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Book review

Contents

Potential Theory, Lester L. Helms. Springer-Verlag London, Ltd., London (2009). xii+441 pp., ISBN: 978-1-84882

Potential theory is a broad area constituted of various subjects related to mathematical analysis, partial differential equations, probability, networks, mathematical physics, and so on. Among these subjects, the Laplace equation and harmonic functions (solutions to the Laplace equation), play central roles. In order to appreciate recent developments in partial differential equations, it is crucial to understand harmonic functions in depth since the Laplace equation is the simplest and the most fundamental partial differential equation, which serves as a prototype for general partial differential equations.

The author wrote an excellent book on potential theory [1], which concentrated on harmonic functions and superharmonic functions (super functions for the Laplace equation) on Euclidean domains. [1] was recognized as a good introduction to potential theory. Prerequisite knowledge was basic analysis in first-year graduate courses and all theorems were presented in self-contained fashion. Unfortunately, [1] was out of print.

This book under review is a revised and expanded version of [1]. Roughly speaking, this book consists of two parts: Chapters 1–6 and Chapters 7–11. The first six chapters of this book correspond to Chapters 1–11 of [1]. The author removes Chapter 12 (Martin boundary) from [1], and adds instead the Neumann problem for a disk in the plane and for a ball in the 3-dimensional

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space. On the other hand, Chapters 7–11 are devoted to an intensive study of classical solutions of general elliptic partial differential equations. So, these two parts have different natures. Let us see these parts in more detail.

Chapters 1–6 are revised versions of [1] and cover the core of potential theory. [1] was known as a friendly introduction to potential theory; arguments are beautifully concise, and yet easy to follow. The first part of this book rearranges the ingredients of [1]. The author changed the order of the subjects. For instance, the Dirichlet problem is solved by the Perron–Wiener–Brelot method in an early stage (Chapter 2) shortly after giving minimum properties of harmonic functions and superharmonic functions. After that, the boundary limit theorem of a harmonic function, the definition of the Green function for a general domain and the Riesz decomposition of a superharmonic function follow. This change would be helpful for the reader to catch the machinery of the Perron–Wiener–Brelot method. Balayage and capacity are discussed in Chapter 4. After Chapter 5 “Dirichlet problem for unbounded regions”, the energy principle and the Wiener criterion for a regular boundary point with respect to the Dirichlet problem are given in Chapter 6. Chapters 1–4 and 6 cover the most fundamental parts of potential theory. The reader who has finished these chapters can proceed to more advanced books and papers in potential theory.

The purposes of the second part, Chapters 7–11, are discussions of classical solutions to general elliptic partial differential equations in smooth domains with various boundary conditions. Chapter 7 “Interpolation and monotonicity” is devoted to Hölder spaces of smooth functions with Hölder continuous derivatives. Very precise norm estimates of derivatives are given. Chapter 8 “Newtonian potential” deals with generalized Newtonian potentials. Actually, Newtonian potentials are introduced in Chapter 3 in connection with the Riesz decomposition. It is observed that a Newtonian potential solves the Poisson equation $\Delta u = f$. Since the main objectives of the second part of this book is a general elliptic partial differential equation, the Poisson equation need to be generalized, and so does a Newtonian potential. Subnewtonian kernels are generalizations of the Newtonian kernel. Precise estimates for the integrals of subnewtonian kernels are given.

In Chapter 8 “Elliptic operators”, the author introduce general elliptic partial differential equations with Hölder continuous coefficients and general boundary conditions, including oblique derivative problems. After giving the fundamental Hopf maximum principle, the Dirichlet problem with respect to an elliptic operator is solved for a ball by the method of continuity. Estimates given in Chapters 6 and 7 are required in this process. By giving the notions of superfunctions and subfunctions for elliptic operators, the Dirichlet problem with respect to an elliptic operator is solved for a bounded domain with the aid of the Perron–Wiener–Brelot method, which was fully explained in Chapter 2 for the Laplace equation. Since superfunctions and subfunctions correspond to superharmonic functions and subharmonic functions, the reader who has grasped the classical Perron–Wiener–Brelot method can easily follow this generalization.

Chapter 10 “A priori bounds” and Chapter 11 “Oblique derivative problem” deal with general boundary conditions such as mixed boundary condition and oblique boundary condition. Again sharp estimates are developed. Some strong assumptions on the geometry of the domain are imposed. The discussions are technically involved.

In the preface the author said: “I have tried to present a clear path from the calculus to classic potential theory and then to recent work on elliptic partial differential equations using potential theory methods. The goal has been to move the reader into a fertile area of mathematical research as quickly as possible.” The first part (Chapters 1–6) corresponds to a path from the calculus to classic potential theory and the second part (Chapters 7–11) to a path from the classic potential theory to recent work on elliptic partial differential equations. The reviewer has an impression that the first part is a very successful short path to the core of classical potential theory, and the second part is also a good path to classical solutions of elliptic partial differential equations with various boundary conditions; very precise boundary estimates are developed. It is, however, impossible for a single monograph to cover all major materials in partial differential equations, since the research on partial differential equations is extremely extensive, For instance, layer potential methods are not discussed in this book. Layer potential methods are useful to solve the Neumann problem for a general domain. See e.g. [2].

In conclusion, this is a very good addition to the mathematical literature. It is recommended particularly to those who are interested in potential theory and elliptic partial differential equations with various boundary conditions.

[1] L. L. Helms, "An Introduction to Potential Theory", Wiley-Interscience, New York, 1969.
[2] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), no. 3, 572–611.

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Log-gases and Random Matrices, Peter J. Forrester. London Mathematical Society Monograph, vol. 34. Princeton University Press, Princeton, NJ (2010). 808 pp., ISBN: 978-0691128290

Random Matrices: High Dimensional Phenomena, Gordon Blower. London Mathematical Society Lecture Notes Series, vol. 367. Cambridge University Press, Cambridge, (2009). 448 pp., ISBN: 9780521133128

Random matrix theory is a rapidly developing area that is on the crossroads of probability theory, mathematical physics, integrable systems, and mathematical analysis. It has profound applications in combinatorics, number theory, statistics, and many parts of physics.

For a long time the main reference for random matrix theory was the influential book of Madan Lal Mehta [4] from 1967 with revised and updated editions from 1991 and 2004. Written from a physics perspective the second edition of Mehta's book presents a fairly comprehensive account of the developments up to the late eighties.

However the field exploded in the nineties with numerous important contributions and directions. Given the many directions that the field branched into it is no longer possible to give a complete account of all important developments in one monograph. Therefore, it is natural that a number of books on random matrices have appeared recently are all remarkably different.

The book of Peter Forrester is a monumental work of almost 800 densely filled pages. It is in spirit closest to the book of Mehta, in that it emphasizes the origins of random matrix theory in models from statistical mechanics. The log-gases from the title refer to joint probability density functions of n particles on the real line having the form

$$\frac{1}{Z_n} \prod_{i<j} |x_j - x_i|^\beta \prod_{j=1}^{n} w(x_j). \tag{1}$$

Eigenvalues of invariant ensembles of random matrices follow such a distribution with the typical values of $\beta = 2$ (for unitary ensembles), $\beta = 1$ (for orthogonal ensembles) and $\beta = 4$ (for symplectic ensembles). These are special cases since for these values of β the eigenvalues are a determinantal (for $\beta = 2$) or Pfaffian (for $\beta = 1, 4$) point process. This allows for many explicit formulas and direct connections with orthogonal polynomials and special functions. The study of gap probabilities in the classical ensembles (that is, with Hermite, Laguerre, or Jacobi weight function $w(x)$) leads to a connection with Painlevé equations and Fredholm determinants, which are treated in depth in Forrester's book.

The book is also special among the recently published monographs in that it presents an extensive introduction to the combinatorial models of non-intersecting lattice paths that give rise to random matrix theory statistics and to parameter dependent ensembles that have a connection with non-intersecting Brownian motion. There is much more in the book than can be mentioned in this review (Jack polynomials, β ensembles, complex random matrices, just to name a few topics). The book contains a wealth of information and is highly recommended.

The book of Gordon Blower emphasizes analytical and probabilistic aspects of random matrices. It centers around the theme of concentration of measure in high dimension, which gives it a distinctive probabilistic flavor. The invariant ensembles of Hermitian and symmetric matrices leading to eigenvalue distributions of the form (1) are also the main random matrix ensembles that are covered in the book. The approach to the distribution of eigenvalues, however, is via transportation and concentration inequalities. Sharp bounds on the constants in these inequalities yield the almost sure convergence of eigenvalues for the case of $w(x) = e^{-nv(x)}$ with a convex potential v. The notion of equilibrium measure in an external field is discussed nicely.

Special functions are treated in the context of integrable operators. The special cases of sine, Airy and Bessel kernels are discussed in some detail. These are related to the local behavior of eigenvalues in unitary matrix ensembles. The Costin–Lebowitz–Soshnikov theorem on fluctuations in determinantal point processes is emphasized as well. The book also covers connections with Lie groups, Young tableaux, and free probability theory.

Besides the works by Blower and Forrester, four other books on random matrices [1–3,6] appeared in the last two years.[1] These are all welcome additions to the literature, since each of them approaches random matrices from its own perspective. For a broad overview one may consult the recent handbook [5].

[1] G.W. Anderson, A. Guionnet and O. Zeitouni, An introduction to random matrices. Cambridge University Press, Cambridge, 2010.

[2] Z. Bai and J. Silverstein, Spectral analysis of large dimensional random matrices, Second edition, Springer, New York, 2010.

[3] P.A. Deift and D. Gioev, Random Matrix Theory: Invariant Ensembles and Universality, Courant Lecture Notes in Mathematics, vol. 18, American Mathematical Society, Providence, RI, 2009.

[4] M. L. Mehta, Random matrices and the statistical theory of energy levels, Academic Press, New York–London, 1967; Random Matrices, second edition, Academic Press Inc., Boston, 1991; Random Matrices, third edition, Elsevier/Academic Press, Amsterdam, 2004.

[5] The Oxford Handbook of Random Matrix Theory, edited by Gernot Akemann, Jinho Baik, and Philippe Di Francesco, Oxford University Press, 2011.

[6] L. Pastur and M. Shcherbina, Eigenvalue distribution of large random matrices, Math. Surveys and Monographs Vol. 171, Amer. Math. Soc. 2011.

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[1] See the reviews of [1,2] below in this compilation.

An Introduction To Random Matrices, G. Anderson, A. Guionnet, O. Zeitouni. Cambridge Studies in Advances Mathematics, vol. 118. Cambridge University Press (2010). xiv+492 pp., Hardcover, ISBN: 978-0-521-19452-5

Random matrix theory is a fast developing subject at the crossroads of mathematics and physics. It provides applications involving special functions, (multiple) orthogonal polynomials and potential theory. Some recent text books are [1,2].[2]

The book by Anderson, Guionnet and Zeitouni is intended for a general mathematical audience. The goal is to provide a rigorous introduction to some of the core issues of modern random matrix theory. The selected issues are quite diverse and sometimes hard to find in the literature. The book occasionally uses special functions, such as Gamma and Beta functions, Hermite and Laguerre polynomials and Painlevé equations (Chapter 3). Compared to other text books like [2], the emphasis is less on special functions and more on probability theory. Some of the presented methods involve logarithmic potential theory with external field (Chapter 2), and Stieltjes transforms and moments of probability measures on the real line (Chapters 2 and 5).

The book consists of five chapters. Chapter 1 is the introduction. Chapter 2 studies real and complex Wigner matrices. These are matrices for which the entries are independent and identically distributed random variables with mean zero, subject to a real or Hermitian symmetry. When the entries are normally distributed with suitable variances then Wigner matrices are known under the name 'Gaussian orthogonal ensemble' (GOE) or 'Gaussian unitary ensemble' (GUE), in the real or complex case respectively.

A classical result by Wigner from 1955 asserts that the normalized eigenvalue counting measure of an appropriately rescaled $N \times N$ real Wigner matrix converges in probability as $N \to \infty$ to the semicircle distribution

$$\frac{1}{2\pi}\sqrt{4-x^2}\,dx, \quad x \in [-2, 2].$$

The second chapter of the book contains several proofs and generalizations of Wigner's theorem. Two major approaches are discussed. The first one is to characterize the limiting behavior of the moments of the eigenvalue counting measures, leading to a combinatorial framework involving the counting of paths, words and sentences encoded by graphs and trees. This method is further extended throughout the chapter to obtain almost sure convergence, bounds on the maximal eigenvalue, central limit theorems, and a similar treatment for complex instead of real Wigner matrices. The second approach for proving Wigner's theorem is to derive a functional equation for the limiting Stieltjes transform of the eigenvalue counting measures of the Wigner matrices. This method is briefly discussed with some pointers to the literature.

A related topic is the 'speed', in function of N, with which the eigenvalues of the GOE, GUE and related random matrix ensembles converge to their limiting distribution for $N \to \infty$. Very precise estimates are presented with the help of large deviation principles. An important role thereby is played by logarithmic potential theory, more precisely equilibrium measures on the real line with an external field (Section 2.6).

Chapter 3 discusses 'gap probabilities', i.e., the probability that a random matrix has no eigenvalues in a certain real interval. By a suitable scaling and letting $N \to \infty$ one obtains differential equations for gap probabilities in terms of the endpoints of the interval, leading to the Painlevé II and Painlevé V equations. The chapter also contains a brief and instructive

[2] See the review of [2] above in this compilation.

introduction to Fredholm determinants and Pfaffians. The chapter is particularly accessible and self-contained and it involves several special functions.

Chapter 4 discusses a number of selected, more advanced topics: Lie group theory and Weyl's theorem; Determinantal point processes; Stochastic analysis of random matrices; Concentration inequalities; Tridiagonal matrix models and the β ensembles. Some of the more involved issues may require more effort or additional background.

Chapter 5 discusses free probability. This is a 'non-commutative analogue' of classical probability theory, intimately related to functional analysis and developed since the pioneering work of Voiculescu in the 1990s. Free probability yields a powerful framework for obtaining e.g. the limiting eigenvalue distribution of sums and products of random matrices, etc. This chapter is one of the most accessible and complete introductions to free probability available in the literature.

Summarizing, the book aims to introduce some of the modern techniques of random matrix theory in a comprehensive and rigorous way. It has a broad range of topics and most of them are fairly accessible. The focus is on introducing and explaining the main techniques, rather than obtaining the most general results. Additional references are given for the reader who wants to continue the study of a certain topic. The writing style is careful and the book is mostly self-contained with complete proofs. This is an excellent new contribution to random matrix theory.

[1] P. Deift, Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert approach. Courant Lecture Notes in Mathematics Vol. 3, Amer. Math. Soc., Providence, RI, 1999.
[2] P.J. Forrester, Log-gases and random matrices. Princeton University Press, Princeton, NJ, 2010.

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Spectral Analysis of Large Dimensional Random Matrices, Zhidong Bai, Jack W. Silverstein. Originally published by Science Press, 2006. Springer Series in Statistics, 2nd ed. (2010). xvi + 552 pp., Hardcover, ISBN: 978-1-4419-0660-1

For many years, there was essentially one reference book concerning random matrix theory. Mehta's *Random Matrices* [5] was the first stop for the 'classical' (*i.e.* pre-1990s) results in the field. Despite being relatively encyclopedic for the early results, Mehta's book is bizarrely organized (the 2004 edition has 53 appendices!) and not always rigorous. The last 20 years has brought something of a renaissance to the subject, and very recently there have been a raft of new books filling various niches, and emphasizing various aspects of the subject. Those dabbling in the field should check out [1–4], though by no means is this list exhaustive.[3] Those with a more statistical bent are directed to *Spectral Analysis of Large Dimensional Random Matrices*, the object of the current review.

In *Spectral Analysis of Large Dimensional Random Matrices*, Zhidong Bai and Jack W. Silverstein provide a glimpse into random matrix theory covering many of the basic theorems in the field with an eye towards results of statistical importance. The book is differentiated from other books in random matrix theory, since the authors avoid all mention of the solvable (*i.e.* determinantal or Pfaffian) structure which leads so eloquently to many deep results for

[3] See the reviews of [2,4] above in this compilation.

ensembles of matrices with a very specific form. Instead, the authors concentrate on results which can be arrived at using more probabilistic methods and which are applicable to a wider array of ensembles. This choice is a good one, since there are other references which address the solvable structure of random matrices, and the authors are certainly writing about the part of the field in which they are experts. However, a reader who picked up only this book with a desire to get a feel for random matrix theory, would miss out on the large (and very beautiful) theory of solvable ensembles. As such, I would hesitate to recommend the book as the sole text for a general graduate-level (mathematics) course in random matrix theory. However, researchers in the field who are interested in the spectral theory of Wigner matrices and their generalizations would find the book useful.

Before giving a very brief overview of the topics considered, one minor critique (which perhaps reflects more on my sensibilities than those of the authors'): throughout the book the authors rely on a number of abbreviations for commonly used terms (RM, RMT, ESD, LSD, CLT, MCT, M-P, TW, GUE, etc.). Some of these abbreviations are so common as to cause no difficulty (*e.g.* RMT, CLT), and some are genuinely entrenched in the literature (*e.g.* GUE) so that their use is expected (if confusing for the initiate). However, the preponderance of abbreviations does nothing to help the exposition. In fact, since I suspect many people will use this book as a reference as opposed to a text, and since there is no table of common abbreviations/symbols included, the abbreviations introduce an unnecessary barrier to understanding. If another edition of the book is released, expanding all abbreviations would undeniably improve the readability of the book, at the expense of only a handful of additional pages.

The book begins with Wigner's classical derivation of the semicircle law for random matrices: if $(\mathbf{M}_n)$ is a sequence of $n \times n$ Hermitian matrices with i.i.d. (up to the obvious dependence imposed by Hermiticity) standard normal entries of mean 0 and variance 1 off the diagonal and variance 2 on the diagonal, and if $\lambda_1^n, \ldots, \lambda_n^n$ are the eigenvalues of $\mathbf{M}_n/\sqrt{n}$, then with probability 1,

$$\frac{\#\{\lambda_m^n < t\}}{n} \longrightarrow \int_{-2}^{t} \frac{1}{2\pi}\sqrt{4-x^2}\,dx \quad \text{for } t \in [-2, 2]. \tag{$*$}$$

This is the famous semicircle law, and it reflects the fact that, loosely speaking, the histogram of the eigenvalues of large Gaussian Hermitian random matrices is semicircular. Since Wigner's original combinatorial proof (which uses the fact that the moments of the semicircle distribution are related to Catalan numbers), new tools have been introduced to prove this theorem and its extensions which strengthen the result either by showing it with a stronger form of convergence or generalize to other types of random matrices (and other limiting distributions). Chapters 2, 3, 4, 6 and 7 deal with these limiting spectral distributions of various ensembles of random matrices and canonical methods, including Wigner's combinatorial approach as well as the Stieltjes transform approach, to deriving them. Chapter 8 provides a discussion on the rate of convergence to the limiting spectral distributions (*e.g* Eq. ($*$)).

The behavior of the largest and smallest eigenvalue, a subject of current interest in the field, is covered somewhat briefly in Chapter 5. The central result in this area, is the fact that, the random variable $n^{2/3}(\lambda_n^n - 2)$ (with λ_n^n being the largest eigenvalue of $\mathbf{M}_n/\sqrt{n}$ above) converges in distribution to a random variable with distribution

$$F_2(x) = \exp\left\{-\int_x^{\infty}(t-x)q^2(t)\,dt\right\},$$

where $q(t)$ is a solution to the Painlevé equation $q'' = tq + 2q^3$ satisfying $q(t) \sim \mathrm{A}i(t)$ as $t \to \infty$. This distribution, the Tracy–Widom law, may look somewhat complicated and/or specialized, however its discovery and subsequent rediscovery in a myriad of other seemingly unrelated contexts, demonstrates its importance as a new type of random variable. In fact, the parallel derivation of this distribution for other ensembles has introduced many new random variables with interesting connections to various Painlevé equations. Unfortunately, since the classical derivation of the Tracy–Widom law relies on the solvable structure of ensembles of Gaussian Hermitian ensembles, it is outside of the purview of the book (though the statement is included for completeness). Nonetheless, the authors are able to prove weaker results for a wider variety of ensembles.

The remainder of the book concentrates on linear statistics in the eigenvalues of random matrices (Chapter 9), eigenvectors of random sample covariance matrices (Chapter 10), the circular law: an analog of the semicircular law for asymmetric random matrices with, in general, complex eigenvalues (Chapter 11) and applications of random matrix theory primarily to wireless communications (Chapter 12).

All in all, this book is a good introduction to the asymptotic theory of eigenvalues of random matrices in non-solvable ensembles. Since the property of being solvable is quite special, the wide applicability of the methods presented here is a valuable addition to the random matrix library. The primary audience is probably researchers in the fields of statistics, random matrix theorists and those interested in the theoretical underpinnings of applications of random matrix theory to wireless communications. The proofs are presented at the level of a research publication, and thus the book does not seem to be particularly well suited as a text for a graduate class. Indeed, at times, the book feels like a collection of the main results published by the two authors. This is not necessarily a bad thing, given the impact the authors have had on the literature in the area.

[1] G. Akemann, J. Baik, and P. Di Francesco. *The Oxford handbook of random matrix theory*. Oxford University Press, Oxford, 2011.

[2] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni. *An Introduction to Random Matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[3] P. A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1999.

[4] Peter Forrester. *Log-gases and Random Matrices*. London Mathematical Society Monographs. Princeton University Press, 2010.

[5] Madan Lal Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.

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Mathematical Methods in Physics. Partial Differential Equations, Fourier Series, and Special Functions, Victor Henner, Tatiana Belorezova, Kyle Forinash. AK Peters Ltd. (2009). xviii+841 pp., Hardcover, ISBN: 978-1-56881-335-6

This text book contains an introduction, nine chapters, four appendices and a very short bibliography. It is implicitly divided into three parts which cover the following topics: Fourier series, partial differential equations (PDEs) and boundary value problems, and special functions.

The first part (Chapter 1) is devoted to Fourier series. It begins with the trigonometric Fourier expansion of a periodic function on the entire x-axis and its range of validity. Then, the Fourier series for non-periodic functions and the Fourier expansions on finite intervals of arbitrary length are briefly considered. The Fourier series in cosine or sine functions are analyzed and many appropriate exercises are solved in detail. Later, the Fourier series in complex form and the Fourier series for functions of several variables are sketched. Then, practical criteria for the uniform convergence of Fourier series are shown, with a closer look at the behavior of the Fourier series near a point of discontinuity of the function. The completeness of a system of trigonometric functions and the Parseval equality are also analyzed. Later the approximation of functions in the mean is defined, the Lyapunov theorem shown and various simple exercises solved. Next, the Fourier series of functions given at a collection of points is explained, along with the means to solve various types of differential equations by use of Fourier series expansions. Finally, the Fourier transform and the famous Fourier integral formula are given and their use illustrated in various exercises and examples.

The second part (Chapters 2, 3, 5, 6 and 7) contains a quite complete description of the partial differential equations of the mathematical physics under different types of boundary conditions. First, it briefly discusses in Chapter 2 the Sturm–Liouville problem and its solutions. Then, it considers the one- and two-dimensional hyperbolic equations (Chapters 3 and 4); the former ones are discussed carefully, analyzing a large variety of physical and engineering problems and phenomena. In the two-dimensional case, the motion of the rectangular and circular membranes are studied. Later, in Chapters 5 and 6 the parabolic equations in one- and higher-dimensional cases are appropriately discussed and applied to various specific physical and engineering problems with emphasis on the heat conduction. Finally, in Chapter 7 the elliptic partial differential equations, particularly the Laplace and Poisson equations, are considered under various boundary conditions. Examples of these equations in several domains are shown and analyzed; the case of a regular domain is considered in a detailed form. Next, a large number of two-dimensional problems with circular or cylindrical symmetry are extensively discussed.

Finally, the third part (Chapters 8 and 9) is devoted to special functions of the mathematical physics and applied mathematics. In fact it considers the Bessel and Legendre functions only. Some of the main properties of these functions are discussed in a very pedagogical way as well as the Fourier–Bessel and the Fourier–Legendre series. However other important properties (e.g., integral representations) are not described. More relevant is to observe that the classical orthogonal polynomials are either only briefly considered in Chapter 2 as eigenfunctions of Sturm–Liouville equations (Hermite, Laguerre, Jacobi) or non-considered at all (Bessel) in the whole book, what is most unfortunate. Neither the hypergeometric functions, other than Legendre's functions, are treated in this book.

In summary, this book is very appropriate for engineers, mathematicians and physicists. The partial differential equations in mathematical physics is significantly more detailed than the Fourier series and the special functions. The reading is not difficult and the topics are generally well described, containing numerous exercises with a detailed solution as well as abundant unsolved problems. Moreover, the attached companion problem-solving software makes the book very useful for self-study.

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Second Order Differential Equations. Special Functions and Their Classification, Gerhard Kristensson. Springer Verlag (2010). xiv + 16 pp., ISBN: 978-1-4419-7019-0

The main focus of this book is to analyze the solutions of second order linear differential equations of the form

$$u'' + \frac{G(z)}{\psi(z)} u' + \frac{H(z)}{\psi^2(z)} u = 0,$$

where $\psi(z)$, $G(z)$, $H(z)$ are polynomials with $\deg(\psi) = n$, $\deg(G) \le n-1$ and $\deg(H) \le 2(n-1)$ (if ∞ is a regular singular point) or $\deg(H) \le 2(n-2)$ (if ∞ is a regular point). These ODEs are called *Fuchsian equations* (or equations of *Fuchsian type*) and n denotes the number of regular singular points in the finite complex plane (plus an additional one at ∞).

Chapters 2 and 3 cover the basic properties of Fuchsian equations and their solutions. Chapter 4 specifically treats the $n = 1, 2, 3$ cases.

Chapter 5 is dedicated to the Fuchsian equation with regular singular points located at $z = 0, 1, \infty$

$$z(z-1)u'' + [(\alpha+\beta+1)z - \gamma]u' + \alpha\beta u = 0.$$

This is the *hypergeometric differential equation*. Its solution is denoted by $F(\alpha, \beta; \gamma; z)$ and is called the *hypergeometric function* (or *Gauss hypergeometric function*, or ${}_2F_1(\alpha, \beta; \gamma; z)$). The *generalized hypergeometric function* ${}_pF_q(\mathbf{a}; \mathbf{b}; z)$ has the series representation

$${}_pF_q(\mathbf{a}; \mathbf{b}; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},$$

where $(a)_n$ is the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Chapter 6 studies the particular cases

$$P_\nu(z) = z^\nu \, {}_2F_1\left(-\frac{\nu}{2}, \frac{1-\nu}{2}; 1; 1 - \frac{1}{z^2}\right),$$

$$Q_\nu(z) = k(\nu) z^{-\nu-1} \, {}_2F_1\left(1 + \frac{\nu}{2}, \frac{1+\nu}{2}; \nu + \frac{3}{2}; \frac{1}{z^2}\right).$$

These are the *Legendre functions* of the first and the second kind. Chapter 7 considers the *Kummer function* ${}_1F_1(\alpha; \gamma; z)$, the *Bessel function,* defined by the author as

$$J_\nu(z) = z^\nu \frac{e^{-\mathrm{i}z}}{2^\nu \Gamma(\nu+1)} \, {}_1F_1\left(\frac{\nu+1}{2}; 2\nu+1; 2\mathrm{i}z\right),$$

and the *Laguerre polynomials*

$$L_n^{(\alpha)}(z) = \binom{n+\alpha}{n} \, {}_1F_1(-n; \alpha+1; z).$$

These are all examples of *confluent hypergeometric functions*.

Chapter 8 reviews some basic properties of the Fuchsian equation with regular singular points located at $z = 0, 1, a, \infty$

$$u'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right) u' + \frac{\alpha\beta(z-h)}{z(z-1)(z-a)} u = 0.$$

This is *Heun's differential equation*. Lamé's and Mathieu's equations and spheroidal functions are briefly named.

Finally, the appendices cover some material on the Gamma function, difference equations, partial fractions, circles and ellipses in the complex plane, and some special functions.

I believe that, while the idea of studying special functions as solutions of ODEs is very valuable, the book is too short to accomplish this goal. Chapter 8 (22 pages) in particular, barely touches the topic of Heun's equation. Comparing for instance with Chapter 3 (66 pages) in Sergei Slavyanov's *Special functions. A unified theory based on singularities,* one gets the feeling that too much material is left out.

As a reference for the researcher working in the area of special functions, it cannot compete with classical texts like Andrews–Askey–Roy's *Special functions*, Temme's *Special functions,* Lebedev's *Special functions and their applications,* Whittaker–Watson's, *A course of modern analysis* or the great Abramowitz–Stegun's *Handbook of mathematical functions* and its modern descendent *NIST handbook of mathematical functions*.[4]

It would, however, be appropriate for an advance undergraduate or graduate course in ODEs. A solution manual is available from the author's webpage.

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The H-Function, A. M. Mathai, R. K. Saxena, H. J. Haubold. Springer Verlag (2010). xiv + 270 pp. ISBN: 978-1-4419-0915-2.

You either love them or you hate them... For some, H-functions are extremely handy tools, much more convenient than other classes of special functions. For others, they are just humongous. Personally, I had the fortune to encounter H-functions early in my graduate studies. Since then I have been using them extensively and promoting them to my students and colleagues.

H-functions are actually so easy to deal with. Integral transforms become mere index manipulation, the series expansion around zero is known, as is the asymptotic behavior. H-functions are particularly useful in the field of fractional calculus, related to dynamics in complex systems. Thus H-functions are the solution of the fractional diffusion equation [1]. Finally, they contain a load of other special functions as limiting cases, for instance, the celebrated Mittag-Leffler function and its generalized form.

When I was a student *the* book on H-functions was that of Mathai and Saxena, published by Wiley Eastern. I used this book during my masters and Ph.D. work, and later borrowed it from Tel Aviv University library during my postdoc period. And very recently, I have managed to secure an old copy through Amazon. After the discontinuation of the Mathai/Saxena book the market was essentially empty, and people needed to scrape together the properties of the H-functions from various sources in the literature.

Now, at long last, the new Springer book by Mathai, Saxena, and Haubold is filling the gap, providing access to the realm of this important special function. This book includes, as far as I can tell, all the information of the old Mathai/Saxena book. There is an historical background, the mathematical definitions in terms of a Mellin–Barnes type integral and the conditions on the parameters, plus a listing of a large number of special cases. The second chapter contains an extensive summary of various kinds of integral transforms of H-functions. After some seventy pages the reader is ready to start using H-functions *in praxi*.

However, the present book is much more than the old one. The authors garnered applications of the H-function in a large variety of fields, including statistics, astrophysics, rheology, or the

[4] See its review below in this compilation.

quickly developing field of fractional calculus and its applications. Particular noteworthy is the extensive bibliography, that makes this book useful for the H-function aficionado, but should also help demonstrate to curious readers the versatility and usefulness of the H-function.

With its very detailed table of contents, the name and subject indices, this book is very practical to use. The first two chapters are for constant referencing, while the remaining chapters provide interesting reading in terms of applications.

I was delighted to see that this book appeared. For around £70 or USD 90 on Amazon it does not come cheap. However, the layout of the book is very nice and the editing has been careful, I did not discover any immediate typos. My weakness for H-functions apart, this is a very useful tool, and I hope that the book will attract a lot of new followers, some of my best friends included.

[1] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*. Phys. Rep. **339**, no. 1 (2000), 77 pp.

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NIST Handbook of Mathematical Functions, F. W. J. Olver (editor-in-chief), D. W. Lozier, R. F. Boisvert, Ch. W. Clarck, (eds.) National Institute of Standards and Technology and Cambridge University Press (2010). 968 pp., 422 color illus., 100 tables, ISBN 978-0521140638 (softcover), 978-0521192255 (hardcover).

Many of us grew up (at least, scientifically) with a Dover's copy of "Abramowitz and Stegun" [1], an impressive piece of work originally published in 1964, and that immediately became a bestseller. But times have changed, and the burden of the years became apparent in many ways: in the large portion of tables, nowadays almost useless if you have access to a computer with a reasonable scientific software; in the lack of information about new classes of functions that have emerged in importance in the last half century, such as the Painlevé transcendents; in the need for better visual representation of the information.[5] So, the update was very necessary, and this project was undertaken by the National Institute of Standards and Technology (NIST) in 1997. The result is the *NIST Digital Library of Mathematical Functions*, a monumental online project [2]. The book under review is its hardcopy companion.

It is a big book (279 × 215 mm). Its almost 1000 pages are printed on a thick and sturdy paper that gives extra weight to the book (almost 3 kg) but at the same time adds quality to the printing. It has large pages with large type, bringing relief to our tired eyes: formulas and text are easy to read now, so you do not have to guess anymore if a subscript is a or α. And it is reasonably affordable too: you can get a softcover copy at Amazon for less than 50 USD.

A natural question is why do we need a hardcopy if we have the online version [2]. Obviously, [2] has apparent advantages with respect to the printed version. The straightforward one is the possibility to update or expand the information, or to correct any misprint in real time. But [2] has many more features:

- there is no need anymore to copy by hand a long identity into your paper: just click on the small letter "i" in the right corner of the formula, and you get access, say, to its TEX version;

[5] I wonder how the 20th edition of the NIST Handbook will look; probably our children will be amused with the flat 2D pictures we are using today.

- there are hyperlinks that take you directly to other sections of the book, or to the MathSciNet/Zentralblatt references to the original source;
- you do not have to chase a notation through the whole book: the cursor over the already mentioned "i" will explain it for you;
- there is a possibility of the interactive display of the graphics of some special functions (but beware of the technical issues, you might need to do some homework before enjoying this feature); however, even the standard full color figures look great.

The implementation of these technical aspects is impressive; the challenge of putting Math on the Web is very well described in [3].

So, why would you need the hardcopy? There might be several reasons, but after spending a good time leafing through the NIST Handbook I realized that I like the one provided by a colleague who visited me in Almería years ago. He wanted to borrow a book from the shelves in my office to have something to read in the evening. I obviously gave him a total freedom of choice, and he took Abramowitz and Stegun! When I asked him if he was actually going to *read* it, he replied that he liked to browse through it and look at the formulas.[6] I am convinced that during his next visit he will take home the *NIST Handbook...*, which will make this experience much more enjoyable.

[1] M. Abramowitz and I. A. Stegun, eds. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, New York: Dover Publications, 1972.
[2] DLMF: NIST Digital Library of Mathematical Functions, v. 1.0.4, 2012-03-23, National Institute of Standards and Technology, http://dlmf.nist.gov/.
[3] R. Boisvert, Ch. W. Clarck, D. Lozier, and F. Olver, *A Special Functions Handbook for the Digital Age*, Notices of the AMS, August 2011, 905–911.

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Interpolation Processes. Basic Theory and Applications, G. Mastroianni, G. V. Milovanović. Springer Monographs in Mathematics. Springer Verlag, Berlin (2008). xiv+ 444, ISBN: 978-3-540-68346-9.

Interpolation of functions is one of the main topics in Approximation Theory. The classical monographs in this area address numerous negative results while the present contribution deals mainly with new results on convergent interpolation processes, with respect to several norms, using algebraic and trigonometric polynomials. The reader interested in the subject can find them basically in specialized journals and, as far as I know, there is no relevant monograph covering both the theoretical point of view and a computational approach. This contribution, by two recognized specialists in Approximation Theory, combine the above two flavors in a successful way. This Italian–Serbian (I follow the alphabetical ordering for the authors) joint venture yields a nice output.

The book is divided into five chapters. A summary of them is presented in next.

In Chapter 1, a basic background on algebraic (in particular Chebyshev) and trigonometric polynomials, best approximation, Fourier series, and moduli of continuity is provided. The

[6] And he knew a lot of them by heart!

interpolation by trigonometric and algebraic polynomials, as well as convergence–divergence results, is carefully analyzed both from a theoretical and numerical point of view with many illustrative examples.

Chapter 2 constitutes an overview of the theory of orthogonal polynomials, with a special emphasis on their asymptotic properties and the distribution of their zeros. Basic properties of the classical orthogonal polynomials (Hermite, Laguerre, and Jacobi) are summarized. I like very much the section devoted to semi-classical orthogonal polynomials, which do not appear in the standard monographs on orthogonal polynomials. In particular, Generalized Gegenbauer, Generalized Jacobi, Generalized Hermite, and Freud weights are very useful examples. The last section of this chapter deals with weighted best approximation on finite and infinite intervals. In the last years, this topic has attracted the interest of people working in Approximation Theory since it allows the approximation of locally continuous functions on finite intervals as well as the approximation of continuous functions on infinite intervals with a given growth condition.

In Chapter 3 the trigonometric approximation by Fourier and Fejér sums, as well as de la Vallée Poussin means is analyzed. Discrete versions of them as well as Marcinkiewicz-type inequalities are studied. The Lagrange interpolation error in L^p spaces as well as in L^1 Sobolev spaces is considered. The weight version of these results is also treated.

Chapter 4 focuses on the interpolation by algebraic polynomials. First, the classical theory of interpolation is summarized, and, as a novelty with respect to similar monographs, the case of weighted interpolation is considered. Notice that this is an important feature taking into account that weighted spaces enlarge the class of functions which can be approximated by operators constructed with the help of interpolation.

The content of Chapter 5 is the most exciting of this book from the perspective of numerical analysis. The standard quadrature rules (Gauss, Gauss–Radau, and Gauss–Lobatto) are considered. Error estimates for the integration of continuous and analytic functions are analyzed. Some interesting numerical examples are discussed. In order to have a more complete framework of the topic, a section on Gauss–Kronrod quadrature rules would be welcome. Other topics like Integral Equations, more precisely, Fredholm integral equations of second kind and a modified version of the Nyström method, Moment Preserving Approximation, with a special emphasis on spline approximation, and Summation of Slowly Convergent Series, with an elegant description of the Laplace transform method, are presented.

Finally, a careful and updated selection of 508 publications is provided.

The book is very well written, with an up-to-date information about the topics covered in its five chapters. It will be of interest both to experts in the field and to people with a basic knowledge in mathematical analysis. I also recommend this book to Ph.D. students interested in an original approach to Approximation Theory.

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The Birth of Numerical Analysis, Adhemar Bultheel, Roland Cools (eds.). World Scientific Publ. Co., Hackensack, NJ., (2009). 240 pp. ISBN: 978-981-283-625-0.

This volume constitutes the proceedings of a two-day symposium held on October 29–30, 2007, at the Department of Computer Science, Catholic University of Leuven. The motivation for organizing this meeting was to commemorate the 60th anniversary of the landmark paper "Numerical inverting of matrices of high order" by John von Neumann and Herman H. Goldstine published 1947 in the Bulletin of the American Mathematical Society, a paper widely regarded as marking the beginning of modern numerical analysis. Included in the proceedings are most of the papers presented at the symposium plus some other papers solicited by the editors. The authors are distinguished researchers that have already been active around the middle of the 20th century and therefore were important players on the early numerical analysis scene.

The contributions vary greatly in their contents. At one end of the spectrum there are highly personal in-depth accounts of the evolution of a substantial subject area of numerical analysis. Such is the case in the paper by M. J. D. Powell on nonlinear optimization since 1959, the paper by K. Atkinson on Fredholm integral equations of the second kind, and the one by J. C. Butcher on numerical methods for ordinary differential equations in the early days. More specialized subjects are considered by J. N. Lyness, dealing with multidimensional extrapolation quadrature, by D. Chen and R. J. Plemmons, reviewing nonnegativity constraints in numerical analysis, and by R. Piessens, discussing the role in numerical analysis of Chebyshev polynomials with special reference to the work done at the Numerical Analysis Division of the Catholic University of Leuven. Several contributions are focused more on individual personalities, such as the contribution by C. Brezinski on some pioneers of extrapolation methods, the contribution by P. J. Davis and A. S. Fraenkel on remembering Philip Rabinowitz, and the interview with H. B. Keller by H. M. Osinga. A distinctly geographic orientation is given by G. A. Watson to his article on the history and development of numerical analysis in Scotland, from which the reviewer learned about the establishment in 1913 by E. T. Whittaker of a “Mathematical Laboratory” in Edinburgh, having always thought that Mauro Picone was the first to have founded (in 1927) such a laboratory in Naples. The latter, however, has grown into a national institute of applied calculus in Rome and is still very active today.

Closely allied with the evolution of numerical analysis is the evolution of computer architectures and computer libraries. These are the topics of several other papers, one by J. J. Dongarra, H. W. Meuer, H. D. Simon, and E. Strohmaier on recent trends in high-performance computing, one by B. Ford on building a general-purpose numerical algorithms library, and the delightful reminiscences of P. J. Davis on his WW2 experiences with scientific computation at the Langley Field Laboratory in Hampton, Virginia.

The volume is attractively produced, containing a number of pictures, name and subject indices, and featuring on the front cover another birth — Botticelli’s birth of Venus.

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Other proceedings and compilations

Approximation Theory XIII: San Antonio 2010. Marian Neamtu, Larry Schumaker, (eds.) Springer Proceedings in Mathematics 13, Springer Verlag, NY, (2012). xvii + 415 pp., 38 illus., 11 in color, ISBN: 978-1-4614-0771-3.

FROM SPRINGER WEBSITE: These proceedings were prepared in connection with the international conference Approximation Theory XIII, which was held on March 7–10, 2010 in San Antonio, Texas. The conference was the thirteenth in a series of meetings in Approximation Theory held at various locations in the United States, and was attended by 144 participants.

The book will be of interest to mathematicians, engineers, and computer scientists working in approximation theory, computer-aided geometric design, numerical analysis, and related application areas. It contains a carefully refereed and edited selection of papers.

Along with the many plenary speakers, the contributors to this proceedings provided inspiring talks and set a high standard of exposition in their descriptions of new directions for research. Many relevant topics in approximation theory are included in this book, such as abstract approximation, approximation with constraints, interpolation and smoothing, wavelets and frames, shearlets, orthogonal polynomials, univariate and multivariate splines, and complex approximation.

Approximation and Computation. In Honor of Gradimir V. Milovanović. Walter Gautschi, Giuseppe Mastroianni, Themistocles M. Rassias (eds.) Springer Optimization and Its Applications 42, Springer Verlag, NY (2011). xxii + 484 p., 47 illus., 16 in color., hardcover. ISBN: 978-1-4419-6593-6.

FROM SPRINGER WEBSITE:

- Includes results from diverse areas of mathematics, engineering and the computational sciences.
- Presents approximation methods in various computational settings including: polynomial and orthogonal systems, analytic functions, and differential equations
- Provides a historical overview of approximation theory and many of its subdisciplines.
- Contains new results from diverse areas of research spanning mathematics, engineering, and the computational sciences

Approximation theory and numerical analysis are central to the creation of accurate computer simulations and mathematical models. The research in these areas can influence the computational techniques used in a variety of mathematical and computational sciences. This collection of contributed chapters, dedicated to the renowned mathematician Gradimir V. Milovanović, represents the recent work of experts in the fields of approximation theory and numerical analysis. These invited contributions describe new trends in these important areas of research including theoretic developments, new computational algorithms, and multidisciplinary applications. Special features of this volume:

- Presents results and approximation methods in various computational settings including polynomial and orthogonal systems, analytic functions, and differential equations.
- Provides a historical overview of approximation theory and many of its subdisciplines.
- Contains new results from diverse areas of research spanning mathematics, engineering, and the computational sciences.

"Approximation and Computation" is intended for mathematicians and researchers focusing on approximation theory and numerical analysis, but can also be a valuable resource to students and researchers in engineering and other computational and applied sciences.

Recent Advances in Orthogonal Polynomials, Special Functions, and Their Applications, Jorge Arvesú Guillermo López Lagomasino (eds.) Contemporary Mathematics 578, American Mathematical Society, Providence, RI (2012). approx. 254 pp., softcover. ISBN: 978-0-8218-6896-6.

FROM THE AMS WEBSITE: This volume contains the proceedings of the 11th International Symposium on Orthogonal Polynomials, Special Functions, and their Applications, held on August 29–September 2, 2011, at the Universidad Carlos III de Madrid in Leganés, Spain.

The papers cover asymptotic properties of polynomials on curves of the complex plane, universality behavior of sequences of orthogonal polynomials for large classes of measures and its application in random matrix theory, the Riemann–Hilbert approach in the study of Padé approximation and asymptotics of orthogonal polynomials, quantum walks and CMV matrices, spectral modifications of linear functionals and their effect on the associated orthogonal polynomials, bivariate orthogonal polynomials, and optimal Riesz and logarithmic energy distribution of points. The methods used include potential theory, boundary values of analytic functions, Riemann–Hilbert analysis, and the steepest descent method.

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