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Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative

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Abstract

In this paper, the solution of a fractional diffusion equation with a Hilfer-generalized Riemann–Liouville time fractional derivative is obtained in terms of Mittag–Leffler-type functions and Fox's H-function. The considered equation represents a quite general extension of the classical diffusion (heat conduction) equation. The methods of separation of variables, Laplace transform, and analysis of the Sturm–Liouville problem are used to solve the fractional diffusion equation defined in a bounded domain. By using the Fourier–Laplace transform method, it is shown that the fundamental solution of the fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative defined in the infinite domain can be expressed via Fox's H-function. It is shown that the corresponding solutions of the diffusion equations with time fractional derivative in the Caputo and Riemann–Liouville sense are special cases of those diffusion equations with the Hilfer-generalized Riemann–Liouville time fractional derivative. The asymptotic behaviour of the solutions are found for large values of the spatial variable. The fractional moments of the fundamental solution of the fractional diffusion equation are obtained. The obtained results are relevant in the context of glass relaxation and aquifer problems.

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1. Introduction

Anomalous diffusion is characterized by deviations from the linear increase with time of the variance of the process. In particular, for anomalous diffusion processes following the law

$$\langle x^2(t)\rangle \simeq K_\mu t^\mu, \tag{1}$$

with the generalized diffusion constant K_μ and the anomalous diffusion exponent μ, we distinguish the cases of subdiffusion ($0 < \mu < 1$) and superdiffusion ($1 < \mu$) [1, 2]. Patterns of the form (1) have been observed in systems as different as tracer dispersion in aquifers [3], bacteria cells moving in biofilms [4], diffusion of lipid granules and biopolymers in living cells [5–8], diffusion in critical percolation networks [9], and charge carrier motion in amorphous semiconductors [10]. Anomalous diffusion is often associated with interesting physical effects such as the inequivalence of time versus ensemble averages [5, 11].

Continuous time random walks (CTRWs) readily generalize the standard random walk concept to describe anomalous diffusion of the form (1) [10]. The associated dynamic equation is the fractional diffusion equation [1, 2, 12, 13]. To see this, consider a subdiffusive CTRW, a process characterized by a distribution of jump lengths with finite variance $\langle\delta x^2\rangle$ and broad distribution of waiting times of the form $\psi(\tau) \simeq (\tau^*)^\mu/\tau^{1+\mu}$ with $0 < \mu < 1$. Consequently, the characteristic waiting time $\int_0^\infty \tau\psi(\tau)\,d\tau$ diverges, and the distribution $\psi(\tau)$ is scale-free. According to CTRW theory [14], this process leads to the form (1) of the mean squared displacement, and the probability density $f(x, t)$ fulfils

$$uf(k, u) - 1 = -K_\mu u^{-\mu}k^2 f(k, u) \tag{2}$$

in Fourier–Laplace space (see below for details). Here we identified the anomalous diffusion constant $K_\mu = \langle\delta x^2\rangle/[2\tau^\mu]$. Fourier and Laplace inversion led to the equivalent, alternative formulations

$$_{RL}D^\mu_{0+}f(x,t) - \frac{\delta(x)}{t^\mu\Gamma(1-\mu)} = K_\mu\frac{\partial^2}{\partial x^2}f(x,t) \tag{3a}$$

$$_{C}D^\mu_{0+}f(x,t) = K_\mu\frac{\partial^2}{\partial x^2}f(x,t) \tag{3b}$$

in the Riemann–Liouville (R–L) and Caputo sense, respectively (compare definitions (7) and (8) below). While in the R–L formulation the initial condition $f(x, t = 0+) = \delta(x)$ is directly incorporated in the dynamic equation, the analogous Caputo version appears closer to the structure of the normal diffusion equation ($\mu = 1$).

Several authors have investigated fractional dynamic equations generalizing the diffusion or wave equations in terms of R–L or Caputo time fractional derivatives, and their fundamental solutions have been represented in terms of the Mittag–Leffler (M–L) functions and their generalizations [13, 15–20]. Similar diffusion-wave equations with the R–L and Caputo time fractional derivatives are considered in [21–28]. A detailed analysis and methods of solving different types of fractional diffusion equations, similar to those considered in this work, may be found in the review articles [1, 2]. Such models are used for the description of the transport dynamics in complex systems. Generalized transport equations of such types are related to the generalized Chapman–Kolmogorov equation discussed by Metzler [29].

Fractional calculus has indeed been studied by a range of celebrated mathematicians and physicists. To name but a few, we mention Leibniz, Euler, Laplace, Lacroix, Fourier, Abel, Liouville, Riemann, Letnikov, etc. Abel in 1823 studied the generalized tautochrone problem and for the first time applied fractional calculus techniques in a physical problem. Later Liouville applied fractional calculus to problems in potential theory. Nowadays fractional calculus receives increasing attention in the scientific community, with a growing number of applications in physics, electrochemistry, biophysics, viscoelasticity, biomedicine, control theory, signal processing, etc [16, 30–37].

At the beginning of the 20th century, the Swedish mathematician Gösta Mittag–Leffler [38] introduced a generalization of the exponential function, today known as the Mittag–Leffler function. The properties of the M–L function and the generalizations by Wiman [39], Agarwal [40], Humbert [41], and Humbert and Agarwal [42] had been totally ignored by the scientific community for a considerable time due to their unknown application in the science. They appear as solutions of differential and integral equations of fractional order. Thus, in 1930 Hille and Tamarkin [43] solved the Abel–Volterra integral equation in terms of the M–L function. The basic properties and relations of the M–L function appeared in the third volume of the Bateman project [44]. A more detailed analysis of the M–L function and their generalizations as well as fractional derivatives and integrals were published later [45–53]. M–L functions are of great interest for modelling anomalous diffusive processes [1, 2, 50, 54–59].

Similarly, Fox's H-function, introduced by Charles Fox [60], is of great importance in solving fractional differential equations and to analyse anomalous diffusion processes [1, 2, 56]. For example, Mainardi *et al* [56] expressed the fundamental solution of the Cauchy problem for the fractional diffusion equation in terms of H-functions, based on their Mellin–Barnes integral representations. A detailed study of these functions as symmetrical Fourier kernels was reported by Srivastava *et al* [61].

Here we consider a fractional diffusion equation with a generalized time fractional differential operator recently derived by Hilfer [50]. We present explicit solutions in both confined and unconfined space. Moreover, fractional moments are derived. The paper is organized as follows. Some generalized differential and integral operators are considered in section 2. In section 3, the exact solution of the generalized fractional diffusion equations in a bounded domain is obtained in terms of M–L functions. The method of separation of variables and the Laplace transform method are applied to solve the equation analytically. In section 4, an infinite domain is considered. The Fourier–Laplace transform method is used to solve the equation analytically, finding exact solutions in terms of H-functions in some special cases. The asymptotic behaviour of the solution is derived, and fractional moments of the fundamental solution obtained. In section 5, a fractional diffusion equation with a singular term is considered. The conclusions are presented in section 5. In the appendix, some properties of the M–L and H-functions are presented.

2. Generalized differential and integral operators

The right-sided R–L fractional integral is defined by [48, 50, 51]

$$\left(I_{a+}^{\mu}f\right)(t)=\frac{1}{\Gamma(\mu)}\int_a^t\frac{f(\tau)}{(t-\tau)^{1-\mu}}\,\mathrm{d}\tau,\qquad t>a,\qquad \Re(\mu)>0. \tag{4}$$

For $\mu=0$, this is the identity operator, $\left(I_{a+}^{0}f\right)(t)=f(t)$. Similarly, the right-sided R–L fractional derivative is defined by [48, 50, 51]

$$\left(D_{a+}^{\mu}f\right)(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\left(I_{a+}^{n-\mu}f\right)(t),\qquad \Re(\mu)>0,\qquad n=[\Re(\mu)]+1, \tag{5}$$

where $[\Re(\mu)]$ denotes the integer part of the real number $\Re(\mu)$. Hilfer generalized the fractional derivative (5) by the following right-sided fractional derivative of order $0 < \mu < 1$ and type $0 \leqslant \nu \leqslant 1$ [50]:

$$\left(D_{a+}^{\mu,\nu} f\right)(t) = \left(I_{a+}^{\nu(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}t}\left(I_{a+}^{(1-\nu)(1-\mu)} f\right)\right)(t). \tag{6}$$

Note that when $0 < \mu < 1$, $\nu = 0$, $a = 0$, the generalized R–L fractional derivative (6) would correspond to the classical R–L fractional derivative [46, 47]

$$\left({}_{RL}D_{0+}^{\mu} f\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(I_{0+}^{(1-\mu)} f\right)(t). \tag{7}$$

Conversely, when $0 < \mu < 1$, $\nu = 1$, $a = 0$, it would reduce to the Caputo fractional derivative [62]

$$\left({}_{C}D_{0+}^{\mu} f\right)(t) = \left(I_{0+}^{(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}t} f\right)(t). \tag{8}$$

The generalized fractional derivative (6) was recently called the Hilfer fractional derivative by Mainardi and Gorenflo [63] (see also [53]). This operator appeared in the theoretical modelling of dielectric relaxation in glass forming materials [64]. The difference between fractional derivatives of different types becomes apparent when we consider their Laplace transform. In [50], it is found for $0 < \mu < 1$ that

$$\mathcal{L}\left[D_{0+}^{\mu,\nu} f(t)\right] = s^{\mu}\mathcal{L}[f(t)] - s^{\nu(\mu-1)}\left(I_{0+}^{(1-\nu)(1-\mu)} f\right)(0+), \tag{9}$$

where the initial-value term $\left(I_{0+}^{(1-\nu)(1-\mu)} f\right)(0+)$ is evaluated in the limit $t \to 0+$, in the space of summable Lebesgue integrable functions

$$L(0,\infty) = \left\{ f : \|f\|_1 = \int_0^{\infty} |f(t)|\mathrm{d}t < \infty \right\}. \tag{10}$$

The Laplace transform of a pure Caputo derivative becomes

$$\mathcal{L}\left[\left({}_{C}D_{0+}^{\alpha} f\right)(t)\right] = s^{\alpha}\mathcal{L}[f(t)] - s^{\alpha-1} f(0+), \quad 0 < \alpha < 1, \tag{11}$$

and thus includes the regular initial value $f(0+)$. This is in contrast with the R–L fractional derivative, for which the Laplace transform

$$\mathcal{L}\left[\left({}_{RL}D_{0+}^{\alpha} f\right)(t)\right] = s^{\alpha}\mathcal{L}[f(t)] - \left(I_{0+}^{(1-\alpha)} f\right)(0+), \quad 0 < \alpha < 1, \tag{12}$$

includes pseudo-initial conditions of fractional order. Both derivatives are equivalent if we consider proper initial conditions:

$$\left({}_{C}D_{0+}^{\alpha} f\right)(t) = \left({}_{RL}D_{0+}^{\alpha} f\right)(t) - f(0+) \cdot \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1, \tag{13}$$

and this ambiguity can be completely avoided in integral formulations of fractional equations [1]. In the light of the composite fractional differential operator (6) and its property (9), the formulation in terms of the composite derivative can be viewed as a convenient abbreviation of complicated initial value terms.

Various operators for fractional integration (involving, for example, kernels with such general classes of functions as the H-function) were investigated systematically by Srivastava and Saxena [65]. Recently, Srivastava and Tomovski [53] introduced an integral operator $\left(\mathcal{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa}\varphi\right)(t)$ defined as

$$\left(\mathcal{E}_{a+;\alpha,\beta}^{\omega;\gamma,\kappa}\varphi\right)(t) = \int_a^t (t-\tau)^{\beta-1} E_{\alpha,\beta}^{\gamma,\kappa}(\omega(t-\tau)^{\alpha})\varphi(\tau)\,\mathrm{d}\tau, \tag{14}$$

where $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ is the generalized four-parameter M–L function (A.8). In the case when $\omega = 0$ the integral operator (14) would correspond to the classical R–L integral operator (4). This operator recently appeared in the expression of the solution of the general time fractional wave equation for a vibrating string [23]. We will see that the generalization (14) will be useful for the solution of the Hilfer-generalized diffusion equations with the fractional derivative (6).

3. Fractional diffusion equation in a bounded domain

The time fractional diffusion equation is obtained from the standard diffusion equation by consistently replacing the first-order time derivative with a given fractional derivative. The main physical purpose adopting and investigating diffusion equations of fractional orders is to describe phenomena of anomalous diffusion, usually met in transport processes through complex and/or disordered media including fractal supports (see, for example, [1, 2, 28, 50, 66]).

In our present investigation, we consider the time fractional diffusion equation

$$D_{0+}^{\mu,\nu}u(x,t) = K_\mu \frac{\partial^2}{\partial x^2}u(x,t) + f(x,t), \quad t > 0, \tag{15}$$

defined in a bounded domain $0 \leqslant x \leqslant l$, with boundary conditions

$$u(x,t)|_{x=0} = h_1(t), \quad u(x,t)|_{x=l} = h_2(t), \tag{16}$$

and initial condition

$$\left(I_{0+}^{(1-\nu)(1-\mu)}u(x,t)\right)(0+) = g(x). \tag{17}$$

Here, $u(x, t)$ represents the probability profile of a given tracer substance, and K_μ denotes the generalized diffusion constant of physical dimension $[K_\mu] = \mathrm{cm}^2\ \mathrm{s}^{-\mu}$. Because of the independence of $[K_\mu]$ on ν we chose the simplified notation with the sole index μ, despite the fact that the numerical value of K_μ depends on the value of ν. The independence of the dimensionality of K_μ of the parameter ν can be directly seen from the dimensional analysis of the composite fractional derivative $D_{0+}^{\mu,\nu}$. Finally, $f(x, t)$ is the density of the sources which transfers the substance into or out of the system as a result of a given reaction (for example, chemical reaction), $D_{0+}^{\mu,\nu}$ is the generalized R–L time fractional derivative (6), and $I_{0+}^{(1-\nu)(1-\mu)}$ is the integral operator (4). Thus, the solution of this problem describes the transition of the solutions of equation (15) in the case of the R–L time fractional derivative ($\nu = 0$) and the Caputo time fractional derivative ($\nu = 1$). Note that equation (15) also represents a heat conduction time fractional differential equation. The proposed equation is a generalization of the classical diffusion equation [67], which can be obtained by using $\mu = 1$ for any value of ν.

A few words concerning equations (15) and (17) are in order. Generalized dynamic equations with the composite fractional derivative or Hilfer-generalized derivative (6) were originally introduced by Hilfer [50]. They arise from fractional time evolutions and in the context of relaxation models lead to versatile solutions that provide excellent description of experimental data over more than ten orders of magnitude, with less parameters than traditional fit functions such as Havriliak–Negami [64, 68]. The seemingly complicated choice (17) for the initial value is for convenience, only. The 'real' initial value is defined by the behaviour of the density $u(x, t)$. Due to the form of the Laplace transform of the fractional R–L derivative, non-integer order derivatives naturally appear. To avoid the complicated notation, we prefer using the function $g(x)$. A similar situation arises in the standard one-parameter fractional diffusion equation (3*a*). Here, the initial value term can be identified as $\left(I_0^{(1-\mu)}u(x,t)\right)(0+) = \delta(x)t^{-\mu}/\Gamma(1-\mu)$. Note that while equation (3*a*) is formulated such that the normalization is conserved, this is generally not the case for the composite-fractional diffusion equation (15), see also section 5. This is legitimate for many problems, in which the dynamic equation (15) would correspond to the real part of the observed quantity, while the imaginary part would lead to damping, for instance, in the description of dielectric or viscoelastic phenomena [64, 69].

Lemma 1. *Let* $0 < \mu < 1$, $0 \leqslant \nu \leqslant 1$ *and* $s, \lambda_n \in R^+$. *Then the following relation holds true:*

$$\mathcal{L}^{-1}\left[\frac{s^{-\nu(1-\mu)}}{s^{\mu}+\lambda_n}\right](t) = t^{-(1-\mu)(1-\nu)} E_{\mu,1-(1-\mu)(1-\nu)}\left(-\lambda_n t^{\mu}\right), \tag{18}$$

where $E_{\mu,1-(1-\mu)(1-\nu)}\left(-\lambda_n t^{\mu}\right)$ *is the two-parameter M–L function (A.2).*

Proof. From relation (A.5), by using $\alpha = \mu$, $\alpha - \beta = -\nu(1-\mu)$ and $a = \lambda_n$, follows the proof of lemma 1. □

Lemma 2. *Let* $0 < \mu < 1$ *and* $s, \lambda_n \in R^+$. *Then the following relation holds true*

$$\mathcal{L}^{-1}\left[\frac{1}{s^{\mu}+\lambda_n}\mathcal{L}\left[\tilde{f}_n(t)\right](s)\right](t) = \left(\mathcal{E}_{0+;\mu,\mu}^{-\lambda_n;1,1}\tilde{f}_n\right)(t), \tag{19}$$

where $\mathcal{E}_{0+;\mu,\mu}^{-\lambda_n;1,1}\tilde{f}_n$ *is the integral operator (14) and* $\tilde{f}_n(t)$ *is a given function.*

Proof. From relation (A.5) it follows that

$$\frac{1}{s^{\mu}+\lambda_n} = \mathcal{L}\left[t^{\mu-1}E_{\mu,\mu}\left(-\lambda_n t^{\mu}\right)\right](s). \tag{20}$$

Thus, by applying the convolution theorem of the Laplace transform one obtains

$$\mathcal{L}^{-1}\left[\frac{1}{s^{\mu}+\lambda_n}\mathcal{L}\left[\tilde{f}_n(t)\right](s)\right](t) = \int_0^t (t-\tau)^{\mu-1}E_{\mu,\mu}\left(-\lambda_n(t-\tau)^{\mu}\right)\tilde{f}_n(\tau)\,\mathrm{d}\tau, \tag{21}$$

from which we obtain the proof of lemma 2. □

Theorem 1. *The time fractional diffusion equation (15) with boundary conditions (16) and an initial condition (17) for* $0 < \mu < 1$, $0 \leqslant \nu \leqslant 1$ *has a summable solution in the space* $L(0,\infty)$ *with respect to t:*

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}\left(\mathcal{E}_{0+;\mu,\mu}^{-\lambda_n;1,1}\tilde{f}_n\right)(t)\sin\left(\frac{n\pi x}{l}\right) + v(x,t), \tag{22}$$

where $x \in [0,l]$,

$$v(x,t) = h_1(t) + \frac{x}{l}\left[h_2(t)-h_1(t)\right], \tag{23}$$

$$a_n(t) = \tilde{c}_n t^{-(1-\mu)(1-\nu)}E_{\mu,1-(1-\mu)(1-\nu)}\left(-K_{\mu}\frac{n^2\pi^2}{l^2}t^{\mu}\right), \tag{24}$$

$$\tilde{c}_n = \frac{2}{l}\int_0^l \tilde{g}(x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x, \tag{25}$$

$$\tilde{f}_n(t) = \frac{2}{l}\int_0^l \tilde{f}(x,t)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x, \tag{26}$$

$$\tilde{f}(x,t) = f(x,t) - D_{0+}^{\mu,\nu}v(x,t) \tag{27}$$

and

$$\tilde{g}(x) = g(x) - \left(I_{0+}^{(1-\nu)(1-\mu)} v(x,t)\right)(0+). \tag{28}$$

Proof. Representing the function $u(x, t)$ in the following way:

$$u(x,t) = U(x,t) + v(x,t), \tag{29}$$

and by the help of the function $v(x, t)$ to satisfy the boundary conditions (16) of equation (15)

$$v(x,t)|_{x=0} = h_1(t), \qquad v(x,t)|_{x=l} = h_2(t), \tag{30}$$

it can be easily obtained that $v(x, t)$ has the form (23). From relations (30) and (29) for the function $U(x, t)$ one finds

$$U(x,t)|_{x=0} = 0, \qquad U(x,t)|_{x=l} = 0. \tag{31}$$

From the initial condition (17) and with relation (29), one obtains

$$\left(I_{0+}^{(1-\nu)(1-\mu)} U(x,t)\right)(0+) = g(x) - \left(I_{0+}^{(1-\nu)(1-\mu)} v(x,t)\right)(0+) = \widetilde{g}(x). \tag{32}$$

Employing

$$U(x,t) = U_1(x,t) + U_2(x,t) \tag{33}$$

from relations (15), (29) and (33), one obtains

$$D_{0+}^{\mu,\nu}\left[U_1(x,t) + U_2(x,t)\right] = \frac{\partial^2}{\partial x^2}\left[U_1(x,t) + U_2(x,t)\right] + \widetilde{f}(x,t), \tag{34}$$

where $\widetilde{f}(x,t)$ is given by (27).

The functions in relation (34) can be separated in the following way:

$$D_{0+}^{\mu,\nu} U_1(x,t) = \frac{\partial^2}{\partial x^2} U_1(x,t), \tag{35}$$

$$U_1(x,t)|_{x=0} = 0, \qquad U_1(x,t)|_{x=l} = 0, \tag{36}$$

$$\left(I_{0+}^{(1-\nu)(1-\mu)} U_1(x,t)\right)(0+) = \widetilde{g}(x) \tag{37}$$

and

$$D_{0+}^{\mu,\nu} U_2(x,t) = \frac{\partial^2}{\partial x^2} U_2(x,t) + \widetilde{f}(x,t), \tag{38}$$

$$U_2(x,t)|_{x=0} = 0, \quad U_2(x,t)|_{x=l} = 0, \tag{39}$$

$$\left(I_{0+}^{(1-\nu)(1-\mu)} U_2(x,t)\right)(0+) = 0. \tag{40}$$

Applying the method of separation of variables in equation (35), i.e. $U_1(x,t) = X(x)T(t)$, one obtains the following equations:

$$D_{0+}^{\mu,\nu} T(t) + \lambda T(t) = 0, \tag{41}$$

$$\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} + \frac{\lambda}{K_\mu} X(x) = 0, \tag{42}$$

where λ is a separation constant and the function $X(x)$ satisfies the following boundary conditions:

$$X(x)|_{x=0} = 0, \quad X(x)|_{x=l} = 0. \tag{43}$$

The eigenfunctions of the Sturm–Liouville problem (42) with the boundary conditions (43) are given by $\lambda_n = K_\mu \frac{n^2\pi^2}{l^2}$ $(n = 1, 2, \ldots)$ [67]. For the eigenfunctions $X_n(x) = \sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right)$ in the Hilbert space $L^2[0, l]$, it is satisfied that

$$\int_0^l \sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right)\sin\left(\sqrt{\frac{\lambda_m}{K_\mu}}x\right)\,\mathrm{d}x = \frac{2}{l}\delta_{nm}, \tag{44}$$

where δ_{mn} is the Kronecker delta.

Equation (41) can be solved by the help of relation (9). Thus, we see that

$$s^\mu \mathcal{L}[T_n(t)](s) - s^{-\nu(1-\mu)}\left(I_{0+}^{(1-\nu)(1-\mu)}T_n\right)(0+) + \lambda_n \mathcal{L}[T_n(t)](s) = 0, \tag{45}$$

so that

$$\mathcal{L}[T_n(t)](s) = \frac{s^{-\nu(1-\mu)}}{s^\mu + \lambda_n}\left(I_{0+}^{(1-\nu)(1-\mu)}T_n\right)(0+). \tag{46}$$

The inverse Laplace transform of relation (46) can be found via relation (A.5) from lemma 1. Thus, we obtain

$$T_n(t) = \left[\left(I_{0+}^{(1-\nu)(1-\mu)}T_n\right)(0+)\right]t^{-(1-\mu)(1-\nu)}E_{\mu,1-(1-\mu)(1-\nu)}(-\lambda_n t^\mu), \tag{47}$$

where $\left(I_{0+}^{(1-\nu)(1-\mu)}T_n\right)(0+) = \frac{2}{l}\int_0^l \tilde{g}(x)\sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right)\mathrm{d}x$ is a Fourier coefficient of $\tilde{g}(x)$. Thus, the solution of equation (35) is given by

$$U_1(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right), \tag{48}$$

where $a_n(t)$ is defined in relation (24).

Equation (38) can be solved by use of the complete set of eigenfunctions $\sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right)$. Thus,

$$U_2(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right) \tag{49}$$

and

$$\widetilde{f}(x,t) = \sum_{n=1}^{\infty} \widetilde{f}_n(t)\sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right), \tag{50}$$

where $\widetilde{f}_n(t)$ is given by (26).

From relations (49), (50), (26) and (38), one obtains

$$\sum_{n=1}^{\infty}\left[D_{0+}^{\mu,\nu}u_n(t) + \lambda_n u_n(t) - \widetilde{f}_n(t)\right]\sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right) = 0, \tag{51}$$

which is satisfied if

$$D_{0+}^{\mu,\nu}u_n(t) + \lambda_n u_n(t) - \widetilde{f}_n(t) = 0, \quad \forall n \in \mathbb{N}. \tag{52}$$

Applying the Laplace transform method, we obtain

$$s^\mu \mathcal{L}[u_n(t)](s) - s^{-\nu(1-\mu)}\left(I_{0+}^{(1-\nu)(1-\mu)}u_n\right)(0+) + \lambda_n\mathcal{L}[u_n(t)](s) - \mathcal{L}[\widetilde{f}_n(t)](s) = 0. \tag{53}$$

From condition (40) it follows that $\left(I_{0+}^{(1-\nu)(1-\mu)}u_n\right)(0+) = 0$, so that recalling the result from lemma 2 we find

$$u_n(t) = \left(\mathcal{E}_{0+;\mu,\mu}^{-\lambda_n;1,1}\widetilde{f}_n\right)(t). \tag{54}$$

Thus, the solution of equation (38) is given by

$$U_2(x,t) = \sum_{n=1}^{\infty} \left(\mathcal{E}_{0+;\mu,\mu}^{-\lambda_n;1,1}\widetilde{f}_n\right)(t) \sin\left(\sqrt{\frac{\lambda_n}{K_\mu}}x\right). \tag{55}$$

Finally, employing relations (29), (33), (48) and (55), we prove theorem 1. □

Corollary 1. *For $\nu = 1$ (Caputo time fractional derivative) and $h_1(t) = h_2(t) = 0$, the solution becomes*

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \left(\mathcal{E}_{0+;\mu,\mu}^{-\lambda_n;1,1} f_n\right)(t) \sin\left(\frac{n\pi x}{l}\right), \tag{56}$$

where

$$a_n(t) = \tilde{c}_n E_\mu(-\lambda_n t^\mu) \tag{57}$$

and

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x, \tag{58}$$

where $\tilde{c}_n$ is given by (25) in which $\tilde{g}(x) = g(x)$.

Remark 1. Solution (56) is equivalent to those obtained by Sandev and Tomovski in theorem 1 from [23] for used $r(x) = 1$, $p(x) = K_\mu$, $q(x) = 0$, $h_1(t) = h_2(t) = 0$, $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, $0 < \alpha < 1$.

Note that if in equation (15) we put $\mu = \nu = 1$ the classical diffusion equation yields [67], whose solution can be obtained directly from theorem 1.

Example 1. The time fractional differential equation

$$D_{0+}^{\mu,\nu} u(x,t) = K_\mu \frac{\partial^2}{\partial x^2} u(x,t), \quad t > 0, \tag{59}$$

with boundary conditions

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=l} = 0, \tag{60}$$

and initial condition

$$\left(I_{0+}^{(1-\nu)(1-\mu)} u(x,t)\right)(0+) = g(x), \tag{61}$$

where $0 < \mu < 1$, $0 \leqslant \nu \leqslant 1$ and $0 \leqslant x \leqslant l$, has a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right). \tag{62}$$

Here, $a_n(t)$ is given by relation (24), in which $\tilde{g}(x) = g(x)$. Indeed, if in theorem 1 we substitute $f(x,t) = 0$ and $h_1(t) = h_2(t) = 0$, we obtain relation (62).

Remark 2. Note that for $\nu = 1$, solution (62) has the form

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{c}_n E_\mu\left(-K_\mu \frac{n^2\pi^2}{l^2} t^\mu\right) \sin\left(\frac{n\pi x}{l}\right), \tag{63}$$

where $\tilde{c}_n$ is given by (25) with $\tilde{g}(x) = g(x)$. This solution can be obtained from corollary 2 in [23] ($a^2 \to K_\mu$, $\alpha \to \mu$, $0 < \alpha < 1$).

For $\mu = \nu = 1$ one obtains the solution of the well-known classical diffusion equation (67):

$$u(x,t) = \sum_{n=1}^{\infty} \tilde{c}_n \, \mathrm{e}^{-K_\mu \frac{n^2\pi^2}{l^2} t} \sin\left(\frac{n\pi x}{l}\right). \tag{64}$$

Example 2. The time fractional diffusion equation

$$D_{0+}^{\mu,\nu} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + at^{\delta-1} E_{\mu,\delta}\left(-bt^\mu\right), \quad t > 0, \tag{65}$$

with boundary conditions

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=l} = 0, \tag{66}$$

and initial condition

$$\left(I_{0+}^{(1-\nu)(1-\mu)} u(x,t)\right)(0+) = g(x), \tag{67}$$

where $0 < \mu < 1$, $0 \leqslant x \leqslant l$, $0 < \delta < 1$, a and $b > 0$ are constants, has a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right) + 4at^{\delta-1} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi} \cdot \frac{E_{\mu,\delta}(-bt^\mu) - E_{\mu,\delta}\left(-\frac{(2n-1)^2\pi^2}{l^2} t^\mu\right)}{\frac{(2n-1)^2\pi^2}{l^2} - b} \sin\left(\frac{(2n-1)\pi x}{l}\right). \tag{68}$$

Here,

$$a_n(t) = \tilde{c}_n t^{-(1-\mu)(1-\nu)} E_{\mu,1-(1-\mu)(1-\nu)}\left(-\frac{n^2\pi^2}{l^2} t^\mu\right) \tag{69}$$

and $\tilde{c}_n$ is given by (25) with $\tilde{g}(x) = g(x)$. Note that an extended source term of the complex form occurring in equation (65) could stem from an anomalously relaxing background ('melting'). This could occur, for instance, in an aquifer backbone, along which small channels feed the backbone stream (subsurface hydrology generally meets anomalous diffusion dynamics [3]).

The solution of this equation can be obtained from theorem 1 if we take $K_\mu = 1$, $f(x,t) = at^{\delta-1} E_{\mu,\delta}(-bt^\mu)$ and $h_1(t) = h_2(t) = 0$. Indeed, the first term of relation (68) is obtained directly from equation (48). From relations (23), (26) and (27), it follows that

$$f_n(t) = \frac{2}{l} \int_0^l at^{\delta-1} E_{\mu,\delta}\left(-bt^\mu\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x = \frac{2\left[1-(-1)^n\right]}{n\pi} at^{\delta-1} E_{\mu,\delta}\left(-bt^\mu\right). \tag{70}$$

By exchanging $f_n(t)$ in relation (55) and recalling relation (A.3), we obtain the second term of (68). If for some value $n = n_0$ the equivalence $b = \frac{n_0^2\pi^2}{l^2}$ is obeyed, then the solution contains a term that can be obtained by using relation (A.4).

Remark 3. Since the asymptotic behaviour of the M–L function for $t \to \infty$ is given by $E_{\mu,\delta}\left(-bt^\mu\right) \sim \frac{1}{b\Gamma(\delta-\mu)} t^{-\mu}$ $(b > 0)$ [54] and $0 < \mu < 1$, $0 < \delta < 1$, then $t^{\delta-1} E_{\mu,\delta}\left(-at^\mu\right) \sim \frac{1}{b\Gamma(\delta-\mu)} t^{-\mu+\delta-1}$ tends to zero for $t \to \infty$. Thus, solution (68) shows a power-law decay in time.

Example 3. The time fractional diffusion equation

$$D_{0+}^{\mu,\nu} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + kt^{\delta-1}, \quad t > 0, \tag{71}$$

with boundary conditions (66) and initial condition (67), where $0 < \mu < 1$, $0 \leqslant x \leqslant l$, $0 < \delta < 1$, k is a constant, has a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right) + 4k\Gamma(\delta)t^{\mu+\delta-1} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi} \cdot E_{\mu,\mu+\delta}\left(-\frac{(2n-1)^2\pi^2}{l^2}t^{\mu}\right) \sin\left(\frac{(2n-1)\pi x}{l}\right). \tag{72}$$

Here, $a_n(t)$ is given by (69).

This result can be proved analogously to corollary 2, and by the help of relation (A.7). Since $0 < \mu < 1$ and $0 < \delta < 1$, then $t^{\mu+\delta-1}E_{\mu,\mu+\delta}\left(-\frac{(2n-1)^2\pi^2}{l^2}t^{\mu}\right) \sim \frac{1}{\frac{(2n-1)^2\pi^2}{l^2}\Gamma(\delta)} \cdot t^{-1+\delta}$ for $t \to \infty$. Solution (72) thus shows a power-law decay, as well.

Remark 4. Note that if $f(x,t) = k \cdot \frac{t^{-\beta}}{\Gamma(1-\beta)}$ $(0 < \beta < 1)$, solution (72) has the following form:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right) + 4kt^{\mu-\beta} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi} \cdot E_{\mu,\mu+1-\beta}\left(-\frac{(2n-1)^2\pi^2}{l^2}t^{\mu}\right) \sin\left(\frac{(2n-1)\pi x}{l}\right), \tag{73}$$

where $a_n(t)$ is given by (69). Thus, in the long time limit $(t \to \infty)$ we have $t^{\mu-\beta}E_{\mu,\mu+1-\beta}\left(-\frac{(2n-1)^2\pi^2}{l^2}t^{\mu}\right) \sim \frac{1}{\frac{(2n-1)^2\pi^2}{l^2}\Gamma(1-\beta)} \cdot t^{-\beta}$. Solution (73) again shows a power-law decay, as it should.

4. Fractional diffusion equation defined in infinite domain

Let us now investigate the following time fractional diffusion equation in the infinite domain $-\infty < x < +\infty$:

$$D_{0+}^{\mu,\nu}u(x,t) = K_{\mu}\frac{\partial^2}{\partial x^2}u(x,t) + f(x,t), \quad t > 0, \tag{74}$$

with the boundary and initial conditions

$$u(\pm\infty, t) = 0, \quad t > 0 \tag{75}$$

$$\left(I_{0+}^{(1-\nu)(1-\mu)}u(x,t)\right)(0+) = g(x), \quad -\infty < x < +\infty. \tag{76}$$

Similar problems with Caputo or R–L time fractional derivatives are considered in [1, 2, 24, 70, 71] and secondary references therein. The classical diffusion equation [67] can be obtained by the choice $\mu = \nu = 1$ in equation (74).

Theorem 2. *The fractional diffusion equation (74) with boundary conditions (75) and initial condition (76) for* $0 < \mu < 1$, $0 \leqslant \nu \leqslant 1$ *has a summable solution*

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{-(1-\nu)(1-\mu)} E_{\mu,1-(1-\nu)(1-\mu)}(-K_{\mu}k^2t^{\mu}) \cdot \hat{g}(k) \cdot \mathrm{e}^{-\imath kx}\,\mathrm{d}k + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\mathcal{E}_{0+;\mu,\mu}^{-K_{\mu}k^2;1,1}F\right)(k,t) \cdot \mathrm{e}^{-\imath kx}\,\mathrm{d}k, \tag{77}$$

where

$$\hat{g}(k) = \mathcal{F}[g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{\imath kx}\,\mathrm{d}x \tag{78}$$

is the Fourier transform of $g(x)$, *and*

$$F(x,s)=\mathcal{L}[f(x,t)],\quad \tilde{F}(k,s)=\mathcal{F}[F(x,s)],\quad F(k,t)=\mathcal{L}^{-1}[\tilde{F}(k,s)]. \tag{79}$$

Proof. Application of the Laplace transform with respect to the time variable t in equation (74) and via the initial condition (76), one obtains

$$s^{\mu}U(x,s)-s^{-\nu(1-\mu)}g(x)=K_{\mu}\frac{\partial^2}{\partial x^2}U(x,s)+F(x,s), \tag{80}$$

where $U(x,s)=\mathcal{L}[u(x,t)]$. If we apply the Fourier transform with respect to the spatial variable x in equation (80) and take into consideration the boundary conditions (75), we obtain

$$s^{\mu}\tilde{U}(k,s)-s^{-\nu(1-\mu)}\hat{g}(k)=-k^2K_{\mu}\tilde{U}(k,s)+\tilde{F}(k,s). \tag{81}$$

Here $\tilde{U}(k,s)=\mathcal{F}[U(x,s)]$, $\tilde{F}(k,s)=\mathcal{F}[F(x,s)]$, and we use the property $\lim_{x\to\pm\infty}\frac{\partial}{\partial x}u(x,t)=0$. Thus, we obtain

$$\tilde{U}(k,s)=\frac{s^{-\nu(1-\mu)}}{s^{\mu}+k^2K_{\mu}}\cdot\hat{g}(k)+\frac{1}{s^{\mu}+k^2K_{\mu}}\cdot\tilde{F}(k,s). \tag{82}$$

Applying an inverse Laplace transform to relation (82) and via results from lemmas 1 and 2, it follows that

$$U(k,t)=t^{-(1-\nu)(1-\mu)}E_{\mu,1-(1-\nu)(1-\mu)}\left(-K_{\mu}k^2t^{\mu}\right)\hat{g}(k)+\left(\mathcal{E}_{0+;\mu,\mu}^{-K_{\mu}k^2;1,1}F\right)(k,t). \tag{83}$$

Finally, by inverse Fourier transform of relation (83), we prove theorem 2.

□

Example 4. The time fractional diffusion equation

$$D_{0+}^{\mu,\nu}u(x,t)=K_{\mu}\frac{\partial^2}{\partial x^2}u(x,t),\quad t>0, \tag{84}$$

with boundary conditions (75) and initial condition (76), where $-\infty<x<+\infty$, has a solution of the form

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^{-(1-\nu)(1-\mu)}E_{\mu,1-(1-\nu)(1-\mu)}\big(-K_{\mu}k^2t^{\mu}\big)\cdot\hat{g}(k)\cdot \mathrm{e}^{-\imath kx}\,\mathrm{d}k. \tag{85}$$

Note that if $g(x)=\delta(x)$, from relation (A.14), solution (85) becomes

$$\begin{aligned}u(x,t)&=\frac{t^{-(1-\nu)(1-\mu)}}{|x|}\cdot H_{1,1}^{1,0}\left[\frac{|x|^2}{K_{\mu}t^{\mu}}\,\middle|\,\begin{matrix}(1-(1-\nu)(1-\mu),\mu)\\(1,2)\end{matrix}\right]\\&=\frac{t^{-(1-\nu)(1-\mu)}}{2|x|}\cdot H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_{\mu}t^{\mu}}}\,\middle|\,\begin{matrix}(1-(1-\nu)(1-\mu),\frac{\mu}{2})\\(1,1)\end{matrix}\right].\end{aligned} \tag{86}$$

Thus, in case of the R–L time fractional derivative ($\nu=0$), this solution (86) reads

$$u(x,t)=\frac{t^{-(1-\mu)}}{2|x|}\cdot H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_{\mu}t^{\mu}}}\,\middle|\,\begin{matrix}\left(\mu,\frac{\mu}{2}\right]\right)\\(1,1)\end{matrix}\right]. \tag{87}$$

In the case of a Caputo time fractional derivative ($\nu=1$), solution (86) assumes the form

$$u(x,t)=\frac{1}{2|x|}\cdot H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_{\mu}t^{\mu}}}\,\middle|\,\begin{matrix}\left(1,\frac{\mu}{2}\right)\\(1,1)\end{matrix}\right]. \tag{88}$$

The time evolution of solution (86) for $\mu=1/2$, $K_{\mu}=1$, and different values of ν is shown in figures 1 and 2. The plots are made by using expansion (A.10) in Mathematica.

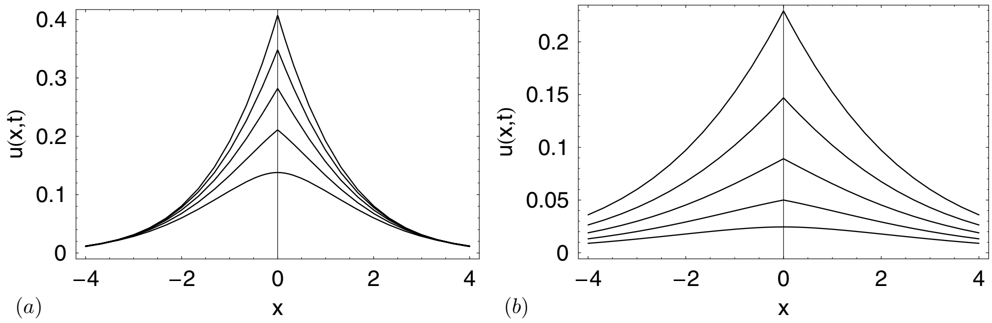

Figure 1. Graphical representation of solution (86) for $\mu = 1/2$, $K_\mu = 1$, $\nu = 0$ (lower line), $\nu = 1/4$, $\nu = 1/2$, $\nu = 3/4$, $\nu = 1$ (upper line); (*a*) $t = 1$; (*b*) $t = 10$.

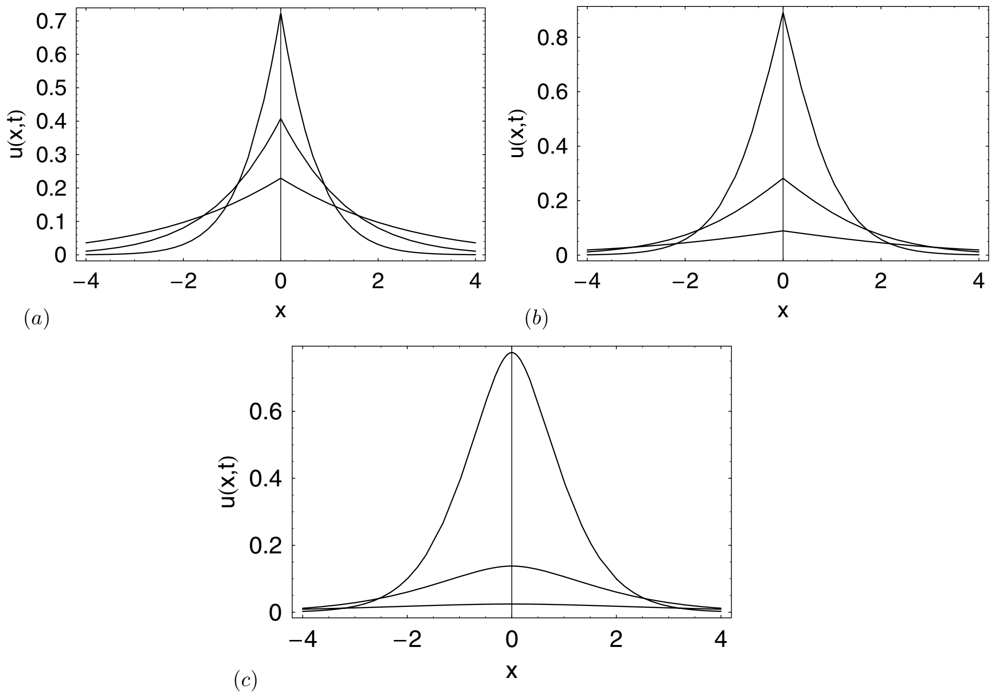

Figure 2. Graphical representation of solution (86) for $\mu = 1/2$, $K_\mu = 1$, $t = 0.1$ (upper line), $t = 1$, $t = 10$ (lower line); (*a*) $\nu = 1$ (see [1]); (*b*) $\nu = 1/2$; (*c*) $\nu = 0$.

Furthermore, if $\nu = \mu = 1$ from relation (86) we obtain the solution of the classical diffusion equation, i.e.

$$u(x,t) = \frac{1}{2|x|} \cdot H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_\mu t}}\,\middle|\,\begin{matrix}(1, \frac{1}{2})\\(1, 1)\end{matrix}\right] = \frac{1}{\sqrt{4\pi K_\mu t}} \cdot \mathrm{e}^{-\frac{x^2}{4K_\mu t}}. \tag{89}$$

This solution for $K_\mu = 1$ is shown in figure 3.

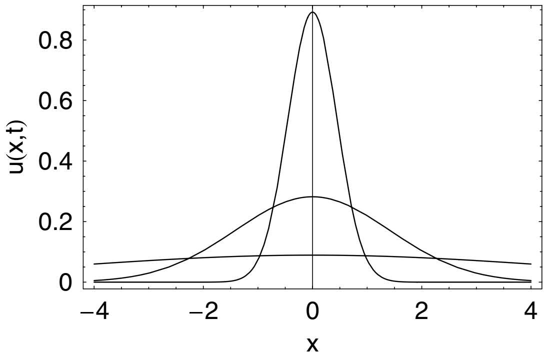

Figure 3. Graphical representation of solution (89) for $\mu = 1$, $K_\mu = 1$, $t = 0.1$ (upper line), $t = 1$, $t = 10$ (lower line).

Remark 5. The asymptotic behaviour of solution (85) follows from relations (A.16), (A.17), (A.18), (A.19), (A.20). Thus, one can obtain

$$u(x,t) \sim \frac{1}{2\sqrt{(2-\mu)\pi}} \cdot \left(\frac{\mu}{2}\right)^{\frac{(1-\mu)(1-2\nu)}{2-\mu}} \cdot |x|^{\frac{(1-\mu)(1-2\nu)}{2-\mu}} \cdot \left(K_\mu t^\mu\right)^{-\frac{(1-\nu)(1-\mu)+1/2}{2-\mu}} \\ \times t^{-(1-\nu)(1-\mu)} \cdot \exp\left[-\frac{2-\mu}{2}\left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}} |x|^{\frac{2}{2-\mu}} \left(K_\mu t^\mu\right)^{-\frac{1}{2-\mu}}\right]. \tag{90}$$

If $\nu = 1$, result (90) is equivalent to that obtained, for example, in [1, 72], i.e.

$$u(x,t) \sim \frac{1}{2\sqrt{(2-\mu)\pi}} \cdot \left(\frac{\mu}{2}\right)^{\frac{\mu-1}{2-\mu}} \cdot |x|^{\frac{\mu-1}{2-\mu}} \cdot \left(K_\mu t^\mu\right)^{-\frac{1}{2(2-\mu)}} \\ \cdot \exp\left[-\frac{2-\mu}{2}\left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}} |x|^{\frac{2}{2-\mu}} \left(K_\mu t^\mu\right)^{-\frac{1}{2-\mu}}\right]. \tag{91}$$

Moreover, if $\mu = 1$, from equation (91) we obtain solution (89) of the classical diffusion equation.

Example 5 (Calculation of fractional moments)**.** The fractional moments of the Cauchy problem of example 4 with $g(x) = \delta(x)$ has the following form:

$$\langle |x|^\xi \rangle = 2\int_0^\infty x^\xi u(x,t)\,\mathrm{d}x = t^{-(1-\nu)(1-\mu)}\left(K_\mu t^\mu\right)^{\xi/2} \cdot \frac{\Gamma(1+\xi)}{\Gamma\left(1-(1-\nu)(1-\mu)+\frac{\mu\xi}{2}\right)}. \tag{92}$$

Indeed, using solution (86) we obtain

$$\langle |x|^\xi \rangle = t^{-(1-\nu)(1-\mu)} \int_0^\infty x^{\xi-1} \cdot H_{1,1}^{1,0}\left[\frac{x}{\sqrt{K_\mu t^\mu}} \middle| \begin{array}{l} (1-(1-\nu)(1-\mu), \frac{\mu}{2}) \\ (1,1) \end{array}\right] \mathrm{d}x. \tag{93}$$

Application of the integral (A.13) produces (92). Note that for $\xi \to 0$ we obtain

$$\lim_{\xi\to 0} \langle |x|^\xi \rangle = \frac{1}{\Gamma(1-(1-\nu)(1-\mu))} \cdot t^{-(1-\nu)(1-\mu)}, \tag{94}$$

i.e. the function u is not normalized (see the discussion in section 3). If $\nu = 0$, we find $\lim_{\xi\to 0}\langle |x|^\xi \rangle = \frac{1}{\Gamma(\mu)} \cdot t^{-(1-\mu)}$, and if $\nu = 1$ we obtain $\lim_{\xi\to 0}\langle |x|^\xi \rangle = 1$, which corresponds to the normalization of the distribution function.

When $\xi \to 2$ we obtain

$$\lim_{\xi\to 2}\langle |x|^{\xi}\rangle = \frac{2}{\Gamma(1+\mu-(1-\nu)(1-\mu))}\cdot K_{\mu}\cdot t^{\mu-(1-\nu)(1-\mu)}; \tag{95}$$

we obtain the regular variance of the anomalous diffusion process. Note that if $\nu = 0$ we obtain $\lim_{\xi\to 2}\left\langle |x|^{\xi}\right\rangle = \frac{2}{\Gamma(2\mu)}\cdot K_{\mu}\cdot t^{-1+2\mu}$ and for $\nu = 1$ it follows that $\lim_{\xi\to 2}\left\langle |x|^{\xi}\right\rangle = \frac{2}{\Gamma(1+\mu)}\cdot K_{\mu}\cdot t^{\mu}$. In the limit $\mu = 1$ the linear time dependence of the mean square displacement is recovered, i.e. $\langle x^2\rangle = 2K_{\mu}t$.

Remark 6. From relation (92) one obtains the general expression

$$\left\langle x^{2n}\right\rangle = t^{n\mu-(1-\nu)(1-\mu)}{K_{\mu}}^{n}\cdot\frac{(2n)!}{\Gamma\left(1+n\mu-(1-\nu)(1-\mu)\right)}, \tag{96}$$

where $n\in\mathbb{N}$. If we divide both sides of relation (96) by $(2n)!$ and sum over n, we obtain the following interesting result:

$$\sum_{n=0}^{\infty}\frac{\langle x^{2n}\rangle}{(2n)!} = t^{-(1-\nu)(1-\mu)}\sum_{n=0}^{\infty}\frac{(K_{\mu}t^{\mu})^{n}}{\Gamma\left(n\mu+1-(1-\nu)(1-\mu)\right)}$$
$$= t^{-(1-\nu)(1-\mu)}E_{\mu,1-(1-\nu)(1-\mu)}\left(K_{\mu}t^{\mu}\right). \tag{97}$$

Note that for $\nu = 1$ the known result $\langle x^{2n}\rangle = (2n)!\frac{{K_{\mu}}^{n}t^{n\mu}}{\Gamma(1+n\mu)}$ yields (see [1, 73]).

5. Fractional diffusion equation with a singular term

Let us now consider the following fractional diffusion equation with a singular term:

$$D_{0+}^{\mu,\nu}u(x,t) = K_{\mu}\frac{\partial^2}{\partial x^2}u(x,t)+\delta(x)\cdot\frac{t^{-\beta}}{\Gamma(1-\beta)},\qquad t>0, \tag{98}$$

with boundary conditions (75) and initial condition

$$\left(I_{0+}^{(1-\nu)(1-\mu)}u(x,t)\right)(0+) = \delta(x),\qquad -\infty<x<+\infty, \tag{99}$$

where $-\infty<x<+\infty$, $\beta>0$.

Corollary 2. *Equation (98) with boundary conditions (75) and initial condition (99) has a solution expressed via the H-function*

$$u(x,t) = \frac{t^{-(1-\nu)(1-\mu)}}{2|x|}\cdot H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_{\mu}t^{\mu}}}\left|\begin{array}{l}(1-(1-\nu)(1-\mu),\frac{\mu}{2})\\(1,1)\end{array}\right.\right]$$
$$+\frac{t^{-(\beta-\mu)}}{2|x|}\cdot H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_{\mu}t^{\mu}}}\left|\begin{array}{l}(1-(\beta-\mu),\frac{\mu}{2})\\(1,1)\end{array}\right.\right]. \tag{100}$$

Proof. This result can be obtained from theorem 2 simply by exchanging $f(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}$, $g(x) = \delta(x)$, and from relations (A.7) and (A.15). □

Remark 7. When $\nu = 1$ and $\beta = \mu$ one obtains

$$u(x,t) = \frac{1}{|x|}\cdot H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_{\mu}t^{\mu}}}\left|\begin{array}{l}(1,\frac{\mu}{2})\\(1,1)\end{array}\right.\right]. \tag{101}$$

Solution (100) for $\beta=\mu=1/2$, $K_{\mu}=1$, and different values of ν is shown in figure 4.

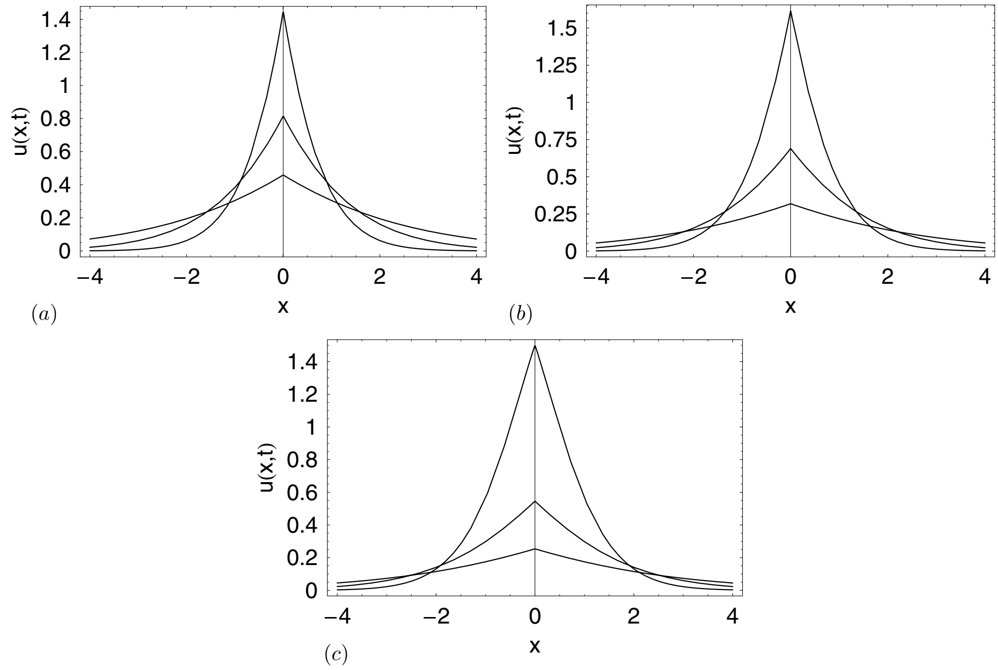

Figure 4. Graphical representation of solution (100) for $\beta = \mu = 1/2$, $K_\mu = 1$, $t = 0.1$ (upper line), $t = 1$, $t = 10$ (lower line); (*a*) $\nu = 1$; (*b*) $\nu = 1/2$; (*c*) $\nu = 0$.

Remark 8. From relations (A.16), (A.17), (A.18), (A.19), (A.20), the asymptotic behaviour of solution (100) follows as

$$u(x,t) \sim \frac{1}{2\sqrt{(2-\mu)\pi}} \cdot \left(\frac{\mu}{2}\right)^{\frac{(1-\mu)(1-2\nu)}{2-\mu}} \cdot |x|^{\frac{(1-\mu)(1-2\nu)}{2-\mu}} \cdot \left(K_\mu t^\mu\right)^{-\frac{(1-\nu)(1-\mu)+1/2}{2-\mu}}$$
$$\times t^{-(1-\nu)(1-\mu)} \cdot \exp\left[-\frac{2-\mu}{2}\left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}} |x|^{\frac{2}{2-\mu}} \left(K_\mu t^\mu\right)^{-\frac{1}{2-\mu}}\right]$$
$$+ \frac{1}{2\sqrt{(2-\mu)\pi}} \cdot \left(\frac{\mu}{2}\right)^{\frac{2\beta-\mu-1}{2-\mu}} \cdot |x|^{\frac{2\beta-\mu-1}{2-\mu}} \cdot \left(K_\mu t^\mu\right)^{-\frac{\beta-\mu+1/2}{2-\mu}} \cdot t^{-(\beta-\mu)}$$
$$\times \exp\left[-\frac{2-\mu}{2}\left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}} |x|^{\frac{2}{2-\mu}} \left(K_\mu t^\mu\right)^{-\frac{1}{2-\mu}}\right]. \qquad (102)$$

If $\nu = 1$ and $\beta = \mu$, result (90) becomes

$$u(x,t) \sim \frac{1}{\sqrt{(2-\mu)\pi}} \cdot \left(\frac{\mu}{2}\right)^{\frac{\mu-1}{2-\mu}} \cdot |x|^{\frac{\mu-1}{2-\mu}} \cdot \left(K_\mu t^\mu\right)^{-\frac{1}{2(2-\mu)}}$$
$$\cdot \exp\left[-\frac{2-\mu}{2}\left(\frac{\mu}{2}\right)^{\frac{\mu}{2-\mu}} |x|^{\frac{2}{2-\mu}} \left(K_\mu t^\mu\right)^{-\frac{1}{2-\mu}}\right]. \qquad (103)$$

Remark 9. What is the condition for which the solution $u(x,t)$ of equation (98) is a probability distribution function with normalization $\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = 1$. Integrating $\int_{-\infty}^{\infty} \ldots \mathrm{d}x$ both sides of equation (98) and using $\lim_{x\to\pm\infty} \frac{\partial}{\partial x} u(x,t) = 0$, one obtains

$$D_{0+}^{\mu,\nu} 1 = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \qquad t > 0. \qquad (104)$$

From relations (6) and (4) it follows that

$$\frac{t^{-\mu}}{\Gamma(1-\mu)} = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t > 0, \tag{105}$$

such that we obtain the requirement $\beta = \mu$. In this case equation (98) includes the initial condition for the R–L part of the fractional operator.

6. Conclusions

Exact solutions of different variants of a time fractional diffusion equation with Hilfer-generalized time fractional derivative with order $0 < \mu < 1$ and type $0 \leqslant \nu \leqslant 1$ were obtained. These solutions are expressed in terms of M–L type functions, *H*-functions, the integral operator (14), and a complete set of eigenfunctions of the Sturm–Liouville problem. It is shown that the solutions of the corresponding classical diffusion equation and fractional diffusion equations with pure Caputo or R–L time fractional derivative are particular cases of the solution of the considered equations. Several special cases are investigated. An inverse-power law decay of the solutions in the long time limit is shown. Fractional moments of the fundamental solution of the fractional diffusion equation with a generalized R–L time fractional derivative are calculated. A fractional diffusion equation with a singular term is considered, as well. The asymptotic behaviour of the solutions are found. Some previously obtained results are recovered.

We believe that the generalized diffusion equation with its solutions in direct and transformed spaces will be helpful for the evaluation of data from complex systems, in particular, in the context of relaxation dynamics in glassy systems or aquifer problems.

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Appendix. Mittag–Leffler and Fox's *H*-functions

The standard M–L function, introduced by Mittag–Leffler, is defined by [38]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{A.1}$$

where ($z \in \mathbb{C}$; $\Re(\alpha) > 0$). Later, Wiman [39], Agarwal [40], Humbert [41], and Humbert and Agarwal [42] introduced and investigated the following two-parameter generalized M–L function:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{A.2}$$

where ($z, \beta \in \mathbb{C}$; $\Re(\alpha) > 0$). The M–L functions (A.1) and (A.2) are entire functions of order $\rho = 1/\Re(\alpha)$ and type 1. Note that $E_{\alpha,1}(z) = E_\alpha(z)$. These functions are generalization of the exponential, hyperbolic and trigonometric functions since $E_{1,1}(z) = \mathrm{e}^z$, $E_{2,1}(z^2) = \cosh(z)$, $E_{2,1}(-z^2) = \cos(z)$ and $E_{2,2}(-z^2) = \sin(z)/z$. For the M–L function (A.2), the following formula is true [74]:

$$\int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}\left(-a\tau^\alpha\right)(t-\tau)^{\beta-1} E_{\alpha,\beta}\left(-b(t-\tau)^\alpha\right) \mathrm{d}\tau = \frac{E_{\alpha,\beta}\left(-bt^\alpha\right) - E_{\alpha,\beta}\left(-at^\alpha\right)}{a-b} t^{\beta-1}, \quad a \neq b. \tag{A.3}$$

When $a = b$, from relation (A.3) one easily obtains that [53]

$$\int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}\left(-a\tau^\alpha\right)(t-\tau)^{\beta-1} E_{\alpha,\beta}\left(-a(t-\tau)^\alpha\right) \mathrm{d}\tau = t^{\alpha+\beta-1} E_{\alpha,\beta}\left(-at^\alpha\right). \tag{A.4}$$

The Laplace transform of the M–L function is given by [49–51]

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(\pm at^\alpha)] = \int_0^\infty \mathrm{e}^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm at^\alpha)\, \mathrm{d}t = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}, \tag{A.5}$$

where $\Re(s) > |a|^{1/\alpha}$.

Prabhakar [45] introduced the three-parameter generalized M–L function

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \tag{A.6}$$

where $\beta, \gamma, z \in \mathbb{C}$, $\Re(\alpha) > 0$, and $(\gamma)_k$ is the Pochhammer symbol. It is an entire function of order $\rho = 1/\Re(\alpha)$ and type 1. Note that $E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z)$. For the M–L function (A.6) the following formula is true [75]:

$$\int_0^t \tau^{\beta-1}(t-\tau)^{\mu-1} E_{\alpha,\mu}^{\gamma}\left(\omega(t-\tau)^\alpha\right) \mathrm{d}\tau = \Gamma(\beta) t^{\mu+\beta-1} E_{\alpha,\mu+\beta}\left(\omega t^\alpha\right). \tag{A.7}$$

Later, Srivastava and Tomovski introduced the four-parameter generalized M–L function [53]

$$E_{\alpha,\beta}^{\gamma,\kappa}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!}, \tag{A.8}$$

where $(z, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > \max\{0, \Re(\kappa) - 1\}; \Re(\kappa) > 0)$ and $(\gamma)_{\kappa n}$ is a notation of the Pochhammer symbol. It is an entire function of order $\rho = \frac{1}{\Re(\alpha-\kappa)+1}$ and type $\sigma = \frac{1}{\rho} big(\frac{\{\Re(\alpha)\}^{\Re(\kappa)}}{\{\Re(\alpha)\}^{\Re(\alpha)}}\big)^\rho$ [53]. Note that $E_{\alpha,\beta}^{\gamma,1}(z) = E_{\alpha,\beta}^{\gamma}(z)$.

The Fox H-function (or simply the H-function) is defined by the Mellin–Barnes integral [60, 76]

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array}\right.\right] = H_{p,q}^{m,n}\left[z \left| \begin{array}{l} (a_p, A_p) \\ (b_q, B_q) \end{array}\right.\right] = \frac{1}{2\pi\imath}\int_\Omega \theta(s) z^s \,\mathrm{d}s, \tag{A.9}$$

where $\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^n \Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j+B_j s) \prod_{j=n+1}^p \Gamma(a_j - A_j s)}$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q$, $a_i, b_j \in \mathbb{C}$, $A_i, B_j \in \mathbb{R}^+$, $i = 1, \ldots, p$, $j = 1, \ldots, q$. The contour Ω starting at $c - \imath\infty$ and ending at $c + \imath\infty$ separates the poles of the function $\Gamma(b_j + B_j s)$, $j = 1, \ldots, m$, from those of the function $\Gamma(1 - a_i - A_i s)$, $i = 1, \ldots, n$. The expansion for the H-function (A.9) is given by [76]

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]$$
$$=\sum_{h=1}^{m}\sum_{k=0}^{\infty}\frac{\prod_{j=1,j\neq h}^{m}\Gamma\left(b_j-B_j\frac{b_h+k}{B_h}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+A_j\frac{b_h+k}{B_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+B_j\frac{b_h+k}{B_h}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-A_j\frac{b_h+k}{B_h}\right)}$$
$$\cdot\frac{(-1)^k z^{(b_h+k)/B_h}}{k!B_h}. \tag{A.10}$$

The H-function has the following property [76]:

$$H_{p,q}^{m,n}\left[z^{\delta}\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{\delta}\cdot H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1/\delta),\ldots,(a_p,A_p/\delta)\\(b_1,B_1/\delta),\ldots,(b_q,B_q/\delta)\end{matrix}\right.\right], \tag{A.11}$$

where $\delta>0$. It is related with the two-parameter M–L function in the following way [1]:

$$E_{\alpha,\beta}(-z)=H_{1,2}^{1,1}\left[z\left|\begin{matrix}(0,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right]. \tag{A.12}$$

The Mellin transform of the H-function is given by

$$\int_0^{\infty}x^{\xi-1}H_{p,q}^{m,n}\left[ax\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]\mathrm{d}x=a^{-\xi}\theta(-\xi), \tag{A.13}$$

where $\theta(-\xi)$ is defined in relation (A.9).

The Mellin-cosine transform of the H-function is given by [76, 77]

$$\int_0^{\infty}k^{\rho-1}\cos(kx)H_{p,q}^{m,n}\left[ak^{\delta}\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]\mathrm{d}k$$
$$=\frac{\pi}{x^{\rho}}H_{q+1,p+2}^{n+1,m}\left[\frac{x^{\delta}}{a}\left|\begin{matrix}(1-b_q,B_q),\left(\frac{1+\rho}{2},\frac{\delta}{2}\right)\\(\rho,\delta),(1-a_p,A_p),\left(\frac{1+\rho}{2},\frac{\delta}{2}\right)\end{matrix}\right.\right], \tag{A.14}$$

where $\Re\left(\rho+\delta\min_{1\leqslant j\leqslant m}\left(\frac{b_j}{B_j}\right)\right)>1, x^{\delta}>0, \Re\left(\rho+\delta\max_{1\leqslant j\leqslant n}\left(\frac{a_j-1}{A_j}\right)\right)<\frac{3}{2}, |\arg(a)|<\pi\theta/2$, $\theta>0, \theta=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j$. Thus, by using relations (A.12) and (A.9), the cosine transform (A.14) of the two-parameter M–L function is given in terms of the H-function, i.e.

$$\int_0^{\infty}\cos(kx)E_{\alpha,\beta}\left(-ak^2\right)\mathrm{d}k=\frac{\pi}{x}H_{3,3}^{2,1}\left[\frac{x^2}{a}\left|\begin{matrix}(1,1),(\beta,\alpha),(1,1)\\(1,2),(1,1),(1,1)\end{matrix}\right.\right]$$
$$=\frac{\pi}{x}H_{1,1}^{1,0}\left[\frac{x^2}{a}\left|\begin{matrix}(\beta,\alpha)\\(1,2)\end{matrix}\right.\right]. \tag{A.15}$$

The asymptotic expansion of the Fox H-function $H_{1,1}^{1,0}(z)$ for large z is [72, 76, 78]

$$H_{1,1}^{1,0}(z)\sim Bz^{\alpha/m}\exp\left(-mC^{1/m}z^{1/m}\right), \tag{A.16}$$

where

$$\alpha=b_1-a_1+\tfrac{1}{2}, \tag{A.17}$$

$$m=B_1-A_1, \tag{A.18}$$

$$C=(A_1)^{A_1}(B_1)^{B_1}, \tag{A.19}$$

$$B=(2\pi)^{-\frac{1}{2}}C^{\alpha/m}m^{-1/2}(A_1)^{-a_1+1/2}(B_1)^{b_1-1/2}. \tag{A.20}$$

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