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Bifurcation, bimodality, and finite variance in confined Lévy flights

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We investigate the statistical behavior of Lévy flights confined in a symmetric, quartic potential well $U(x)\propto x^4$. At stationarity, the probability density function features a distinct bimodal shape and decays with power-law tails which are steep enough to give rise to a finite variance, in contrast to free Lévy flights. From a δ-initial condition, a *bifurcation* of the unimodal state is observed at $t_c>0$. The nonlinear oscillator with potential $U(x)=ax^2/2+bx^4/4$, $a,b>0$, shows a crossover from unimodal to bimodal behavior at stationarity, depending on the ratio a/b.

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Lévy flights (LFs) in a homogeneous environment constitute a Markovian random process whose probability density function (PDF) is a Lévy stable law, $f(x,t)$ of index $0<\alpha<2$, defined in terms of the characteristic function $\hat{f}(k,t)=\int_{-\infty}^{\infty}f(x,t)e^{ikx}dx=e^{-D|k|^{\alpha}t}$, where D of dimension $\mathrm{cm}^{\alpha}/\mathrm{sec}$ is the generalized diffusion constant [1–5]. The most prominent property of LFs is the clustered (fractal) nature of their points of visitation, intimately related to the power-law asymptotic behavior $f(x,t)\sim Dt/|x|^{1+\alpha}$ and the ensuing divergence of the variance $\langle x^2(t)\rangle$. LFs fall into the basin of attraction of the generalized central limit theorem [1–4], and have been recognized as the signature of a variety of systems ranging from turbulent plasma dynamics to spectral diffusion in single molecule spectroscopy, from bacterial motion to the albatross flight, see, e.g., Refs. [6,7], and references therein, or to impulsive noise in signal processing [8].

An important point in understanding a random process is its behavior in external fields. Although the stage has been set for the study of such properties of LFs, only very limited information is available. Thus, LFs have been studied, both analytically and numerically in the framework of Langevin and Fokker-Planck equations [5,9–16]. For the case of an harmonic potential $U(x)=ax^2/2$, one finds that the stationary PDF is given by a (unimodal) Lévy stable law; in particular, $\langle x^2(t)\rangle\to\infty$ [12,15]. A question arises on the behavior in steeper potentials. In the present communication, we investigate the dynamic evolution of the stationary PDF under the nonlinear oscillator potential

$$U(x)=\frac{a}{2}x^2+\frac{b}{4}x^4,\quad a,b>0, \tag{1}$$

and show that this process exhibits, depending on the ratio of a and b, hitherto unknown bifurcations between the unimodal initial condition and a final bimodal state. The potential (1) combines the famed harmonic form of the Ornstein-Uhlenbeck potential exerting a restoring linear force on the test particle, with the quartic term. After a brief introduction to fractional Fokker-Planck equations, we start off with the discussion of LFs in the quartic potential $U(x)=bx^4/4$, the simplest form for which the unusual properties of the PDF can be recovered, before moving on to the general case (1) in which the relative strength of harmonic and quartic terms, a/b, can be tuned.

For stochastic processes whose underlying statistics satisfies the conditions of the central limit theorem, the continuum description in the external force field $F(x)=-dU(x)/dx$ is given through the Fokker-Planck equation (FPE) [17], whose stationary solution corresponds to the Boltzmann distribution. In contrast, the spatial correlations underlying LFs can be described by a Langevin equation for an overdamped test particle driven by white Lévy stable noise [9–12,15]. On the level of the corresponding deterministic equation, the PDF $f(x,t)$ is determined by the fractional Fokker-Planck equation (FFPE) [5,10–13,15,16]

$$\frac{\partial f}{\partial t}=\left(-\frac{\partial}{\partial x}\frac{F(x)}{m\gamma}+D\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}\right)f(x,t), \tag{2}$$

where γ is the friction constant, m is the mass of the test particle, D is a measure for the intensity of the Lévy noise, and the Riesz fractional derivative of $f(x,t)$ is defined through its Fourier transform as $\mathcal{F}(\partial^{\alpha}f(x,t)/\partial|x|^{\alpha})\equiv-|k|^{\alpha}\hat{f}(k,t)$ [18]. (Note that this is just a formal way of writing the fractional spatial derivative; it reduces to the standard second derivative $\partial^2/\partial x^2$ in the limit $\alpha=2$ but does *not* correspond to $\partial/\partial x$ for $\alpha=1$.) Equation (2) is linear in f and reduces to the standard FPE in the limit $\alpha=2$ [19].

Introducing dimensionless variables $x\to x/x_0$, $t\to t/t_0$, with $x_0=(m\gamma D/b)^{1/(2+\alpha)}$ and $t_0=x_0^{\alpha}/D$, and $a\to at_0/m\gamma$, Eq. (2) is transformed into the equation for the characteristic function $\hat{f}(k,t)$,

$$\frac{\partial}{\partial t}\hat{f}+|k|^{\alpha}\hat{f}=\mathsf{U}_k\hat{f},\quad \mathsf{U}_k=k\frac{\partial^3}{\partial k^3}-ak\frac{\partial}{\partial k}. \tag{3}$$

The initial condition $\hat{f}(k,t=0)=1$ corresponds to a δ condition in x space. The solution of Eq. (3) is a real even function obeying the following boundary conditions: (i) $\hat{f}(0,t)=1$, (ii) $\partial\hat{f}(0)/\partial k=0$, and (iii) $\hat{f}(k\rightarrow\pm\infty,t)=0$.

Consider first the stationary quartic Cauchy oscillator, $a=0$, $\alpha=1$, and $\partial\hat{f}/\partial t=0$, for which Eq. (3) can be readily integrated. By inverse Fourier transform, we obtain the stationary PDF

$$f(x)=\pi^{-1}(1-x^2+x^4)^{-1}. \tag{4}$$

This PDF combines the distinct steep asymptotic power-law behavior $f(x)\propto x^{-4}$, and therefore finite variance, with a bimodal structure: there is a local minimum at $x=0$ and two global maxima at $x_m=\pm 1/\sqrt{2}$. Let us now show that the steep power-law asymptotics and the bimodality are inherent for the stationary PDF for all Lévy noise exponents $1\leqslant\alpha<2$ of the quartic oscillator $(a=0)$. Due to symmetry, we consider the positive semiaxis, $k\geqslant 0$. With the transformation $\eta(\xi)\equiv k^{(\alpha-1)/3}\hat{f}(k)$, where $\xi=k^{(\alpha+2)/3}$, the stationary Eq. (3) can be converted into an equation for $\eta(\xi)$, whose asymptotics for large ξ can be found by known methods [20,21]. The leading term of the asymptotics is obtained from the equation $d^3\eta/d\xi^3=\lambda\eta$, where $\lambda=27/(2+\alpha)^3$. We obtain for the characteristic function

$$\hat{f}(k)\approx Ck^{-(\alpha-1)/3}\exp\left(-\frac{3}{2(2+\alpha)}k^{(\alpha+2)/3}\right)\times\cos\left(\frac{3\sqrt{3}}{2(2+\alpha)}k^{(\alpha+2)/3}-\theta\right) \tag{5}$$

for $k\rightarrow\infty$. C and θ are unknown constants, since we make use of the boundary condition at infinity, only. For small k, the stationary solution $\hat{f}(k)$ of Eq. (3) for $a=0$ can be represented in the form of a series as

$$\hat{f}(k)=S_1-Ak^2S_2, \tag{6}$$

where $S_1=\Sigma_{j=0}^{\infty}a_j|k|^{j(\alpha+2)}$ and $S_2=\Sigma_{j=0}^{\infty}b_j|k|^{j(\alpha+2)}$, and the coefficients a_j and b_j are determined by the recurrent relations $a_j j(\alpha+2)(j\alpha+2j-1)(j\alpha+2j-2)=a_{j-1}$, and $b_j j(\alpha+2)(j\alpha+2j+1)(j\alpha+2j+2)=b_{j-1}$ $(j\geqslant 1,\ a_0=b_0=1)$ [22]. The asymptotics of the PDF at $x\rightarrow\pm\infty$ are determined by the first nonanalytical term in Eq. (6), i.e., by $a_1|k|^{\alpha+2}$. By inverse Fourier transformation, using the Abel method of summation for the improper integral [23], we obtain

$$f(x)\approx C_\alpha|x|^{-\alpha-3},\quad |x|\rightarrow\infty, \tag{7}$$

where $C_\alpha=\sin(\pi\alpha/2)\Gamma(\alpha)/\pi$ [24]. This is consistent with Eq. (4) for $\alpha=1$. Although the Lévy noise has a diverging variance, the stationary PDF has a steep power-law tail, and hence the variance $\langle x^2\rangle$ is finite. Thus, the effect of the quartic potential is to *confine* the flights and lead to a finite variance PDF. The nature of confined Lévy flights is, of course, different from truncated Lévy flights [25] which have finite moments.

To construct the characteristic function numerically, we use solution (6), which is continued with the asymptotics (5) for large k. The PDF is then obtained by inverse Fourier transformation. In Fig. 1, the profiles of the stationary PDFs are shown for the different Lévy indices in the range $1\leqslant\alpha<2$ and for $\alpha=2$, the bimodality being most pronounced for $\alpha=1$. With the Lévy index increasing, the bimodal profile smoothes out, and, finally, it turns into the unimodal one at $\alpha=2$, that is, for the Boltzmann distribution. Besides analytical estimates, we use two methods of numerical simulation: one, based on numerical solution of the Langevin equation, with the subsequent construction of the PDF, and another one based on numerical solution of Eq. (2), where the fractional derivative is expressed through Grünwald-Letnikov operators [26]. Both methods produce comparable results. In Fig. 2, we show a comparison of analytical and numerical results for the stationary PDF of the quartic oscillator, demonstrating a good agreement. Qualitatively, the occurrence of the bimodal structure can be understood as a trade-off between the relatively high probability for large amplitude of the Lévy noise, and the sharp increase in the slope $\propto|x|^4$ of the quartic potential relative to the harmonic case.

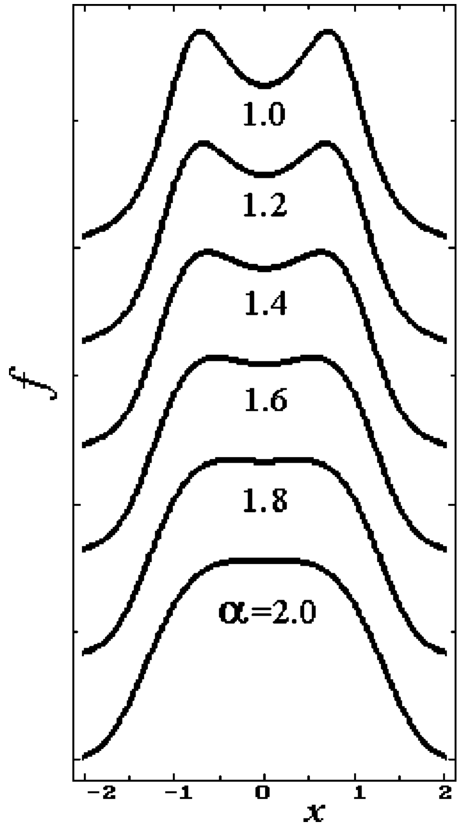

FIG. 1. Profiles of the stationary PDFs of the quartic oscillator for different Lévy indices, from $\alpha=1$ (top) to $\alpha=2$ (bottom).

Since the harmonic Lévy oscillator has one hump at the origin, and its quartic counterpart exhibits two humps, one expects that the unimodal-bimodal crossover occurs when the ratio a/b is varied. Let a_c be the critical value, in the rescaled coordinates of Eq. (3), which we determine from the condition $d^2f(0)/dx^2=0$ at $a=a_c$. Equivalently, $J(a_c)=0$, where $J(a)=\int_0^\infty dk k^2\hat{f}(k)$. If $J>0$, the stationary PDF

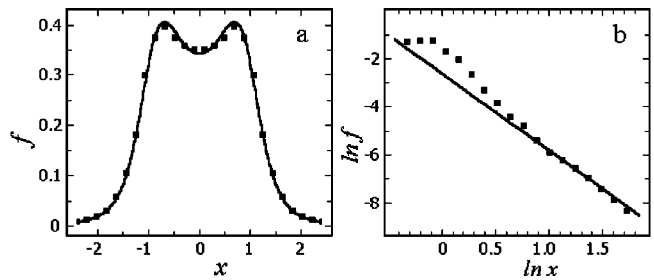

FIG. 2. The dots show the stationary PDF for the quartic oscillator obtained from numerical evaluation of the underlying Langevin equation for $\alpha = 1.5$. Left: the solid line indicates the PDF obtained from Eqs. (5) and (6) after inverse Fourier transformation. Right: the solid line indicates the asymptotics $C_\alpha |x|^{-\alpha-3}$ on a log-log scale.

is unimodal; if $J<0$, it is bimodal. We consider the transition for the anharmonic Cauchy oscillator, $\alpha=1$. For this particular case, the stationary solution of Eq. (3) is $\hat{f}(k)=(ze^{z^*k}-z^*e^{zk})(z-z^*)^{-1}$, z being the complex root of the characteristic equation $z^3-az-1=0$, i.e., $z=-(u_+ + u_-)/2 + i\sqrt{3}(u_+ - u_-)/2$, where $u_\pm = (1 \pm \sqrt{1-4a^3/27})/2^{1/3}$. Inserting these expressions into $J(a)$, we get $\text{sgn}\,(J) = \text{sgn}(z^2+z^{*2}) = -\text{sgn}(u_+^2 + u_-^2 - 4a/3)$. Defining $\zeta = 4^{1/3}a_c/3$, we obtain $4\zeta = (1+\sqrt{1-\zeta^3})^{2/3} + (1-\sqrt{1-\zeta^3})^{2/3}$, from which $\zeta = 0.420$, and therefore $a_c = 0.794$, follow. For $a>a_c$, the quadratic term in the potential energy function prevails, and the stationary PDF has one maximum at the origin. In contrast, for $a<a_c$, the quartic term dominates and dictates the shape of the PDF. As a result, the bimodal stationary PDF appears with the local minimum at the origin. Returning to the dimensional variables, we can rewrite the condition of transition in terms of a critical value b_c of the quartic term amplitude: $b_c = a^3/0.794^3(m\gamma D)^2$. This relation implies that increasing noise requires smaller anharmonicity to cause the bimodal stationary PDF. Thus, the bimodality results indeed from the combination of the Lévy character of the noise and the anharmonicity of the potential well. In Fig. 3, the profiles of the stationary PDFs are shown for the anharmonic Cauchy oscillator for different values of the dimensionless coefficient a, the bimodality being most pronounced for $a=0$.

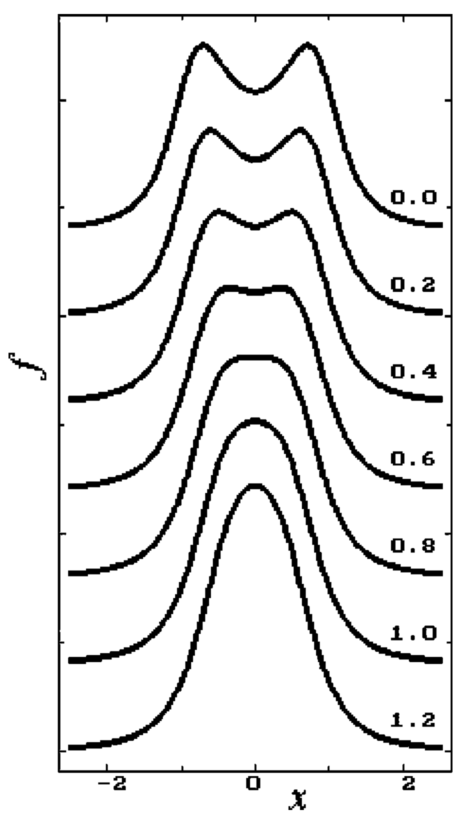

FIG. 3. Stationary PDFs of the anharmonic Cauchy oscillator for different values of the dimensionless parameter a in the potential energy function ($a_c = 0.794$).

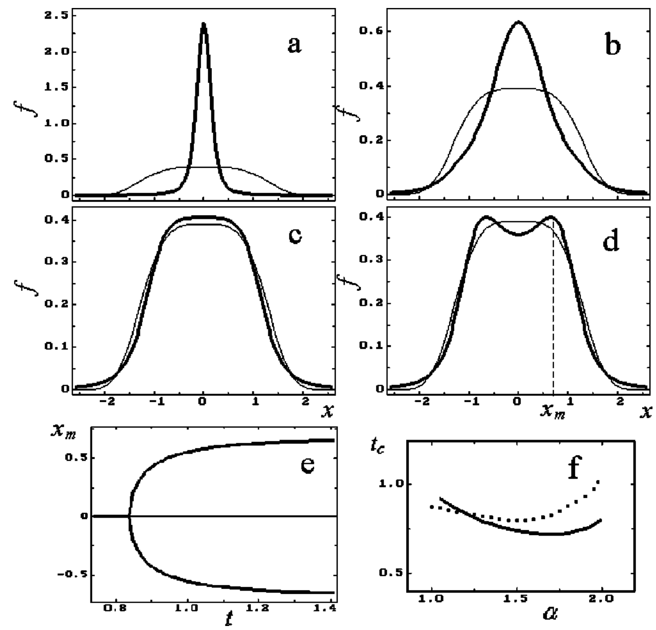

FIG. 4. (a)–(d): The thick lines show the time evolution of the PDF obtained from numerical solution of the FFPE (2) using the Grünwald-Letnikov representation of the fractional Riesz derivative. The thin lines indicate the Boltzmann distribution. Dimensionless times: $t_a = 0.06$, $t_b = 0.39$, $t_c = 0.83$ (at the bifurcation time), and $t_d = 1.33$. (e) Locations x_m of the two maxima of the PDF vs time. (f) Transition time t_c versus order of the Lévy exponent. The solid line represents the theoretical curve, which is in qualitative agreement to the results of the numerical solution of the FFPE indicated by the dots.

Let us now turn to the nonstationary properties. The formal solution of Eq. (3) can be obtained after rewriting it in the equivalent integral form,

$$\hat{f}(k,t) = \hat{p}_\alpha(k,t) + \int_0^t d\tau \hat{p}_\alpha(k,t-\tau) \mathsf{U}_k \hat{f}(k,\tau), \qquad (8)$$

where $\hat{p}_\alpha(k,t) \equiv \exp(-|k|^\alpha t)$ is the characteristic function of the Lévy stable process. Equation (8) can be solved by iterations,

$$\hat{f}(k,t) = \sum_{n=0}^{\infty} \hat{p}_{\alpha}(*\mathsf{U}_k \hat{p}_{\alpha})^n, \tag{9}$$

where * implies convolution. The bimodality of the stationary PDF stems from a unimodal-bimodal bifurcation in time, if the initial condition is given by the δ function at the origin. In Figs. 4(a)–4(d), the time evolution of the propagator is obtained by numerical solution of Eq. (2) for the quartic oscillator, $\alpha = 1.2$. The initial state disperses and, at $t_c = 0.833$, the transition occurs. Figure 4(e) shows the location $|x_m|$ of the two maxima of the PDF. After the transition, a valley is formed between the maxima, and the PDF approaches the stationary state. The bifurcation time t_c can be determined from the condition $\partial^2 f(0,t=t_c)/\partial x^2 = 0$, which implies the appearance of the inflection point during time evolution. Introducing $J(t) = \int_0^\infty dk k^2 \hat{f}(k,t)$, this is equivalent to the condition $J(t_c) = 0$, from which one can get successive approximations to t_c by inserting iterative approximations $\hat{f}_1(k,t)$, $\hat{f}_2(k,t)$, ... from Eq. (9). In Fig. 4(f), the solid line demonstrates the second approximation to t_c vs α for the quartic oscillator, using Maple6.

In summary, we have investigated some interesting and *a priori* unexpected statistical properties of systems driven by Lévy noise. In particular, we have shown that Lévy noise can be confined by a quartic external potential and that the stationary distribution is characterized by a bimodality which occurs at a critical time. We suggest that external potentials of the form $|x|^c$, $c > 2$, confine Lévy noise, leading to bimodality and to a finite variance of the stationary PDF.

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