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Barrier crossing of a Lévy flight

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Abstract. – We consider the barrier crossing in a bistable potential for a random-walk process that is driven by Lévy noise of stable index α. It is shown that the survival probability decays exponentially, but with a power law dependence $T_c(\alpha, D) = C(\alpha)D^{-\mu(\alpha)}$ of the mean escape time on the noise intensity D. Here C is a constant, and the exponent μ varies slowly over a large range of the stable index $\alpha \in [1, 2)$. For the Cauchy case, we explicitly calculate the escape rate.

Introduction. – The escape of a particle from a potential well is a generic problem investigated by Kramers [1] that is often used to model chemical reactions, nucleation processes, or the escape from an external, confining potential of finite height [2]. Despite the outstanding role of the central-limit theoreom giving rise to the Gaussian nature of the propagator of traditional Brownian motion, a large number of stochastic processes have been identified, whose behaviour is more general and leaves the basin of attraction of Gaussian processes [3]. Instead, these processes are governed by the generalised central-limit theorem, leading to the Lévy stable nature of their propagators [4, 5]. External noise, that is Lévy stable, is typical for systems such as fluctuations in plasmas [6], random walks along a polymer chain in solution such that the walker can jump across contact points created by polymer looping [7], or fluctuations in the kinetic energy in optical lattices [8]. It is therefore a natural question to ask how the Lévy stable nature of such processes generalises the barrier crossing behaviour of the classical Kramers problem. An interesting example is given by the α-stable noise-induced barrier crossing in long paleoclimatic time series [9]; another potential application is given by the escape from traps in optical or plasma systems, see, for instance, [10].

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In what follows, we investigate barrier crossing of processes in some coordinate $x(t)$ that are governed by a Langevin equation of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\frac{1}{m\gamma}\frac{\mathrm{d}V(x)}{\mathrm{d}x} + D^{1/\alpha}\xi_\alpha(t), \tag{1}$$

where $\xi_\alpha(t)$ denotes white Lévy noise such that the quantity $L(\Delta t) = \int_t^{t+\Delta t}\xi_\alpha(\tau)\mathrm{d}\tau$ is a symmetric, α-stable process with probability density $p(L, \Delta t)$ defined in terms of the characteristic function $p(k, \Delta t) = \exp[-D|k|^\alpha \Delta t]$ $(0 < \alpha \leq 2)$ [11]. D, of dimension $[\text{length}]^\alpha/\text{time}$, is the strength of the noise, m the mass of the diffusing particle, and γ is a friction constant. The external potential is $V(x)$, for which we choose the rather generic double-well shape

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4; \tag{2}$$

compare, for instance, ref. [12]. For convenience, we turn to dimensionless variables $t \to t/t_0$ and $x \to x/x_0$ with $t_0 = m\gamma/a$ and $x_0^2 = 1/(bt_0)$ and dimensionless noise strength $D \to Dt_0^{1/\alpha}/x_0$ (by $\xi_\alpha(t_0 t) \to t_0^{1/\alpha-1}\xi_\alpha(t)$) [6], such that we arrive at the stochastic equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \left(x - x^3\right) + D^{1/\alpha}\xi_\alpha(t). \tag{3}$$

Here, we restrict our discussion to $1 \leq \alpha < 2$.

In regular Brownian motion corresponding to the limit $\alpha = 2$, the survival probability $\mathscr{S}$ of a particle whose motion at time $t = 0$ is initiated in one of the potential minima $x_{\min} = \pm 1$, follows an exponential decay $\mathscr{S}(t) = \exp\left[-t/T_c\right]$ with the mean escape time T_c, such that the probability density function $p(t) = -\mathrm{d}\mathscr{S}/\mathrm{d}t$ of the barrier crossing time t becomes

$$p(t) = T_c^{-1}\exp\left[-t/T_c\right]. \tag{4}$$

The mean crossing time (MCT) follows the exponential law

$$T_c = C\exp\left[h/D\right], \tag{5}$$

where h is the barrier height (equal to 1/4 for the potential (2)) in rescaled variables, and the prefactor C includes details of the potential [2]. We want to find out how the presence of Lévy stable noise modifies the laws (4) and (5).

Numerical solution. – The Langevin equation (3) was integrated numerically following the procedure developed in ref. [13]. From this, we obtained the trajectories of the particle as displayed in fig. 1. In the Brownian limit, we reproduce qualitatively the behaviour found in ref. [12]. Accordingly, the fluctuations around the positions of the minima are localised in the sense that their width is clearly smaller than the distance between minima and barrier. In contrast, for progressively smaller stable index α, characteristic spikes become visible, and the individual sojourn times in one of the potential wells decrease. In particular, we note that single spikes can be of the order of or larger than the distance between the two potential minima. This reflects the slowly decaying probability density $p(L, \Delta t)$ of the magnitude of the Lévy stable noise ξ_α, as a function of L.

From such single trajectories we determine the individual barrier crossing times as the time interval between a jump into one well across the zero line $x = 0$ and the escape across $x = 0$ back to the other well. In fig. 2, we demonstrate that on average, the crossing times are distributed exponentially, and thus follow the same law (4) known from the Brownian case.

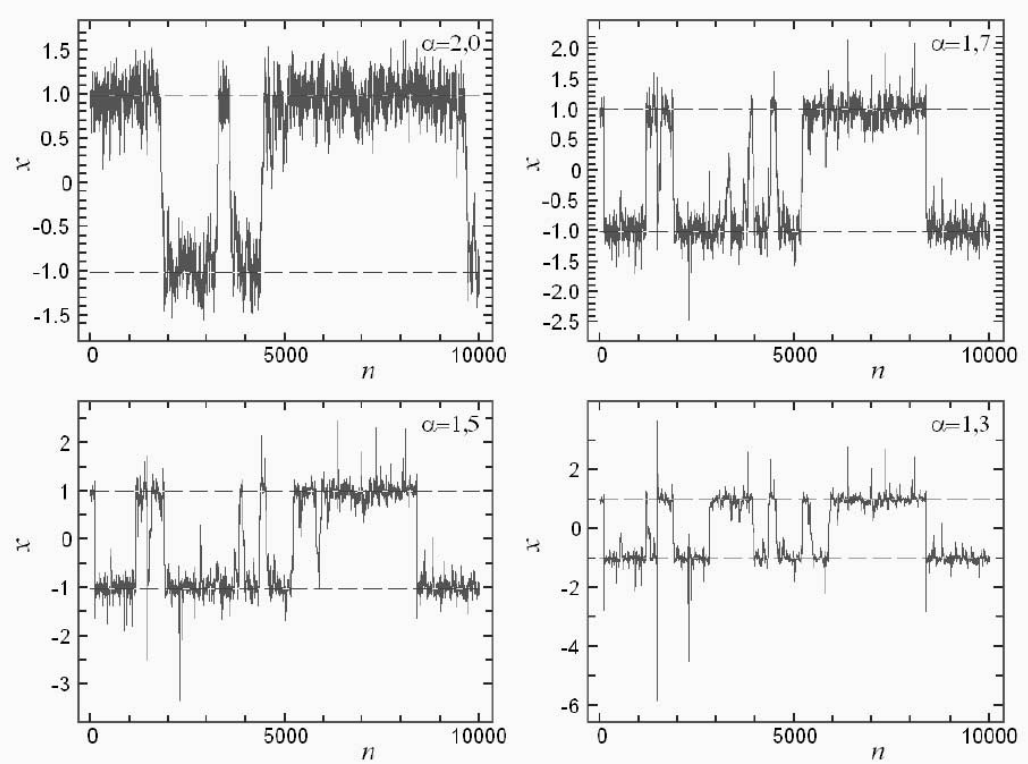

Fig. 1 – Typical trajectories for different stable index α obtained from numerical integration of the Langevin equation (3). The dashed lines represent the potential minima at ± 1. In the Brownian case $\alpha = 2$, previously reported behaviour is recovered [12]. In the Lévy stable case, occasional long jumps of the order of or larger than the separation of the minima can be observed. Note the different scales.

Such a result was reported in a previous study of Kramers' escape driven by Lévy noise [14]. In fact, this observed exponential decay of the survival probability $\mathscr{S}$ in a Lévy flight is not surprising, given the Markovian nature of the process. Due to the Lévy stable properties of the noise ξ_α, the Langevin equation (3) produces occasional long jumps, by which the particle can cross the barrier. Large enough values of the noise ξ_α thus occur considerably more frequently than in the Brownian case with Gaussian noise ($\alpha = 2$), causing a lower MCT.

The numerical integration of the Langevin equation (3) was repeated for various stable indices α, and for a range of noise strengths D. From these simulations we obtain the detailed dependence of the MCT $T_c(\alpha, D)$ on both parameters, α and D. As expected, for decreasing noise strength, the MCT increases. For sufficiently large values of $1/D$ and fixed α, a power law trend in the double-logarithmic plot is clearly visible. These power law regions, for the investigated range of α are in very good agreement with the analytical form

$$T_c(\alpha, D) = \frac{C(\alpha)}{D^{\mu(\alpha)}}, \tag{6}$$

over a large range of D. Equation (6) is the central result of this study. It is clear from fig. 3, that this relation is appropriate for the entire α-range sweeped over in our simulations. For larger noise strength, we observe a breakdown of the power law trend, and the curves seem

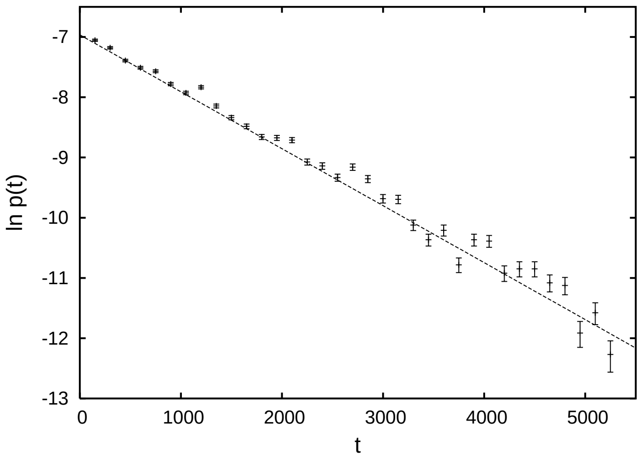

Fig. 2 – Probability density function $p(t)$ of barrier crossing times for $\alpha = 1.0$ and $D = 10^{-2.5} \approx 0.00316$. The dashed line is a fit to eq. (4) with MCT $T_c = 1057.8 \pm 17.7$.

to approach the MCT behaviour of the Brownian process ($\alpha = 2$) as a common envelope. A more thorough numerical analysis of this effect will be necessary to be more precise about its nature. What we want to focus on here is the behaviour (6). We note from fig. 3 that for α ranging roughly between the Cauchy case $\alpha = 1$ and the Holtsmark case $\alpha = 3/2$, the exponent μ is almost constant, *i.e.*, the corresponding lines in the log-log plot are almost parallel. The behaviour of both the scaling exponent μ and the prefactor C as a function of stable index α becomes clearer in fig. 4. There, we recognise a slow variation of μ for values of α between 3/2 and slightly below 2, before a steeper rise in close vicinity of 2. This apparent divergence must be faster than any power, so that in the Gaussian noise limit $\alpha = 2$, the

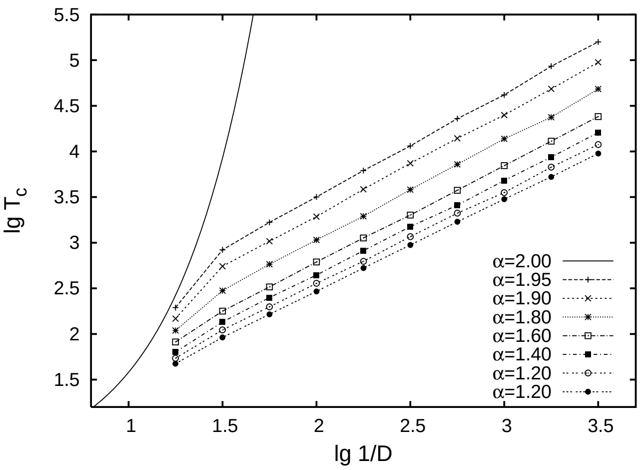

Fig. 3 – Escape time T_c as a function of noise strength D for various α's. Above roughly $\lg 1/D = 1.5$, a power law behaviour is observed that corresponds to eq. (6). The curve (5) for $\alpha = 2.0$ appears to represent a common envelope.

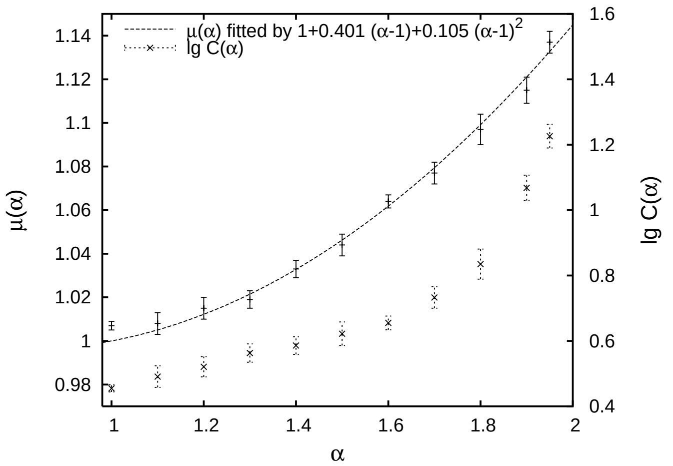

Fig. 4 – Scaling exponent μ as function of stable index α. The constant behaviour $\mu(\alpha) \approx 1$ over the range $1 \leq \alpha \lessapprox 1.6$ is followed by an increase above 1.6, and it eventually shows an apparent divergence close to $\alpha = 2$, where eq. (5) holds. Corresponding to the right ordinate, we also plot the decadic logarithm of the amplitude $C(\alpha)$.

activation follows the exponential law (5) instead of the scaling form (6). The $\mu(\alpha)$ results are fitted with the parabola indicated in the plot where, for the analytical results derived below, we forced the fit function to go through the point $\mu(1) = 1$.

Analytical approximation for the Cauchy case. – In the Cauchy limit $\alpha = 1$, we can find an approximate result for the MCT as a function of noise strength D. To this end, we start with the rescaled fractional Fokker-Planck equation [3,5,15–17], corresponding to eq. (3),

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(-x + x^3\right) f(x,t) + D\frac{\partial^\alpha}{\partial |x|^\alpha} f(x,t), \tag{7}$$

where the fractional Riesz-Weyl derivative is defined in terms of the Fourier transform through $\mathscr{F}\left\{\partial^\alpha f(x,t)/\partial|x|^\alpha\right\} \equiv -|k|^\alpha f(k,t)$, and where $\mathscr{F}\{f(x,t)\} = \int_{-\infty}^{\infty} f(x,t)e^{ikx}\mathrm{d}x$; see refs. [3, 18] for explicit definitions and solution methods. Rewriting eq. (7) in the continuity form $\partial f(x,t)/\partial t + \partial j(x,t)\partial x = 0$, that is equivalent to $\partial f(k,t)\partial t = ikj(k,t)$ in Fourier space, we obtain for the flux the expression

$$j(k) = \left(-\frac{\partial^3}{\partial k^3} - i\frac{\partial}{\partial k} + iD\,\mathrm{sign}(k)|k|^{\alpha-1}\right) f(k,t). \tag{8}$$

To obtain an approximate expression for the MCT, we follow the standard steps [19] and for large values of $1/D$ make the constant flux approximation assuming that the flux across the barrier is a constant, j_0, corresponding to the existence of a stationary solution $f_{\mathrm{st}}(x)$. By integration of the continuity equation, it then follows that eq. (4) is fulfilled, and $T_c = 1/j_0$. Due to the low noise strength, we also assume that for all relevant times the normalisation $\int_{-\infty}^{0} f_{\mathrm{st}}(x) = 1$ is fulfilled.

In this constant flux approximation, we obtain from eq. (8) the relation

$$\frac{\mathrm{d}^3 f_{\mathrm{st}}(x)}{\mathrm{d}k^3} + \frac{\mathrm{d}f_{\mathrm{st}}(x)}{\mathrm{d}k} - D\,\mathrm{sign}(k) f_{\mathrm{st}}(x) = 2\pi i j_0 \delta(k) \tag{9}$$

in the Cauchy case $\alpha = 1$. With the ansatz $f_{\rm st}(k) = C_1 e^{z^{\pm}k} + C_2 e^{(z^*)^{\pm}k}$ for $k \gtrless 0$, we find the characteristic equation $(z^{\pm})^3 + z^{\pm} \mp D = 0$ solved by the Cardano expressions $z^{\pm} = -\frac{1}{2}(u_{\pm} + v_{\pm}) + \frac{1}{2}i\sqrt{3}(u_{\pm} - v_{\pm})$, with $u_+^3 = D\left(1 + \sqrt{1 + 4/[27D^2]}\right)/2 = -v_-^3$ and $v_+^3 = D\left(1 - \sqrt{1 + 4/[27D^2]}\right)/2 = -u_-^3$. Matching the left and right solutions at $k = 0$, requiring that $f_{\rm st}(k) \in \mathbb{R}$, and assuming that $f_{\rm st}(k)$ in the constant flux approximation is far from the fully relaxed ($t \to \infty$) solution, we obtain the shifted Cauchy form

$$f_{\rm st}(k) = \frac{j_0}{2\zeta_+\zeta_-}\frac{\zeta_+}{(x+\zeta_-)^2 + \zeta_+^2}, \qquad \therefore\ \zeta_+ = \frac{1}{2}(u_+ + v_+), \qquad \zeta_- = \frac{\sqrt{3}}{2}(u_+ - v_+). \quad (10)$$

With the normalisation $\int_{-\infty}^0 f_{\rm st}(x){\rm d}x = 1$, we arrive at the MCT

$$T_c = \frac{\pi}{4\zeta_+\zeta_-}\left(1 + \frac{2}{\pi}\arctan\frac{\zeta_-}{\zeta_+}\right). \quad (11)$$

For $D \ll 1$, $\zeta_+ \approx D/2$ and $\zeta_- \approx 1$, so that $T_c \approx \pi/D$. In comparison with the numerical result corresponding to fig. 2 with $T_c = 1057.8$ for $D = 0.00316$, we calculate from our approximation $T_c \approx 994.2$, within 6% of the numerical result. This good agreement also corroborates that the constant flux approximation appears to pertain to Lévy flights.

Discussion. – This is the first detailed and systematic analysis of the problem of escape over a potential barrier for a process governed by Lévy stable noise, *i.e.*, a Lévy flight, over the range of stable indices $1 \le \alpha < 2$. We observe from numerical simulations an exponential decrease of the survival probability $\mathscr{S}(t)$ in the potential well, in whose bottom we initialise the process. Moreover, we find that the MCT assumes the scaling form (6) with the scaling exponent μ being approximately constant in the range $1 \le \alpha \lessapprox 1.6$, followed by an increase before the apparent divergence at $\alpha = 2$, that leads back to the exponential form prevalent in the Brownian case, eq. (5). An analytic calculation in the Cauchy limit $\alpha = 1$ reproduces, consistently with the constant flux approximation commonly applied in the Brownian case, the scaling $T_c \sim 1/D$ and, within a few percent error, the numerical value of the MCT T_c.

Employing scaling arguments, we can restore dimensionality into expression (6) for the MCT. From our model potential (2), in whose coefficients we choose to absorb the friction factor $m\gamma$ such that $a \to a/(m\gamma)$ and $b \to b/(m\gamma)$, we find the location of the minima, $x_{\min} = \pm\sqrt{a/b}$ and the barrier height $\Delta V = a^2/(4b)$. In terms of the rescaled prefactors a and b with dimensions $[a] = {\rm s}^{-1}$ and $[b] = {\rm s}^{-1}{\rm cm}^{-2}$, we can now re-introduce the dimensions through $t_0 = 1/a$ and $x_0^2 = b/a$. In the domain where $T_c \sim 1/D$ (*i.e.*, $\mu(\alpha) \approx 1$), we then come up with the scaling

$$T_c \sim \frac{x_0^\alpha}{D} = \frac{(a/b)^{\alpha/2}}{D} = \frac{|x_{\min}|^\alpha}{D}, \quad (12)$$

in analogy to the result reported in ref. [14]. However, we remind the two caveats based on our results: i) The linear behaviour in $1/D$ does not appear valid over the entire α-range. For larger values, $\alpha \gtrapprox 1.6$, the scaling exponent $\mu(\alpha)$ assumes non-trivial values; in that case, the simple scaling used to establish eq. (12) has to be modified. It is not immediately obvious how this should be done systematically. ii) From relation (12) it cannot be concluded that the MCT is independent of the barrier height ΔV, despite the sole dependence of T_c on the distance $|x_{\min}|$ from the barrier. This latter statement is obvious from the expressions for $x_{\min}$ and ΔV derived for our model potential: the location of the minima relative to the barrier, and barrier height, namely, are in fact coupled. Therefore, a random walker subject to Lévy noise does feel the influence of the potential barrier and does not simply move across it with the characteristic time given by the free mean-squared displacement. However, as we could see, the activation for the MCT as a function of noise strength D varies only in the form of a power law instead of the standard exponential behaviour.

The time dependence of the probability density $-\mathrm{d}\mathscr{S}(t)/\mathrm{d}t$ for first barrier crossing time of a Lévy flight process is exponential, exactly as in the standard Brownian case. This can be understood qualitatively from the fact that the process is Markovian. From the governing dynamical equation (7), it is clear that the relaxation of modes is exponential, compare ref. [5]. For low-noise strength D, the barrier crossing will be dominated by the slowest time-eigenmode $\simeq e^{-\lambda_1 t}$ with eigenvalue λ_1. This is similar to the first passage time problem of a Lévy flight in a semi-infinite geometry [20].

We finally note that throughout this work we use the term Lévy flight in the sense of a Lévy stable process in which the drift exerted on the test particle by the external field enters additively and linearly. This corresponds to the continuous-time random-walk picture derived in ref. [21] that leads to the dynamical equation (7). There exist alternative ways to include the external field giving rise to a different behaviour in a double-well potential, see, for instance, ref. [22].

REFERENCES

[1] Kramers H. A., *Physica A*, **7** (1940) 284.
[2] Hänggi P., Talkner P. and Bokrovec M., *Rev. Mod. Phys.*, **62** (1990) 251.
[3] Metzler R. and Klafter J., *J. Phys. A*, **37** (2004) R161.
[4] Bouchaud J.-P. and Georges A., *Phys. Rep.*, **88** (1990) 127.
[5] Metzler R. and Klafter J., *Phys. Rep.*, **339** (2000) 1.
[6] Chechkin A., Gonchar V. Y. and Szydlowsky M., *Phys. Plasmas*, **9** (2002) 78.
[7] Sokolov I. M., Mai J. and Blumen A., *Phys. Rev. Lett.*, **79** (1997) 857.
[8] Katori H., Schlipf S. and Walther H., *Phys. Rev. Lett.*, **79** (1997) 2221.
[9] Ditlevsen P. D., *Geophys. Res. Lett.*, **26** (1999) 1441.
[10] Fajans J. and Schmidt A., *Nucl. Instrum. Methods Phys. Res. A*, **521** (2004) 318.
[11] Samorodnitsky G. and Taqqu M. S., *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance* (Chapman and Hall, New York) 1994.
[12] Hanggi P., Mroczkowski T. J., Moss F. and McClintock P. V. E., *Phys. Rev. A*, **32** (1985) 695.
[13] Chechkin A. V. and Gonchar V. Yu., *Physica A*, **27** (2000) 312.
[14] Ditlevsen P. D., *Phys. Rev. E*, **60** (1999) 172.
[15] West B. J. and Seshadri V., *Physica A*, **113** (1982) 203.
[16] Fogedby H. C., *Phys. Rev. Lett.*, **73** (1994) 2517.
[17] Jespersen S., Metzler R. and Fogedby H. C., *Phys. Rev. E*, **59** (1999) 2736.
[18] Chechkin A. V., Gonchar V. Yu., Klafter J., Metzler R. and Tanatarov V., *J. Stat. Phys.*, **115** (2004) 1505.
[19] Klimontovich Yu. L., *Statistical Theory of Open Systems*, Vol. **1** (Kluwer Academic Publishers, Dordrecht) 1995.
[20] Chechkin A. V., Metzler R., Klafter J., Gonchar V. Yu. and Tanatarov L. V., *J. Phys. A*, **36** (2003) L537.
[21] Metzler R., Barkai E. and Klafter J., *Europhys. Lett.*, **46** (1999) 431.
[22] Brockmann D. and Sokolov I. M., *Chem. Phys.*, **284** (2002) 409.