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Correlated continuous-time random walks—scaling limits and Langevin picture

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Abstract. In this paper we analyze correlated continuous-time random walks introduced recently by Tejedor and Metzler (2010 *J. Phys. A: Math. Theor.* **43** 082002). We obtain the Langevin equations associated with this process and the corresponding scaling limits of their solutions. We prove that the limit processes are self-similar and display anomalous dynamics. Moreover, we extend the model to include external forces. Our results are confirmed by Monte Carlo simulations.

Keywords: stochastic processes (theory), diffusion

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Contents

1. Introduction

The theory of continuous-time random walks (CTRW) was originally introduced almost 50 years ago by Montroll and Weiss [1]. It was epitomized as a physical tool to model anomalous diffusion in amorphous semiconductors by Scher and Montroll [2]. Since then CTRWs have been established as powerful machinery for the description of various physical systems, especially those displaying anomalous dynamics [3]–[5]. There are numerous examples of applications of CTRWs in the modeling of real-life phenomena: subsurface tracer dispersion [6], electron transfer [7], noise in plasma devices [8], models in gene regulation [9] and chemical reactions [10, 11], dispersion in turbulent systems [12], and transport of light in certain optical materials [13], to name only a few.

CTRWs are stochastic processes described uniquely by consecutive jumps of the walker with variable jump lengths, and waiting times of immobilization periods (trapping) between them. Each pair of jump lengths and waiting times is drawn from some probability distribution. In the simplest setting, jump lengths and waiting times are assumed independent. This is the so-called uncoupled case. Assuming that the jumps in the uncoupled CTRW have finite second moment, while the waiting times have finite first moment, we arrive at the Brownian motion in the scaling limit. On the other hand, if the jumps are heavy-tailed, we obtain the class of Lévy flights [14, 15]. The main problem with Lévy flights is the diverging mean square displacement (MSD), a property which is not physical for a particle with finite mass, although there exist exceptions: for instance, due to the topology of space [9]. Lévy flights may be cut off physically by dissipative nonlinearities [16]. In order to avoid a diverging MSD and to keep the heavy-tailed distribution of the jumps, coupled CTRWs need to be introduced [17]–[21]. A class of coupled CTRWs, which is particularly important in physics, are Lévy walks [17]–[19]. They proved to be appropriate models in the description of various phenomena: anomalous and super-diffusion dynamics in complex systems [18, 19], fluid

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flow in a rotating annulus [12], human travel [22, 23], epidemic spreading [24, 25], and foraging patterns of animals [26]–[28].

Uncoupled CTRWs with heavy-tailed distributions of waiting times give rise to the celebrated fractional Fokker–Planck equation [3, 29, 30]. In the past decade, this equation has become a standard tool in the description of subdiffusive dynamics, especially for the case of complex systems displaying aging and weak ergodicity breaking [31]–[35].

The Langevin picture of CTRWs corresponding to time-fractional dynamics was originally derived by Fogedby [36]. The idea was based on the subordination method, in which the usual Langevin dynamics is subordinated to the random operational time (counting process) of the system. The concept of subordination and coupled Langevin equations was further developed in [37]–[39] in the context of fractional Klein–Kramers equations. The equivalence between the fractional Fokker–Planck equation and subordinated Langevin equations was studied in detail in [40]–[46]. Some recent mathematical advances in this field can be found in [47]–[51]. Langevin equations with fractional derivatives were the subject of study in [52]–[54].

Both uncoupled and coupled CTRW are renewal processes, i.e., after each jump a new, independent pair of jump length and waiting time is drawn from the corresponding distributions. This so-called semi-Markovian property is not always justified, as underlined, for instance, by observations of active biological movements [55], human motion patterns [56], or in financial market dynamics [57]. It is therefore imperative to study the behavior of correlated CTRW processes in which the values of the jump lengths and/or the waiting times at the nth step depend on the corresponding values at the $n-1$th step. Progress in this direction has recently been obtained in [58]–[60] using different approaches. Here we build on the idea of creating correlations in jump lengths and waiting times by letting the two quantities diffuse in the respective space of jump lengths and waiting times [60]. In what follows we derive the continuous-time limit of this model and find the complementary Langevin picture. The paper is structured as follows: section 2 is devoted to CTRWs with correlated waiting times. We analyze its scaling limit process and come up with the corresponding system of coupled Langevin equations. This allows us to extend the model to include external forces. Also, we consider the heavy-tailed random walk of waiting times. In section 3 we study scaling limits of CTRWs with correlated jumps. We find a link between such CTRWs and the celebrated Feynman–Kac formula. Section 4 concludes the paper.

2. CTRW with correlated waiting times

2.1. Scaling limit

We begin by recalling the general framework for CTRW theory. Let $T_i, i = 1, 2, \ldots,$ be the sequence of nonnegative random variables representing the waiting times between successive jumps of a test particle. The number of jumps that this particle has performed up to time $t > 0$ equals

$$N(t) = \max\left\{n \geq 0 : \sum_{i=1}^{n} T_i \leq t\right\}. \quad (1)$$

The process $N(t)$ is usually referred to as the renewal or counting process.

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Next, we denote by J_i, $i = 1, 2, \ldots$, the sequence of consecutive jump lengths of the particle. The jump lengths and waiting times are assumed to be independent of each other. The position of the particle at time t is given by

$$R(t) = \sum_{i=1}^{N(t)} J_i. \tag{2}$$

The process $R(t)$ is called CTRW. It is fully characterized by the sequences of waiting times and jump lengths.

In what follows, we analyze a particular type of CTRW with correlated waiting times, which has been recently introduced in [60]. More specifically, we assume that the jumps J_i are independent, identically distributed (iid) symmetric random variables with finite second moment (for simplicity, we will assume a unit second moment). On the other hand, the waiting times T_i are correlated in the following manner: each waiting time is equal to

$$T_i = |\xi_1 + \cdots + \xi_i|, \tag{3}$$

where ξ_j are iid symmetric with finite second moment (we assume for simplicity that their second moment is equal to 1). Thus, the sequence of waiting times can be viewed as a standard random walk reflected at the origin. As shown in [60], CTRW with such waiting times are subdiffusive with exponent 2/3. In the next theorem, we derive the corresponding scaling limit. This result will allow for further detailed study of the model. In particular, we will write down the corresponding Langevin equations, derive the distributional properties of the limit process and extend the model to include external forces.

Theorem 1. *Let J_i, $i = 1, 2, \ldots$, be the sequence of iid symmetric random variables with finite second moment. Let T_i, $i = 1, 2, \ldots$, be defined by (3) and independent of J_i, $i = 1, 2, \ldots$. Then, the corresponding CTRW process $R(t)$ satisfies*

$$\frac{R(nt)}{n^{1/3}} \xrightarrow{d} X(t) = B_1(S^{-1}(t))$$

as $n \to \infty$ (time variable t is fixed). Here

$$S(t) = \int_0^t |B_2(u)|\,\mathrm{d}u, \tag{4}$$

moreover, $B_1(t)$ and $B_2(t)$ are two independent Brownian motions with Fourier transforms $\mathbb{E}[\exp(\mathrm{i}zB_i(t))] = \exp(-\frac{1}{2}tz^2)$, $i = 1, 2$, whereas $S^{-1}(t)$ is the inverse of $S(t)$, i.e.

$$S^{-1}(t) = \inf\{\tau > 0 : S(\tau) > t\}. \tag{5}$$

Proof. See appendix A. □

In the above theorem, '$\xrightarrow{d}$' means convergence in distribution[5]. Some remarks clarifying the structure of the limit process $X(t)$ are in order. Brownian motion $B_1(t)$

[5] In appendices A and B we prove stronger results—convergence in the Skorohod $\mathbb{J}_1$ topology, see [61]. In particular, this convergence implies convergence of all finite-dimensional distributions.

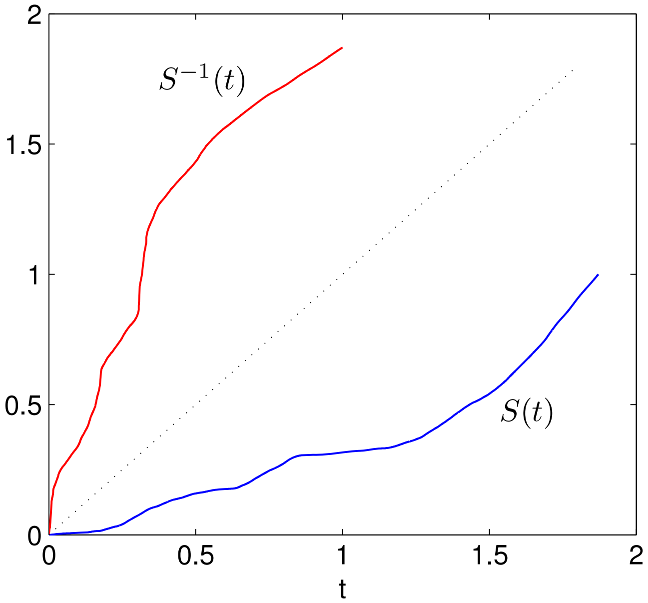

Figure 1. Typical trajectories of the process $S(t)$ and its inverse $S^{-1}(t)$, defined in formulas (4) and (5), respectively. Note that the trajectories are symmetric with respect to the identity function (dotted line). Moreover, they are smooth and strictly increasing, in contrast to the step-like behavior observed in CTRWs with diverging characteristic waiting times.

appears here naturally as the limit of the cumulated and scaled jumps J_i. Moreover, the cumulated and scaled waiting times (3) converge to the process $S(t)$. Consequently, the inverse process $S^{-1}(t)$ is the scaling limit of the corresponding counting process $N(t)$. Finally, the scaling limit of the correlated CTRW is obtained as a superposition of $B_1(t)$ and $S^{-1}(t)$ and has the form $X(t) = B_1(S^{-1}(t))$.

The typical trajectories of $S(t)$ and its inverse $S^{-1}(t)$ are shown in figure 1. These appear smooth and strictly increasing, in contrast to the step-like behavior observed in CTRWs with diverging characteristic waiting times.

Let us show that all processes $S(t)$, $S^{-1}(t)$ and $X(t)$ are self-similar. Recall that Brownian motion is 1/2-self-similar. Consequently, for $a, t > 0$ we have

$$S(at) = \int_0^{at} |B_2(u)|\,\mathrm{d}u = a\int_0^{t} |B_2(au)|\,\mathrm{d}u \stackrel{d}{=} a^{3/2}\int_0^{t} |B_2(u)|\,\mathrm{d}u = a^{3/2}S(t).$$

Thus, $S(t)$ is 3/2-self-similar. Moreover

$$\begin{aligned}\Pr[S^{-1}(at) \le \tau] &= \Pr[at \le S(\tau)] = \Pr[t \le S(\tau)/a] \\ &= \Pr[t \le S(\tau a^{-2/3})] = \Pr[S^{-1}(t) \le \tau a^{-2/3}] = \Pr[a^{2/3}S^{-1}(t) \le \tau].\end{aligned}$$

Therefore, $S^{-1}(at) \stackrel{d}{=} a^{2/3}S^{-1}(t)$, so $S^{-1}(t)$ is 2/3-self-similar. Finally, multiplying self-similarity exponents of $B_1(t)$ and $S^{-1}(t)$, we find that $X(t)$ is 1/3-self-similar. In figure 2 we show typical trajectories of $X(t)$ with the corresponding nine quantile lines. Recall that a p-quantile line, $p \in (0, 1)$, for a stochastic process $X(t)$ is a function $q_p(t)$ given by the relationship $\Pr(X(t) \le q_p(t)) = p$. Quantile lines visualize nicely the evolution in time of the process $X(t)$.

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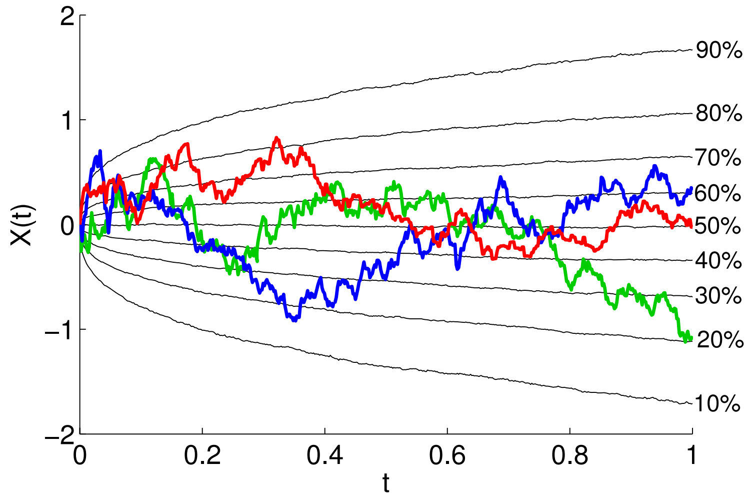

Figure 2. Three trajectories of the process $X(t) = B_1(S^{-1}(t))$ (red, green, blue lines) and the estimated nine quantile lines of $X(t)$ (thin black lines). Each of the quantile lines has the form $c_i t^{1/3}$, where c_i are the appropriate constants. This confirms that $X(t)$ is 1/3-self-similar.

The above result allows us to analyze the mean-square displacement of $X(t)$. We obtain

$$\mathbb{E}[X^2(t)] = \mathbb{E}[B_1^2(S^{-1}(t))] = \mathbb{E}[S^{-1}(t)] = t^{2/3}\mathbb{E}[S^{-1}(1)].$$

Thus, $X(t)$ displays subdiffusive dynamics with subdiffusion exponent 2/3, which is in perfect agreement with the result derived in [60].

The finiteness of the mean-square displacement of $X(t)$, or equivalently, of the expected value $\mathbb{E}[S^{-1}(1)]$ is by no means obvious. It can be proved in the following way:

$$\mathbb{E}[S^{-1}(1)] = \int_0^\infty \Pr[S^{-1}(1) > u]\,\mathrm{d}u = \int_0^\infty \Pr[S(1) < u^{-3/2}]\,\mathrm{d}u$$
$$\leq \int_0^\infty \Pr\left[\left|\int_0^1 B_2(y)\,\mathrm{d}y\right| < u^{-3/2}\right]\,\mathrm{d}u < \infty.$$

Finiteness of the last integral follows from the fact that $\Pr[|\int_0^1 B_2(y)\,\mathrm{d}y| < u^{-3/2}] \leq \min\{1, cu^{-3/2}\}$ for appropriate constant $c > 0$.

It is also useful to derive an explicit integral representation of the probability density function (pdf) of the process $X(t) = B_1(S^{-1}(t))$. Denote the pdf of $X(t)$ by $p(x,t)$. Then, by the total probability formula, we have

$$p(x,t) = \int_0^\infty f(x,\tau)g(\tau,t)\,\mathrm{d}\tau.$$

Here, $f(x,\tau) = (1/\sqrt{2\pi\tau})\exp(-x^2/2\tau)$ is the pdf of $B_1(\tau)$, and $g(\tau,t)$ is the pdf of $S^{-1}(t)$. Moreover, using the self-similarity property, we see that

$$g(\tau,t) = \frac{3/2t}{\tau^{5/2}}h\left(\frac{t}{\tau^{3/2}}\right),$$

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where $h(x)$ is the pdf of $S(1)$. The last density can be represented as [62]

$$h(x) = \frac{\sqrt{3}}{x\sqrt{\pi}} \sum_{j=1}^{\infty} C_j \mathrm{e}^{-v_j} v_j^{2/3} U(1/6, 4/3, v_j).$$

Here

$$U(1/6, 4/3, x) = \frac{1}{\Gamma(1/6)} \int_0^{\infty} \mathrm{e}^{-tx} t^{-5/6} (1+t)^{1/6} \, \mathrm{d}t$$

is a confluent hypergeometric function, and

$$C_j = \frac{1 + 3\int_0^{a'_j} Ai(-u) \, \mathrm{d}u}{3a'_j Ai(-a'_j)},$$

where

$$Ai(z) = \frac{1}{\pi} \int_0^{\infty} \cos(t^3/3 + tz) \, \mathrm{d}t$$

is the Airy integral. The $z = -a'_j$, $j = 1, 2, 3, \ldots$, are the zeros of $Ai'(z)$ arranged such that $0 < a'_1 < a'_2 < \cdots$, and $v_j = 2(a'_j)^3/(27x^2)$. The above results can be applied to approximate numerically the pdfs $h(x)$ and $p(x, t)$. For further properties of $h(x)$ (Laplace transform, moments, series expansion, details of numerical evaluation, etc), see [62].

2.2. Langevin picture and extension to external forces

Now, let us present the process $X(t) = B_1(S^{-1}(t))$ derived in theorem 1 in the equivalent Langevin form. Using the notation of Fogedby [36], we get the following set of coupled Langevin equations for the position x and time t of the considered process

$$\dot{x}(s) = \Gamma_1(s), \qquad \dot{t}(s) = |y(s)|, \qquad \dot{y}(s) = \Gamma_2(s). \tag{6}$$

Here, $\Gamma_1(s)$ and $\Gamma_2(s)$ are two independent white noises corresponding to Brownian motions B_1 and B_2, respectively (formally $\Gamma_i(s) = \mathrm{d}B_i(s)/\mathrm{d}s$, $i = 1, 2$).

The first equation in (6) is interpreted as the usual Langevin equation in the operational time s. The relationship between the operational time s and the physical time t is described by the second and third equation.

To solve the system of equations (6), one first solves the first equation and obtains the driving process $x(s)$. Here s is the operational time of the system. Next, one needs to solve the second and third equation to obtain the process $t(s)$. Consequently, one finds the process $s(t)$, which is inverse to $t(s)$, cf figure 1. Finally, in the last step, one assembles both processes $x(s)$ and $s(t)$ to obtain the solution of (6), which has the subordination form $X(t) = x(s(t))$.

To approximate numerically the trajectories of the solution of (6), it is enough to simulate two independent Gaussian white noises Γ_1 and Γ_2, which is a well known procedure, see [14] for the details. The typical trajectories of the solution of (6) are shown in figure 2.

The above-derived Langevin picture of CTRW with correlated waiting times can be easily extended to include an external force $F(x)$ acting on the particle. This is particularly important from the point of view of applicability of the considered model,

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since many real-life phenomena take place under the influence of external forces. The set of coupled Langevin equations (6) in the presence of the force $F(x)$ takes on the form

$$\dot{x}(s) = F(x(s)) + \Gamma_1(s), \qquad \dot{t}(s) = |y(s)|, \qquad \dot{y}(s) = \Gamma_2(s). \tag{7}$$

The interpretation of the above system as well as the method of finding its solution is analogous to the force-free case (6). Equivalently, the solution to the above set of equations can be written in the form of subordination

$$X(t) = x(S^{-1}(t)), \tag{8}$$

where $x(s)$ satisfies the following stochastic differential equation

$$\mathrm{d}x(s) = F(x(s))\,\mathrm{d}s + dB_1(s)$$

and $S^{-1}(t)$ is given by (5). For the details of simulating subordinated Langevin equations, see [40], [42]–[44].

An immediate consequence of equation (8) is that the stationary solution of (7) has the Gibbs–Boltzmann distribution $p(x) = c\exp(-2V(x))$, where $V(x)$ is the potential $(\mathrm{d}V(x)/\mathrm{d}x = -F(x))$, and $c > 0$ is the normalizing constant. This follows from the fact that the processes x and S^{-1} are independent and that $S(t) \to \infty$ as $t \to \infty$.

The detailed study of the correlated CTRW in the presence of external forces will be the subject of a forthcoming paper.

2.3. Heavy-tailed case

Let us now consider an extension of the correlated CTRW to the heavy-tailed Lévy-stable case. We assume that the jumps J_i of the walker are iid α-stable random variables with Fourier transform $\mathbb{E}[\exp(\mathrm{i}zJ_i)] = \exp(-\frac{1}{2}|z|^\alpha)$, $0 < \alpha \le 2$. Moreover, the correlated waiting times T_i are given by

$$T_i = |\xi_1 + \cdots + \xi_i|, \tag{9}$$

where ξ_j are iid β-stable random variables with Fourier transform $\mathbb{E}[\exp(\mathrm{i}z\xi_j)] = \exp(-\frac{1}{2}|z|^\beta)$, $0 < \beta \le 2$. As before the ξ_i are symmetric such that the value of T_i may increase or decrease in a given step. Note that for $\alpha = \beta = 2$ we recover the CTRW analyzed in previous sections.

In the next theorem, we derive the scaling limit for the stable case.

Theorem 2. *Let J_i, $i = 1, 2, \ldots$, be the sequence of iid α-stable random variables with $\mathbb{E}[\exp(\mathrm{i}zJ_i)] = \exp(-\frac{1}{2}|z|^\alpha)$. Let T_i, $i = 1, 2, \ldots$, be defined by (9) and independent of J_i, $i = 1, 2, \ldots$. Then, the corresponding CTRW process $R(t)$ satisfies*

$$\frac{R(nt)}{n^{\beta/\alpha(\beta+1)}} \xrightarrow{d} X(t) = L_\alpha(S_\beta^{-1}(t))$$

as $n \to \infty$. Here

$$S_\beta(t) = \int_0^t |L_\beta(u)|\,\mathrm{d}u, \tag{10}$$

whereas $L_\alpha(t)$ and $L_\beta(t)$ are two independent Lévy-stable motions with Fourier transforms $\mathbb{E}[\exp(\mathrm{i}zL_\alpha(t))] = \exp(-\frac{1}{2}t|z|^\alpha)$, $\mathbb{E}[\exp(\mathrm{i}zL_\beta(t))] = \exp(-\frac{1}{2}t|z|^\beta)$, respectively. Moreover

$S_\beta^{-1}(t)$ *is the inverse of* $S_\beta(t)$*, i.e.*

$$S_\beta^{-1}(t) = \inf\{\tau > 0 : S_\beta(\tau) > t\}. \tag{11}$$

Proof. See appendix A. □

The interpretation of the above result is analogous to the Brownian case. The Lévy flight process $L_\alpha(t)$ appears here as a limit of the cumulated and scaled heavy-tailed jumps J_i. Moreover, the cumulated and scaled sequence of waiting times (9) converges to the process $S_\beta(t)$. Thus, the inverse process $S_\beta^{-1}(t)$ is the scaling limit of the corresponding counting process $N(t)$. Finally, the scaling limit of the correlated CTRW is obtained as a superposition of $L_\alpha(t)$ and $S_\beta^{-1}(t)$ and has the form $X(t) = L_\alpha(S_\beta^{-1}(t))$.

Similarly as before, one can show that $S_\beta(t)$ is $(\beta+1)/\beta$-self-similar, $S_\beta^{-1}(t)$ is $\beta/(\beta+1)$-self-similar, whereas $X(t) = L_\alpha(S_\beta^{-1}(t))$ is $\beta/\alpha(\beta+1)$-self-similar. The second moment of the process $X(t)$ is finite only if $\alpha = 2$. Then, it is equal to

$$\mathbb{E}[X^2(t)] = t^{\beta/(\beta+1)}\mathbb{E}[S_\beta^{-1}(1)].$$

Thus, $X(t)$ displays subdiffusive behavior with exponent $\beta/\beta+1$, which is in agreement with the result derived in [60]. Interestingly this exponent ranges exclusively between zero and 2/3. Time-correlated CTRW can therefore not span the range between 2/3 and 1.

The Langevin equations corresponding to $X(t)$ in the presence of external force $F(x)$ take the form

$$\dot{x}(s) = F(x(s)) + \Gamma_\alpha(s), \qquad \dot{t}(s) = |y(s)|, \qquad \dot{y}(s) = \Gamma_\beta(s). \tag{12}$$

Here, $\Gamma_\alpha(s)$ and $\Gamma_\beta(s)$ are two independent stable noises corresponding to L_α and L_β, respectively (formally $\Gamma_\alpha(s) = \mathrm{d}L_\alpha(s)/\mathrm{d}s$, $\Gamma_\beta(s) = \mathrm{d}L_\beta(s)/\mathrm{d}s$). Methods of solving and simulating the above system of equations are analogous to the previously analyzed Brownian case.

An equivalent representation of (12) in the form of subordination reads

$$X(t) = x(S_\beta^{-1}(t)), \tag{13}$$

where $x(s)$ satisfies the stochastic differential equation

$$\mathrm{d}x(s) = F(x(s))\,\mathrm{d}s + \mathrm{d}L_\alpha(s)$$

and $S_\beta^{-1}(t)$ is given by (11).

3. CTRW with correlated jumps

Let us now consider a CTRW process in which the jump lengths are correlated. More precisely, each jump J_i is equal to

$$J_i = \phi_1 + \cdots + \phi_i, \tag{14}$$

where ϕ_j are iid symmetric with finite second moment (we assume for simplicity that the second moment is equal to 1). Moreover, we assume that each waiting time T_i is equal to 1. This kind of CTRW was first introduced in [60]. The next theorem establishes its continuous-time limit.

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Theorem 3. *Let J_i, $i = 1, 2, \ldots$, be defined by (14) and T_i, $i = 1, 2, \ldots$, all equal to 1. Then, the corresponding CTRW process $R(t)$ satisfies*

$$\frac{R(nt)}{n^{3/2}} \xrightarrow{d} Z(t)$$

as $n \to \infty$ (at t fixed). Here

$$Z(t) = \int_0^t B(u)\,\mathrm{d}u \tag{15}$$

and $B(t)$ is the standard Brownian motion.

Proof. See appendix B. □

It follows immediately that for fixed $t > 0$, the random variable $Z(t)$ is normally distributed, $Z(t) \sim N(0, t^3/3)$. The factor 1/3 in the variance of $Z(t)$ is the consequence of integration. Thus, the mean-square displacement of $Z(t)$ equals

$$\mathbb{E}[Z^2(t)] = t^3/3.$$

This shows that $Z(t)$ is super-diffusive, which is in agreement with the result obtained for the underlying CTRW in [60].

The process $Z(t)$ is called the random acceleration process in the physical literature and has been studied by many authors, see [63] for a review on this subject.

The results of theorems 1 and 3 unveil a link between correlated CTRWs and the celebrated Feynman–Kac formula. Namely, the scaling limits derived in these theorems lead to so-called Brownian functionals. These functionals have the general form

$$Q(t) = \int_0^t f(B(u))\,\mathrm{d}u, \tag{16}$$

where $B(u)$ is a standard Brownian motion and $f(x)$ is a specified deterministic function. In the case of the process $S(t)$ derived in (4) we have $f(x) = |x|$, whereas for $Z(t)$ in (15) we have $f(x) = x$. Thus, both processes $S(t)$ and $Z(t)$ are particular examples of Brownian functionals. It was Kac [64] who discovered that many properties of Brownian functionals can be studied well by using the path integral formalism devised by Feynman in his PhD thesis. This observation led to the discovery of the Feynman–Kac formula. Since the result of Kac, Brownian functionals have become an important mathematical tool in diverse fields ranging from probability [64, 65] to finance [66] and physics [67]. For some recent results concerning the fractional Feynman–Kac formula for non-Brownian functionals, see [68]–[70].

The above observation implies that processes $S(t)$ and $Z(t)$, which appear in a natural way in the context of correlated CTRWs, can be successfully analyzed in the framework of the Feynman–Kac formalism. In particular, if we denote by $p(q, t|x_0)$ the pdf of $Q(t)$ with condition $B(0) = x_0$, then its Laplace transform $\hat{p}(k, t, x_0) = \mathbb{E}_{x_0}[\mathrm{e}^{-kQ(t)}]$ satisfies the equation

$$\frac{\partial \widehat{p}}{\partial t} = \frac{1}{2}\frac{\partial^2 \widehat{p}}{\partial x_0^2} - kf(x_0)\widehat{p}.$$

For further applications of the Feynman–Kac formula in the analysis of Brownian functionals, see [67].

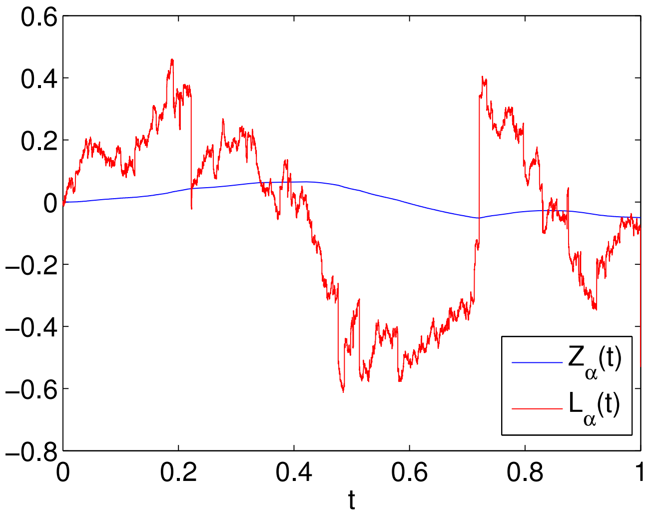

Figure 3. Sample paths of $L_\alpha(t)$ and the corresponding $Z_\alpha(t) = \int_0^t L_\alpha(u)\,\mathrm{d}u$ with $\alpha = 1.5$. Although both processes have α-stable marginal distributions, the trajectories are very different. The sample path of $Z_\alpha(t)$ has no jumps. This is due to the smoothing effects of the integral $\int_0^t L_\alpha(u)\,\mathrm{d}u$.

Another useful observation is that the Brownian functional $Q(t)$ (and, in particular, the processes $S(t)$ and $Z(t)$) can also be studied in the framework of diffusion processes. Although $Q(t)$ is not a diffusion process (it is not even Markovian), the two-dimensional process $(Q(t), B(t))$ belongs to the family of diffusion processes. It follows from the fact that $(Q(t), B(t))$ can be written in the form of stochastic differential equations,

$$\mathrm{d}Q(t) = f(B(t))\,\mathrm{d}t \qquad \mathrm{d}B(t) = \Gamma(t)\,\mathrm{d}t,$$

where $\Gamma(t)$ is the standard white noise. This in turn implies that the pdf $n(q, b, t)$ of $(Q(t), B(t))$ satisfies the Fokker–Planck equation

$$\frac{\partial n}{\partial t} = -f(b)\frac{\partial n}{\partial q} + \frac{1}{2}\frac{\partial^2 n}{\partial b^2}.$$

A more detailed analysis of correlated CTRWs via the Feynman–Kac or diffusion approach is the subject of our further studies.

Analogous considerations can be performed for the heavy-tailed case, i.e. when the random variables ϕ_i in (14) are iid α-stable with Fourier transform $\mathbb{E}[\exp(\mathrm{i}z\phi_j)] = \exp(-\frac{1}{2}|z|^\alpha)$. Then, the scaling limit of the corresponding CTRW has the form

$$Z_\alpha(t) = \int_0^t L_\alpha(u)\,\mathrm{d}u.$$

Here, $L_\alpha(t)$ is the α-stable Lévy motion with $\mathbb{E}[\exp(\mathrm{i}zL_\alpha(t))] = \exp(-\frac{1}{2}t|z|^\alpha)$. Proof of this result is analogous to the proof of theorem 3. It is easy to verify that $Z_\alpha(t)$ has α-stable marginal distributions. The typical trajectories of $L_\alpha(t)$ and $Z_\alpha(t)$ are shown in figure 3.

The process $Z_\alpha(t)$ is the generalization of the random acceleration process $Z(t)$. In fact it reduces to $Z(t)$ for $\alpha = 2$. Clearly, $Z_\alpha(t)$ describes the area under the α-stable Lévy process $L_\alpha(t)$. This process (and related, more complicated Lévy bridges) were recently studied in [71].

4. Conclusions

In this paper we have presented a detailed study of the asymptotic behavior of correlated CTRWs. We have derived the corresponding scaling limits, which allowed us to find the Langevin equations describing correlated dynamics. Two cases of correlated waiting times and jumps were analyzed in the general heavy-tailed setting. The obtained results allowed us to find some interesting distributional properties of the correlated model and to elucidate its relationship with the Feynman–Kac formula. We believe that the introduced Langevin approach to correlated CTRW is quite intuitive and allows for a natural extension to the case of external forces. Correlated CTRW processes are good candidates for the stochastic modeling of non-renewal processes such as search strategies in movement ecology, human motion patterns, financial market dynamics, as well as various physical processes such as the motion in materials under stress.

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Appendix A.

The structure of this appendix is as follows: first we introduce the necessary notation, then formulate and prove lemma 1 which we use in the proof of theorem 2. Finally we conclude that theorem 1 is a special case of theorem 2.

Let $\mathbb{D}([0,\infty),\mathbb{R})$ be the space of all functions defined on the $[0,\infty)$ and taking values in $\mathbb{R}$ which are right-continuous and have limits from the left. We equip this space with Skorohod $\mathbb{J}_1$ topology (see [61, 72]) and denote it by $(\mathbb{D}([0,\infty),\mathbb{R}),\mathbb{J}_1)$. The subspace of all functions from $\mathbb{D}([0,\infty),\mathbb{R})$ that are nonnegative, nondecreasing and unbounded from above we denote by $\mathbb{D}_{u,\uparrow}$.

Let x^- denote the lcrl (left-continuous with right-hand side limits) version of the rcll (right-continuous with left-hand side limits) function x, y^+ stands for the *rcll* version of the *lcrl* function y and $f\circ g$ is the composition of functions $f\in\mathbb{D}([0,\infty),\mathbb{R})$ and $g\in D_{u,\uparrow}$.

We write $\{X_n(t)\}_{t\geq 0}\xrightarrow{d}\{X(t)\}_{t\geq 0}$, as $n\to\infty$, to denote the weak convergence sequence of stochastic processes X_n to the limiting process X in Skorohod $\mathbb{J}_1$ topology (see [61, 72]), while $X_n(t)\xrightarrow{d}X(t)$, as $n\to\infty$, means the weak convergence of one-dimensional distributions for any fixed $t\geq 0$.

For any $r>0$ we define the functional $f_r:\mathbb{D}([0,\infty),\mathbb{R})\mapsto\mathbb{R}$ as follows

$$f_r(x)\stackrel{df}{=}\int_0^r |x(s)|\,\mathrm{d}s$$

and the mapping $F:\mathbb{D}([0,\infty),\mathbb{R})\mapsto\mathbb{D}([0,\infty),\mathbb{R})$ given by formula

$$F(x)(t)\stackrel{df}{=}\int_0^t |x(s)|\,\mathrm{d}s=f_t(x),\qquad x\in\mathbb{D}\left([0,\infty),\mathbb{R}\right),\qquad t\geq 0.\qquad\text{(A.1)}$$

As a matter of fact, $F(x)(t)$ is a continuous function of $t\in[0,\infty)$ for any $x\in\mathbb{D}([0,\infty),\mathbb{R})$.

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Lemma 1. *Let $x_n, x \in \mathbb{D}([0,\infty),\mathbb{R})$ and assume that $x_n \longrightarrow x$ in $\mathbb{J}_1$ topology. Then for any $r > 0$ the following convergences hold*

$$f_r(x_n) \longrightarrow f_r(x) \qquad \text{as } n \to \infty \tag{A.2}$$

and

$$\sup_{0\le t\le r} |F(x_n)(t) - F(x)(t)| \to 0 \qquad \text{as } n \to \infty. \tag{A.3}$$

Proof of lemma 1. Fix arbitrary $r \ge 0$ and take $z > r$ being a continuity point of x. To simplify notation, for the rest of this proof x_n, x are to mean their restrictions to interval $[0,z]$. Since $x_n \longrightarrow x$ in $\mathbb{J}_1$ topology it follows that there exists sequence a sequence λ_n of continuous and strictly increasing functions mapping $[0,z]$ onto $[0,z]$ such that

$$\sup_{0\le t\le z} |\lambda_n(t) - t| \to 0 \tag{A.4}$$

and

$$\sup_{0\le t\le z} |x_n(t) - x(\lambda_n(t))| \to 0. \tag{A.5}$$

Observe that

$$\begin{aligned}\sup_{0\le t\le z} |x_n(t) - x(t)| &\le \sup_{0\le t\le z} |x_n(t) - x(\lambda_n(t))| + \sup_{0\le t\le z} |x(\lambda_n(t)) - x(t)| \\ &\le \sup_{0\le t\le z} |x_n(t) - x(\lambda_n(t))| + 2\sup_{0\le t\le z} |x(t)|.\end{aligned}$$

By (A.5) and the fact that x is bounded on interval $[0,z]$, it follows that there exists constant $M > 2\sup_{0\le t\le z}|x(t)|$ such that for all $n \ge 1$

$$\sup_{0\le t\le z} |x_n(t) - x(t)| \le M. \tag{A.6}$$

The obvious inequality $||a| - |b|| \le |a-b|$, $a, b \in \mathbb{R}$, yields that

$$\begin{aligned}|f_r(x_n) - f_r(x)| &= \left| \int_0^r |x_n(s)|\,\mathrm{d}s - \int_0^r |x(s)|\,\mathrm{d}s \right| \le \int_0^r |x_n(s) - x(s)|\,\mathrm{d}s \\ &\le \int_0^z |x_n(s) - x(s)|\,\mathrm{d}s.\end{aligned} \tag{A.7}$$

Convergence $x_n \to x$ in $(D([0,z],\mathbb{R}),\mathbb{J}_1)$ implies that $x_n(t) \to x(t)$ for all t being continuity points of x. Hence $|x_n(t) - x(t)| \to g(t) = 0$ for all t being continuity points of x. Since the set of discontinuity points of x is countable, its Lebesgue measure is zero. Thus (A.6) and Lebesgue dominated convergence theorem imply that

$$\lim_{n\to\infty} \int_0^z |x_n(s) - x(s)|\,\mathrm{d}s = \int_0^z \lim_{n\to\infty} |x_n(s) - x(s)|\,\mathrm{d}s = \int_0^z 0\,\mathrm{d}s = 0. \tag{A.8}$$

Then convergence (A.2) follows from (A.8) and (A.8).

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To show convergence (A.3) observe that

$$\sup_{0\le t\le r}|F(x_n)(t)-F(x)(t)| = \sup_{0\le t\le r}\left|\int_0^t |x_n(s)| - \int_0^t |x(s)|\,\mathrm{d}s\right|$$
$$= \sup_{0\le t\le r}\left|\int_0^t (|x_n(s)|-|x(s)|)\,\mathrm{d}s\right| \le \sup_{0\le t\le r}\int_0^t ||x_n(s)|-|x(s)||\,\mathrm{d}s$$
$$\le \sup_{0\le t\le r}\int_0^t |x_n(s)-x(s)|\,\mathrm{d}s \le \int_0^z |x_n(s)-x(s)|\,\mathrm{d}s.$$

Using (A.8) to the above we get convergence (A.3), which completes the proof of the lemma. □

Proof of theorem 2. Define sequences of processes

$$S_n(t) \overset{d}{=} n^{-1/\alpha}\sum_{i=1}^{[nt]} J_i, \qquad t\ge 0, \qquad \Xi_n(t) \overset{d}{=} n^{-1/\beta}\sum_{i=1}^{[nt]} \xi_i, \qquad t\ge 0,$$

and

$$W_n(t) \overset{d}{=} n^{-(1+1/\beta)}\sum_{i=1}^{[nt]} T_i, \qquad t\ge 0,$$

where $[a]$ denotes the integer part of $a\ge 0$. To clarify any ambiguities in the notation we note that processes $S_n(t)$ and $S_\beta(t)$ are not related. The assumptions of theorem 2. together with theorem 4.1 [47] yield joint convergence

$$\{(S_n(t),\Xi_n(t))\}_{t\ge 0} \overset{d}{\longrightarrow} \{(L_\alpha(t),L_\beta(t))\}_{t\ge 0}. \tag{A.9}$$

Observe that for any fixed $n\ge 1$ and any $t\ge 0$ we have that

$$W_n(t) = \frac{1}{n}\sum_{i=1}^{[nt]} n^{-1/\beta}T_i = \frac{1}{n}\sum_{i=1}^{[nt]}\left|n^{-1/\beta}\sum_{j=1}^{i}\xi_j\right| = \frac{1}{n}\sum_{q\in Q}\left|n^{-1/\beta}\sum_{j=1}^{nq}\xi_j\right| = \frac{1}{n}\sum_{q\in Q}|\Xi_n(q)|,$$

where $Q=\{q=i/n, i=1,2,\ldots,[nt]\}$. Since the trajectories of Ξ_n are the step functions, it follows that

$$W_n(t) = \frac{1}{n}\sum_{q\in Q}|\Xi_n(q)| = \int_0^t |\Xi_n(s)|\,\mathrm{d}s = F(\Xi_n)(t),$$

where F is defined by (A.1). Note also that $S_\beta(t)=F(L_\beta)(t)$ for any $t\ge 0$. Since uniform topology is finer than Skorohod $\mathbb{J}_1$ topology (see [61] p 150), by lemma 1. it follows that mapping F is $\mathbb{J}_1$-continuous. Then (A.9) and theorem 5.1 [61] imply joint convergence

$$\{(S_n(t),W_n(t))\}_{t\ge 0} \overset{d}{\longrightarrow} \{(L_\alpha(t),S_\beta(t))\}_{t\ge 0}. \tag{A.10}$$

Next we define the auxiliary sequence of correlated CTRWs

$$X_n(t) \overset{df}{=} n^{-1/\alpha}\sum_{i=1}^{N(n^{(1+1/\beta)}t)} J_i = S_n\left(\frac{N(n^{(1+1/\beta)}t)}{n}\right), \qquad t\ge 0.$$

Observe that

$$W_n^{-1}(t) = \inf\{s > 0 : W_n(s) > t\} = \inf\left\{s > 0 : \sum_{i=1}^{[ns]} T_i > n^{-(1+1/\beta)}t\right\}.$$

By definition of the counting process $N(t)$ (see (1)) we have that

$$nW_n^{-1}(t) = N(n^{-(1+1/\beta)}t) + 1$$

or equivalently

$$W_n^{-1}(t) = \frac{1}{n}N(n^{-(1+1/\beta)}t) + \frac{1}{n}.$$

Using lemma 3.5 [50] one easily checks that

$$X_n(t) = S_n(W_n^{-1}(t) - 1/n) = \left(S_n{}^{-} \circ (W_n^{-1})^{-}\right)^{+}(t).$$

Define the mapping $\Phi : \mathbb{D}([0,\infty),\mathbb{R}) \times \mathbb{D}_{u,\uparrow} \longmapsto \mathbb{D}([0,\infty),\mathbb{R})$ as follows

$$\Phi(x,y)(t) \stackrel{df}{=} (x^{-} \circ (y^{-1})^{-})^{+}(t).$$

By proposition 2.3 [50] mapping Φ is continuous in $\mathbb{J}_1$ topology at points (x,y), such that y is a strictly increasing function. Convergence (A.10) together with theorem 5.1 [61] yields that

$$\{X_n(t)\}_{t\geq 0} = \{\Phi(S_n, W_n)(t)\}_{t\geq 0} \xrightarrow{d} \{\Phi(L_\alpha, S_\beta)(t)\}_{t\geq 0} \equiv \{X(t)\}_{t\geq 0}.$$

Sample paths of $S_\beta(t)$ are continuous and strictly increasing and so are the trajectories of $S_\beta^{-1}(t)$. Thus

$$X(t) = \Phi(L_\alpha, S_\beta)(t) = \left(L_\alpha^{-} \circ (S_\beta^{-1})^{-}\right)^{+}(t) = L_\alpha(S_\beta^{-1}(t)).$$

Finally we show that the sequence of auxiliary CTRWs $X_n(t)$ and the sequence of scaled CTRWs $R(nt)/(n^{\beta/(\alpha(\beta+1))})$ both converge to the same limit process. Indeed

$$\frac{R(nt)}{n^{\beta/(\alpha(\beta+1))}} = (n^{\beta/(\beta+1)})^{-1/\alpha} \sum_{i=1}^{N(nt)} J_i = (n^{\beta/(\beta+1)})^{-1/\alpha} \sum_{i=1}^{N((n^{\beta/(\beta+1)})^{(1+1/\beta)}t)} J_i = X_{(n^{\beta/(\beta+1)})}(t)$$

and since $(n^{\beta/(\beta+1)}) \to \infty$ as $n \to \infty$, we have that both sequences converge to the same limit process. Hence

$$\{X_n(t)\}_{t\geq 0} \xrightarrow{d} \{L_\alpha(S_\beta^{-1}(t))\}_{t\geq 0}$$

which completes the proof of theorem 2. □

We conclude appendix A with the proof of theorem 1.

Proof of theorem 1. By the assumptions of theorem 1 we have that

$$\{(S_n(t), \Xi_n(t))\}_{t\geq 0} \stackrel{df}{=} \left\{\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} J_i, \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} \xi_i\right)\right\}_{t\geq 0} \xrightarrow{d} \{(B_1(t), B_2(t))\}_{t\geq 0}.$$

Brownian motions $B_1(t)$ and $B_2(t)$ are two-stable processes, thus the proof of theorem 1 follows exactly as the proof of theorem 2 with $\alpha = \beta = 2$, $L_\alpha(t) = B_1(t)$ and $L_\beta(t) = B_2(t)$. □

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Appendix B.

Define the mapping $G : \mathbb{D}([0,\infty),\mathbb{R}) \mapsto \mathbb{D}([0,\infty),\mathbb{R})$ as follows

$$G(x)(t) \stackrel{df}{=} \int_0^t x(s)\,\mathrm{d}s, \qquad t \ge 0. \tag{B.1}$$

Lemma 2. *Let $x_n, x \in \mathbb{D}([0,\infty),\mathbb{R})$ and assume that $x_n \longrightarrow x$ in $\mathbb{J}_1$ topology. Then for any fixed $r > 0$*

$$\sup_{0\le t\le r} |G(x_n)(t) - G(x)(t)| \to 0 \tag{B.2}$$

Proof of lemma 2. Fix arbitrary $r > 0$ and take $z > r$ being a continuity point of x. Let x_n, x stands for their restrictions to interval $[0, z]$ and let mappings λ_n be as in the proof of lemma 1. Then

$$\sup_{0\le t\le r} |G(x_n)(t) - G(x)(t)| = \sup_{0\le t\le r} \left| \int_0^t x_n(s) - \int_0^t x(s)\,\mathrm{d}s \right|$$
$$\le \sup_{0\le t\le r} \int_0^t |x_n(s) - x(s)|\,\mathrm{d}s \le \int_0^z |x_n(s) - x(s)|\,\mathrm{d}s.$$

Using (A.8) to the above we get convergence (B.2), which completes the proof of the lemma. □

Proof of theorem 3. Define the sequence of processes

$$S_n(t) \stackrel{df}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \phi_i, \qquad t \ge 0.$$

Since $T_i \equiv 1$ it follows that $N(nt) = [nt]$. Then

$$\frac{R(nt)}{n^{3/2}} = \frac{1}{n} \sum_{i=1}^{[nt]} \frac{1}{\sqrt{n}} \sum_{j=1}^{i} \phi_j = \frac{1}{n} \sum_{i=1}^{[nt]} S_n(i/n) = \int_0^t S_n(s)\,\mathrm{d}s = G(S_n)(t).$$

Moreover $Z(t) = G(B)(t)$. Lemma 2 implies $\mathbb{J}_1$-continuity of mapping G. The assumptions of theorem 3 yield convergence

$$\{S_n(t)\}_{t\ge0} \xrightarrow{d} \{B(t)\}_{t\ge0}\,.$$

Thus using 5.1 [61] we obtain convergence

$$\left\{ \frac{R(nt)}{n^{3/2}} \right\}_{t\ge0} = \{G(S_n)(t)\}_{t\ge0} \xrightarrow{d} \{G(B)(t)\}_{t\ge0} = \{Z(t)\}_{t\ge0}$$

and the proof of theorem 3 is complete. □

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