]

$$E^{\delta}_{\alpha,\beta}(z) = \frac{(-z)^{-\delta}}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)}{\Gamma(\beta-\alpha(\delta+n))}\frac{(-z)^{-n}}{n!}, \quad |z| > 1. \tag{4.7}$$

Thus, for large z one obtains

$$E^{\delta}_{\alpha,\beta}(z) \sim O\left(|z|^{-\delta}\right), \quad |z| > 1. \tag{4.8}$$

When $\delta \to 1$ series (4.7) reduces to the asymptotic expansion for the regular generalized M-L function (4.2), and by resummation the index n goes from 1 to infinity, i.e.

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty}\frac{(-z)^{-n}}{\Gamma(\beta-\alpha n)}, \quad |z| > 1. \tag{4.9}$$

4.2. Appendix II: Fractional derivatives and integrals

The R-L fractional integral of order $\mu > 0$ is defined by [22]:

$$\left(I^{\mu}_{a+}f\right)(t) = \frac{1}{\Gamma(\mu)}\int_a^t \frac{f(t')}{(t-t')^{1-\mu}}\mathrm{d}t', \quad t > a, \quad \Re(\mu) > 0. \tag{4.10}$$

For $\mu = 0$, this is the identity operator, $\left(I_{a+}^{0} f\right)(t) = f(t)$. For the R-L integral (4.10) the following formula is true [45]:

$$\left(I_{0+}^{\mu}\left[t^{\beta-1} E_{\alpha,\beta}^{\delta}\left(\omega t^{\alpha}\right)\right]\right)(x) = x^{\mu+\beta-1} E_{\alpha,\mu+\beta}^{\delta}\left(\omega x^{\alpha}\right). \tag{4.11}$$

There are different types of fractional derivatives. The most used are the R-L fractional derivative of order μ defined by [22]:

$$\left({}_{RL}D_{a+}^{\mu} f\right)(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(I_{a+}^{(n-\mu)} f\right)(t) \tag{4.12}$$

and the Caputo fractional derivative [8]:

$$\left({}_{C}D_{0+}^{\mu} f\right)(t) = \left(I_{a+}^{(n-\mu)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f\right)(t), \tag{4.13}$$

where $n = [\Re(\alpha)] + 1$ is the smallest integer larger than α. R-L fractional derivative is a left inverse of R-L fractional integral, i.e. ${}_{RL}D_{a+}^{\mu}\,{}_{RL}I_{a+}^{\mu} f(t) = f(t)$. Note that if we consider proper initial conditions the R-L and Caputo fractional derivatives are equivalent since [22]

$$\left({}_{RL}D_{a+}^{\mu} f\right)(t) = \left({}_{C}D_{a+}^{\mu} f\right)(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{k-\mu}}{\Gamma(k-\mu+1)} f^{(k)}(a+). \tag{4.14}$$

For the three parameter M-L function (4.4) the following formula is true [40, 45] (see relation (4.11)):

$${}_{RL}D_{0+}^{\mu}\left[z^{\beta-1} E_{\alpha,\beta}^{\delta}(a z^{\alpha})\right](x) = x^{\beta-\mu-1} E_{\alpha,\beta-\mu}^{\delta}(a x^{\alpha}), \tag{4.15}$$

where $\Re[\beta-\mu] > 0$, $\mu > 0$, $\Re[\delta] > 0$, $a \in C$.

The Laplace transform of the R-L and Caputo fractional derivatives are given by, respectively, [22, 38]:

$$\mathcal{L}\left[{}_{RL}D_{0+}^{\mu} f(t)\right](s) = s^{\mu}\mathcal{L}\left[f(t)\right](s) - \sum_{k=0}^{n-1}\left({}_{RL}D_{0+}^{\mu-k-1} f\right)(0+)s^{k}, \tag{4.16}$$

$$\mathcal{L}\left[{}_{C}D_{0+}^{\mu} f(t)\right](s) = s^{\mu}\mathcal{L}\left[f(t)\right](s) - \sum_{k=0}^{n-1} f^{(k)}(0+)s^{\mu-1-k}. \tag{4.17}$$

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