Comment on "Anomalous Heat Conduction and Anomalous Diffusion in One-Dimensional Systems"

A relation between anomalous diffusion, in which the mean squared displacement grows in time like $\langle(\Delta x)^2\rangle = 2D_\alpha t^\alpha$ $(0 < \alpha \leq 2)$, and anomalous heat conduction was recently derived through a scaling approach by Li and Wang [1] (LW). This model assumes that heat transport in a 1D channel is due solely to the flow of noninteracting particles: those entering the channel from the left have a different average kinetic energy than those entering from the right. The energies of the particles at both ends of the channel are defined through the Boltzmann distributions that correspond to the temperatures of two heat baths coupled to either end. LW are correct in stating that different billiard models discussed in literature belong to this class of processes. However, in this Comment we point out certain crucial inconsistencies of the LW model with the physical picture of random processes leading to anomalous diffusion.

First, consider the collision-free heat transport between the two heat baths. This situation corresponds to ballistic transport, $\alpha = 2$, and the mean first passage time (MFPT) acquires the scaling $T \propto L/v$. In this case, the model of LW reproduces the original result [2] for the heat conductivity, $\kappa \propto L$. The first inconsistency becomes apparent already in this limiting case: since the typical velocities of particles entering the channel from the left and from the right are different, the corresponding left and right MFPT necessarily differ, as well. The equality of both first passage times invoked in LW can be fulfilled only if the particles are thermalized *within* the channel; however, under this assumption, the ballistic nature is lost and the whole model no longer holds. Partially, this problem may be circumvented by taking $(T_L - T_R) \to 0$ in Eq. (4).

The crucial flaw in LW is the fact that Eq. (1) does not necessarily imply Eq. (2) in the range $0 < \alpha \leq 2$ and vice versa. Thus, although it is tempting to argue that if the typical displacement of the particle grows like $\langle(\Delta x)^2\rangle^{1/2} \propto t^{\alpha/2}$ then the *typical* time for traveling a distance L will scale like $\tau \propto L^{2/\alpha}$, one cannot conclude what exactly this time τ defines: it may well differ from the MFPT T. The latter may even diverge while τ exists.

To explain this need for caution, let us first address subdiffusion, which corresponds to a long-tailed waiting time distribution $\psi(t) \sim (t/t_0)^{-1-\alpha}/t_0$ $(0 < \alpha < 1)$ [3]. In this case, it was shown in Ref. [4] that the temporal eigenfunctions for a finite geometry are given by Mittag-Leffler functions, and therefore the survival probability decays like $t^{-\alpha}$. Thus, the associated MFPT diverges: $T = \infty$, corresponding to the dominance of the probability of *not* moving in subdiffusion [3]. (We should note that in one of the three references, Ref. [5], cited in LW to support their scaling relation, the result for the first passage time distribution is based on an integral expression, which diverges for a waiting time distribution of long-tailed nature, and is therefore wrong.) Without an external bias, the conductivity of a subdiffusive system vanishes [4,6]. The other case in which the approach presented in LW fails are Lévy flights [3]. Their MFPT exists and is finite, as can be shown using the methods described in Ref. [7]; however, their $\langle(\Delta x)^2\rangle$ *diverges* [8].

It must therefore be concluded that the model proposed in LW is by far less general than assumed there and, due to the combination of two *a priori* unrelated equations, contains a crucial flaw in the foundations such that erroneous results ensue for both subdiffusion and Lévy flights. We also note that in a related context a model studied in Ref. [9] provides results for the heat conductivity consistent with our objections.

Finally, we point out that the interpretation in terms of the finite-time measurement in the case of subdiffusion, brought forth in the Reply of LW [10], would lead to a correct result. However, it would cause an explicitly cutoff time-dependent MFPT and would therefore be different from the original model of Ref. [1].

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