Fractional diffusion and Lévy stable processes

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(Received 30 May 1996)

Anomalous diffusion in which the mean square distance between diffusing quantities increases faster than linearly in ''time'' has been observed in all manner of physical and biological systems from macroscopic surface growth to DNA sequences. Herein we relate the cause of this nondiffusive behavior to the statistical properties of an underlying process using an exact statistical model. This model is a simple two-state process with long-time correlations and is shown to produce a random walk described by an exact fractional diffusion equation. Fractional diffusion equations describe anomalous transport and are shown to have exact solutions in terms of Fox functions, including Lévy α-stable processes in the superdiffusive domain $(1/2<H<1)$. [S1063-651X(97)03301-1]

PACS number(s): 05.40.+j, 05.45.+b, 05.60.+w

I. INTRODUCTION

It is not quite two decades since Mandelbrot introduced the term fractal into the scientists' lexicon. This term took cognizance of the fact that there was a large class of physical, biological, and physiological phenomena that traditional statistical physics was not equipped to describe, much less to explain. The typical features of these phenomena is that they are complex, nonlinear, and appear to fluctuate randomly in space and/or time. The spectral properties of such systems, rather than being dominated by a narrow band of frequencies, spread themselves into a broadband spectrum, so that correlations persist from very-short- to very-long-time scales; see, e.g., [1]. Such spectra, when they are inverse power law, indicate fractal random time series and could be generated either by colored noise or by the chaotic solutions to low-dimensional deterministic nonlinear dynamical equations. Of course, chaotic dynamical systems often have exponentially decaying rather than inverse power-law spectra. On the other hand, the statistics of the fluctuations are often found to deviate strongly from that normally expected using the central limit theorem (CLT); for example, the second moment diverges. A generalized version of the CLT yields Lévy stable distributions to describe the statistical fluctuations in these systems; see, for example, [2]. Subsequently, it has been found that both the inverse power-law spectra and the Lévy statistical distribution are a consequence of scaling and fractals; see [3]. Herein we restrict our discussion of dynamical systems to those that can be characterized by either an inverse power-law spectrum, Lévy statistics, or both.

The understanding of these phenomena, and some previous ones as well, has only come about through the development and implementation of alternative modeling strategies. For example, the Maxwell-Zener standard constitutive equations relating stress to strain have been generalized to fractional order differential equations in time; cf. [4] and [5]. Glöckle and Nonnenmacher [6] pointed out some relations of fractional differential equations to continuous-time random walks (CTRW's) of trapping type leading to the identification of the fractional order parameter with the index of the inverse power-law waiting-time distribution function and the Lévy index. Another example is anomalous diffusion, where the anomaly can be either the time dependence of the variance of a process, i.e., the variance $\sim t^{2H}$, where t is the time and $H\neq 1/2$, or the statistics of the variate. The former anomaly is often described by fractional Brownian motion, whereas the latter uses Lévy stable distributions. Both these types of anomalies arise in the modeling of DNA sequences (see for example, [7]) and in a variety of other biomedical phenomena, including interbeat interval distribution of mammalian heartbeats and ion-channel gating; see [8] for an overview.

Herein we develop the exact equation of evolution for a dichotomous process having correlated fluctuations. In Sec. II we argue that the general equation reduces to normal diffusion when the microscopic correlation time scale is finite. However, when there is no separation of the microscopic from the macroscopic time scales the diffusion is anomalous. In Sec. III we examine anomalous diffusion using an inverse power-law correlation function and demonstrate that the evolution of such a process can be represented by a fractional diffusion equation. Seshadri and West [9] showed that a Lévy stable process is described by a fractional diffusion equation. We show that anomalous diffusion, where the variance does not increase linearly with time, is not described by fractional Brownian motion, that is, the statistics are not Gaussian. A method for solving such fractional diffusion equations using Fox functions is presented in Sec. IV. The Fox function method has been championed in relaxation processes by Glöckle and Nonnenmacher [6] and was introduced into the study of anomalous diffusion processes by Schneider [10], where he derived a Fox function representation of Lévy stable distribution functions. Explicit solutions of fractional wave and diffusion equations were given by Schneider and Wyss [11] and, more recently, by Metzler *et al.* [12] in terms of Fox functions. The fractional diffusion equation derived herein is formally different from that derived by Schneider [10] and is shown to reduce to the even

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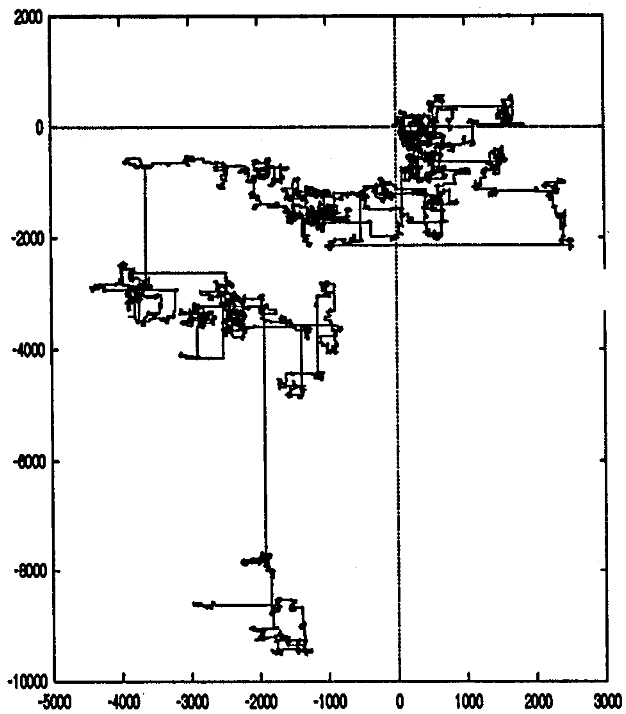

FIG. 1. Two-dimensional random walk given by (1) with an inverse power-law correlation function (15), depicted with $w=1$ and the power-law index $\beta=0.5$. Note the clustering of the walk so typical of a Lévy process.

earlier integro-differential equation for Lévy processes developed by Seshadri and West [9]. In Sec. V we apply this method to our model process and show that dichotomous random fluctuations with an inverse power-law correlation function can have Lévy statistics. In Sec. VI we draw some general conclusions.

II. TWO-STATE MODEL

We consider one of the simplest of stochastic differential equations, that being

$$\frac{dx(t)}{dt}=\xi(t), \tag{1}$$

where $\xi(t)$ is a two-state process taking the values $\pm w$ and is depicted in Fig. 1. If $\varphi(x,\xi,t)dxd\xi$ is the probability that the dynamical variables $x(t)$ and $\xi(t)$ have values in the intervals $(x,x+dx)$ and $(\xi,\xi+d\xi)$ in general, then the phase-space equation of evolution corresponding to the dynamical equation (1) is given by

$$\frac{\partial}{\partial t}\varphi(x,\xi,t)=\left(-\hat{\xi}\frac{\partial}{\partial x}+\hat{\Gamma}\right)\varphi(x,\xi,t). \tag{2}$$

Here $\hat{\Gamma}$ is an operator characterizing the dynamics of the ξ process and $\hat{\xi}$ is an operator having the eigenvalues $\pm w$. The underlying process generating $\xi(t)$ is not known and need not be specified except insofar as it provides the appropriate fluctuations driving Eq. (1). An example of such a generator is the chaotic map used by Allegrini *et al.* [7] to model DNA sequences with long-range correlations.

The reduced probability density for the random walk variable $x(t)$ is determined by the projection operator $\mathcal{P}^2=\mathcal{P}$ such that

$$\sigma_0(x,t)\equiv\mathcal{P}\varphi(x,\xi,t). \tag{3}$$

The orthogonal complement to this reduced probability is determined by

$$\sigma_1(x,t)\equiv\mathcal{Q}\varphi(x,\xi,t), \tag{4}$$

where $\mathcal{P}+\mathcal{Q}=1$. Using these two distributions allows us to partition Eq. (2) into the two equations

$$\frac{\partial\sigma_0(x,t)}{\partial t}=-\mathcal{P}\hat{\xi}\frac{\partial\sigma_1(x,t)}{\partial x}, \tag{5a}$$

$$\frac{\partial\sigma_1(x,t)}{\partial t}=-\mathcal{Q}\hat{\xi}\frac{\partial\sigma_0(x,t)}{\partial x}+\mathcal{Q}\hat{\Gamma}\sigma_1(x,t), \tag{5b}$$

where we have used the operator relations $\mathcal{P}\hat{\Gamma}=\hat{\Gamma}\mathcal{P}=0$, indicating that the dynamical operator for the velocity fluctuations couples only to the excited state $\sigma_1(x,t)$ and not to the ground state $\sigma_0(x,t)$. On the other hand, the operator $\hat{\xi}$ connects $\sigma_1(x,t)$ to $\sigma_0(x,t)$ and has no diagonal matrix elements so that $\mathcal{Q}\hat{\xi}\mathcal{Q}=\mathcal{P}\hat{\xi}\mathcal{P}=0$. Thus we can integrate Eq. (5b) to obtain

$$\sigma_1(x,t)=-\int_0^t\mathcal{Q}e^{-\hat{\Gamma}(t'-t)}\mathcal{Q}\frac{\partial\sigma_0(x,t)}{\partial x}dt', \tag{6}$$

which when inserted into Eq. (5a) yields

$$\frac{\partial\sigma_0(x,t)}{\partial t}=\int_0^t\mathcal{P}\hat{\xi}\mathcal{Q}e^{-\hat{\Gamma}(t'-t)}\mathcal{Q}\hat{\xi}\frac{\partial^2\sigma_0(x,t')}{\partial x^2}dt'. \tag{7}$$

It is a simple matter to prove that the coefficient of the second derivative in the integrand is just the two-time correlation function

$$\mathcal{P}\hat{\xi}\mathcal{Q}e^{-\hat{\Gamma}(t'-t)}\mathcal{Q}\hat{\xi}\equiv\langle\xi(t)\xi(t')\rangle, \tag{8}$$

so that Eq. (7) can be rewritten

$$\frac{\partial\sigma_0(x,t)}{\partial t}=\int_0^t\langle\xi(t)\xi(t')\rangle\frac{\partial^2\sigma_0(x,t')}{\partial x^2}dt'. \tag{9}$$

Note that Eq. (9) is an *exact* equation of evolution for a two-state process having the correlation function given by Eq. (8).

Normal diffusion is a natural consequence of the existence of a microscopic time scale, defined by

$$\tau=\int_0^\infty\frac{\langle\xi(0)\xi(t')\rangle}{\langle\xi^2\rangle}dt'. \tag{10}$$

If the correlation function $\langle\xi(0)\xi(t')\rangle$ decays quickly enough to make τ finite, we can explore the random-walk process for times t very large compared to τ. The time scale separation between the random-walk process and the velocity fluctuations allows the CLT to work, thereby reaching a Gaussian diffusion process for the two-state model. On the other hand, when $\tau \to \infty$ there is no time scale separation between the macroscopic (diffusion) and the microscopic process (fluctuations of the velocity variable ξ) and the resulting statistics are not Gaussian in general. In the next section we turn our attention to a realization of such a process.

III. FRACTIONAL DIFFUSION EQUATION

It was not stressed in the preceding section, so let us do so now, and emphasize that Eq. (8) implies that the two-point correlation $\langle \xi(t')\xi(t'')\rangle$ depends only on the time difference $|t'-t''|$ and the process is therefore stationary. Let us introduce the equilibrium correlation function $\Phi_\xi(t)$ defined by

$$\Phi_\xi(t) \equiv \frac{\langle \xi(0)\xi(t)\rangle}{\langle \xi^2\rangle}, \tag{11}$$

which is the function used in the definition of the microscopic time rate (10). Geisel *et al.* [13] established a connection between the stationary correlation function (11) and another important statistical function, the waiting-time distribution $\psi(t)$ used in CTRW models. This latter function determines the probability that $\xi(t)$ has made a transition between states in a time t. In the specific case where the variable ξ is a dichotomous process, as in the case of interest here, this connection between $\Phi_\xi(t)$ and $\psi(t)$ is exact and is given by

$$\Phi_\xi(t) = \frac{\int_t^\infty (t'-t)\psi(t')dt'}{\int_0^\infty t'\psi(t')dt'}. \tag{12}$$

For this relation we consider the case of an inverse power-law waiting-time distribution

$$\psi(t) \sim \frac{1}{t^{1+\gamma}}, \quad \gamma > 0 \tag{13}$$

with

$$1 < \gamma < 2. \tag{14}$$

Inserting Eq. (13) into Eq. (12), the restriction on the index (14) yields

$$\Phi_\xi(t) \sim \frac{A}{t^\beta}, \tag{15}$$

with

$$0 < \beta < 1, \tag{16}$$

since

$$\beta = \gamma - 1. \tag{17}$$

Thus we see that the functional form $\psi(t)$ (13) with the index in the range (14) generates the inverse power-law behavior of $\Phi_\xi(t)$ and hence the breakdown of the condition of a finite microscopic time scale τ (10) for normal diffusion.

The form of the two-point correlation function (15) allows us to rewrite Eq. (9) in the form

$$\frac{\partial \sigma_0(x,t)}{\partial t} = \int_0^t \langle \xi^2\rangle \frac{A}{(t-t')^\beta} \frac{\partial^2 \sigma_0(x,t')}{\partial x^2} dt'. \tag{18}$$

Introducing the Riemann-Liouville fractional derivative in time

$$\frac{\partial^\beta f(t)}{\partial t^\beta} \equiv \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f(t')dt'}{(t-t')^\beta}, \tag{19}$$

we can rewrite Eq. (18) as the fractional diffusion equation

$$\frac{\partial \sigma_0(x,t)}{\partial t} = C \frac{\partial^2}{\partial x^2} \frac{\partial^\beta \sigma_0(x,t)}{\partial t^\beta}, \tag{20}$$

where C is the appropriate collection of constants. Equation (20) was previously obtained by Compte [14] using a CTRW formalism that summarized the research of a number of investigators who had obtained equivalent results for factorable space and time transition probabilities with inverse power-law memory functions, see for example, [15] and [6]. The form of the fractional diffusion equation (20) is not very useful for practical calculations, however, so we now turn our attention to how one actually solves equations expressed in terms of fractional derivatives.

IV. SOLUTIONS TO FRACTIONAL DIFFUSION EQUATIONS

The most direct way to solve fractional diffusion equations is by means of Fox H functions. The Fox functions arise as a consequence of applying Laplace-Mellin transform techniques to fractional operator equations. Let us briefly review this relation following Glöckle and Nonnenmacher [6]. The fractional Riemann-Liouville operator is defined by

$${}_aD_t^{-\mu} f(t) = \int_a^t \frac{(t-t')^{\mu-1}}{\Gamma(\mu)} f(t')dt' \tag{21}$$

for $\mu > 0$, which represents a fractional integration. For $\nu = -\mu > 0$ the fractional differential operator ${}_aD_t^\nu$ is considered to be composed of a fractional integration of the order $n-\nu$ $(n-1 < \nu \leqslant n)$ followed by an ordinary differentiation of the order n, i.e.,

$${}_aD_t^\nu f(t) = \left(\frac{d}{dt}\right)^n {}_aD_t^{\nu-n} f(t). \tag{22}$$

The fractional derivatives and integrals of a Fox function are calculated by formally manipulating the parameters in the H function as

$$\begin{aligned} &{}_0D_z^\nu \left\{ z^\alpha H_{p,q}^{m,n}\left((az)^\beta \middle| \begin{matrix} (a_j, \alpha_j) \\ (b_j, \beta_j) \end{matrix} \right) \right\} \\ &\quad = z^{\alpha-\nu} H_{p+1,q+1}^{m,n+1}\left((az)^\beta \middle| \begin{matrix} (-\alpha,\beta),(a_j,\alpha_j) \\ (b_j,\beta_j),(\nu-\alpha,\beta) \end{matrix} \right) \end{aligned} \tag{23}$$

for arbitrary ν, for $\alpha,\beta > 0$ and $\alpha + \beta \min(b_j/\beta_j) > -1$ $(1 \leqslant j \leqslant m)$. See the Appendix for definitions and some formal properties of H functions. The most important property of the Fox functions for our present purposes has to do with their Laplace transforms and inverse Laplace transforms. Using the notation

$$H(t)=H^{m,n}_{p,q}\left(t\left|\begin{matrix}(a_j,\alpha_j)\\(b_j,\beta_j)\end{matrix}\right.\right), \tag{24}$$

the Laplace transform of this Fox function can be expressed in terms of another Fox function

$$\widetilde{H}(s)=\mathcal{L}H(t)=\frac{1}{s}H^{n+1,m}_{q,p+1}\left(s\left|\begin{matrix}(1-b_j,\beta_j)\\(1,1),(1-a_j,\alpha_j)\end{matrix}\right.\right) \tag{25}$$

for $0\leqslant\mu\leqslant1$ and

$$\widetilde{H}(s)=\frac{1}{s}H^{m,n+1}_{p+1,q}\left(\frac{1}{s}\left|\begin{matrix}(0,1)(a_j,\alpha_j)\\(b_j,\beta_j)\end{matrix}\right.\right) \tag{26}$$

for $\mu\geqslant1$, respectively. The parameter μ is defined in the Appendix. On the other hand, if we are given

$$\widetilde{H}(s)=H^{m,n}_{p,q}\left(s\left|\begin{matrix}(a_j,\alpha_j)\\(b_j,\beta_j)\end{matrix}\right.\right), \tag{27}$$

the inverse Laplace transform is given by

$$H(t)=\mathcal{L}^{-1}\widetilde{H}(s)=\frac{1}{t}H^{n,m}_{q,p+1}\left(t\left|\begin{matrix}(1-b_j,\beta_j)\\(1-a_j,\alpha_j),(1,1)\end{matrix}\right.\right) \tag{28}$$

for $0\leqslant\mu\leqslant1$ and

$$H(t)=\frac{1}{t}H^{m,n}_{p+1,q}\left(\frac{1}{t}\left|\begin{matrix}(a_j,\alpha_j),(0,1)\\(b_j,\beta_j)\end{matrix}\right.\right) \tag{29}$$

for $\mu\geqslant1$, respectively. The relations (25)–(29) hold for $\lambda>0$ and for

$$\max_{1\leqslant j\leqslant n}\mathrm{Re}\left(\frac{a_j-1}{\alpha_j}\right)<\min_{1\leqslant j\leqslant m}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right), \tag{30}$$

where Re denotes the real part of a complex number.

Now consider the Laplace-Fourier transform $(x,t\rightarrow k,s)$ of Eq. (18),

$$\hat{\sigma}_0(k,s)=\frac{1}{s+\widetilde{\Phi}(s)k^2}, \tag{31}$$

where $\widetilde{\Phi}(s)$ is the Laplace transform of the stationary correlation function. For the inverse power law (15) we have

$$\widetilde{\Phi}(s)=A\langle\xi^2\rangle s^{\beta-1}. \tag{32}$$

To make use of the Fox functions we require only the Laplace transform of the probability density. Therefore we consider the inverse Fourier transform of Eq. (31) to obtain

$$\widetilde{\sigma}_0(x,s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikx}dk}{s+\widetilde{\Phi}(s)k^2}. \tag{33}$$

The poles of the integral occurs at $k=i[s\widetilde{\Phi}^{-1}(s)]^{1/2}$ for $x<0$ and at $k=-i[s\widetilde{\Phi}^{-1}(s)]^{1/2}$ for $x>0$, so Eq. (33) is directly integrated to yield

$$\widetilde{\sigma}_0(x,s)=\frac{\exp\{-|x|\sqrt{s\widetilde{\Phi}^{-1}(s)}\}}{2\sqrt{s\widetilde{\Phi}(s)}}, \tag{34}$$

which for the inverse power-law correlation function yields

$$\widetilde{\sigma}_0(x,s)=\frac{s^{H-1}}{2\sqrt{A\langle\xi^2\rangle}}\exp\left\{-\frac{|x|s^H}{\sqrt{A\langle\xi^2\rangle}}\right\}, \tag{35}$$

where

$$H=1-\beta/2. \tag{36}$$

It is a simple matter to express Eq. (35) in terms of the Fox function (see also [12]) and then use the inverse Laplace transform relations to obtain the solution to the fractional diffusion equation (20). We obtain

$$\widetilde{\sigma}_0(x,s)=\frac{1}{\sqrt{4A\langle\xi^2\rangle}H|\overline{x}|^{(H-1)/H}}H^{1,0}_{0,1}\left(|\overline{x}|^{1/H}s\left|\left(\frac{H-1}{H},\frac{1}{H}\right)\right.\right), \tag{37}$$

where $\overline{x}=x/\sqrt{A\langle\xi^2\rangle}$. Since $0<\beta<1$ we have from Eq. (36) that $1/2<H<1$, so that the inverse Laplace transform of Eq. (37) is

$$\sigma_0(x,t)=\frac{1}{\sqrt{4A\langle\xi^2\rangle}Ht^H}H^{1,0}_{1,1}\left(\left[\frac{|x|}{\sqrt{A\langle\xi^2\rangle}t^H}\right]^{1/H}\left|\begin{matrix}(1-H,1)\\(0,1/H)\end{matrix}\right.\right). \tag{38}$$

A closed-form asymptotic expression for Eq. (38), with $|x|^{1/H}/t\gg1$, is

$$\sigma_0(x,t)\cong C\left(\frac{|x|^{2H-1}}{\sqrt{4A\langle\xi^2\rangle}t^H}\right)^{1/(2-2H)}\exp\left\{-(1-H)H^{H/(1-H)}\times\left(\frac{|x|}{\sqrt{A\langle\xi^2\rangle}t^H}\right)^{1/(1-H)}\right\}, \tag{39}$$

where the prefactor C is given by

$$C=\frac{2^{H/(2-2H)}H^{(2H-3/2)/(1-H)}}{2\sqrt{\pi}\sqrt{1/H}\sqrt{1/H-1}}. \tag{40}$$

Both Eqs. (38) and (39) for $H=1/2$ reduce to the familiar Gaussian distribution

$$\sigma_0=\frac{1}{\sqrt{4\pi A\langle\xi^2\rangle t}}\exp\left\{-\frac{x^2}{4A\langle\xi^2\rangle t}\right\}. \tag{41}$$

Note that in Eq. (39) the dimensions of $\sqrt{A\langle\xi^2\rangle}$ are (length)/(time)^H, so that the exponent is dimensionless and the prefactor has dimension of (length)$^{-1}$, as it should for a probability density in one dimension.

In Fig. 2 the probability density is depicted in the scaled variable x/t^H for $x>0$. The dependence of the tail of the

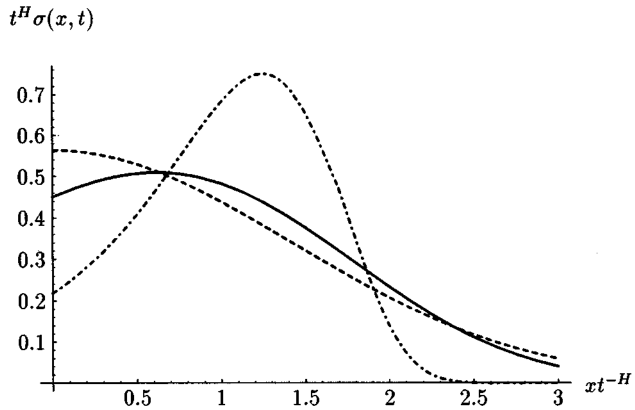

FIG. 2. Probability density $\sigma_0(x,t)$ that is the solution to the fractional diffusion equation (20) for $H=0.5$ (- - -), $H=0.6$ (—), and $H=0.8$ $(-\cdot-)$, plotted in scaled variables.

distribution on H is evident, with the higher H decaying more rapidly. This behavior is more clearly seen in Fig. 3 on a log-log plot. The Gaussian distribution has a rapidly decaying shoulder and is flat in the vicinity of $x=0$, whereas $H>1/2$ gives rise to a hump prior to its decay. The full range of the symmetric distribution is depicted in Fig. 4. The superdiffusive process $H>1/2$ is bimodal with the symmetric peaks revealing the tendency for a walker to continue walking in a direction over and above that determined by pure randomness. In Fig. 5 the peaks in this latter process are seen to separate with increasing time and the distribution flattens out. Note that the apparent cusp in the bimodal distribution is an illusion of scale, the distribution is actually flat in the vicinity of $x=0$, as seen in Fig. 5.

It is also worthwhile to use Eq. (39) to calculate the mean-square displacement of the random walker at time t:

$$\langle x^2(t)\rangle=\int_{-\infty}^{\infty}x^2\sigma_0(x,t)dx, \tag{42}$$

from which we obtain

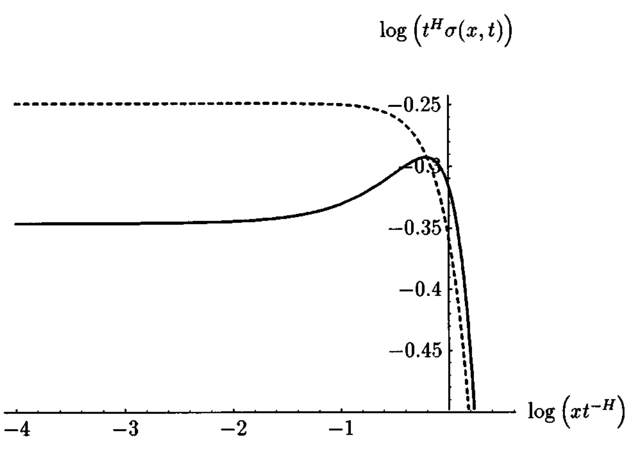

FIG. 3. Probability density $\sigma_0(x,t)$ depicted in Fig. 2 for $H=0.5$ (- - -) and $H=0.6$ (—), replotted on a decadic log-log scale to emphasize the behavior of the distribution in the neighborhood of $x=0$.

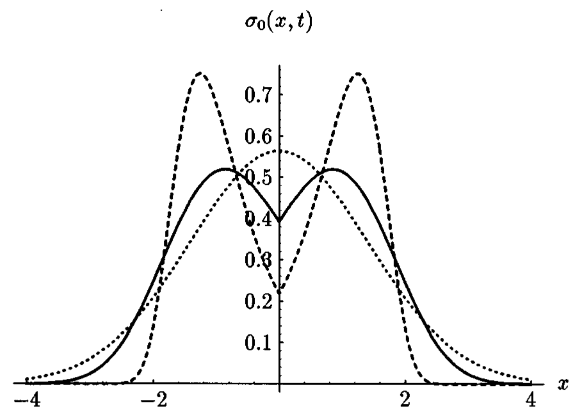

FIG. 4. Full range of the symmetric probability density $\sigma_0(x,t)$, depicted at the single time $t=1$ for $H=0.5$ $(\cdots)$, $H=0.65$ (—), and $H=0.8$ (- - -).

$$\langle x^2(t)\rangle=Kt^{2H}, \tag{43}$$

where K is a constant. Because $1>H\geq 1/2$ we have anomalous diffusion, since from Eq. (36)

$$\langle x^2(t)\rangle=Kt^{2-\beta},\quad 0<\beta<1 \tag{44}$$

so that the process is superdiffusive and non-Gaussian. This same result could, of course, also have been obtained using the Fox function (38) rather than its asymptotic form (39) with a little more work.

V. LÉVY STABLE PROCESSES

It was shown by Zumofen and Klafter [16], using a CTRW, and by Trefán *et al.* [17], using a master equation approach, that if unavoidable dynamical truncations are ignored, the diffusion generated by the correlated, dichotomous process ξ results in a characteristic function for a symmetric Lévy stable process with the Lévy index $\alpha=\gamma$. This means that we are observing an α-stable Lévy process with an index in the interval $1<\alpha\leq 2$. Notice that, in principle, the α-stable Lévy process concerns the wider range

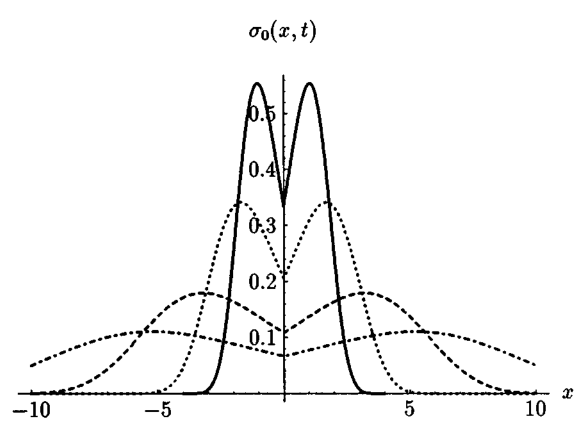

FIG. 5. Full range of the symmetric probability density $\sigma_0(x,t)$, depicted for $H=0.7$ at four times $t=1$ (—), $t=2$ $(\cdots)$, $t=5$ (- - -), and $t=10$ (- · -).

$0<\alpha\leqslant 2$. However, the condition $\alpha\leqslant 1$ refers to processes faster than ballistic diffusion and so is incompatible with the dynamical nature of the process described by Eq. (1). In the preceding section we observed that taking the asymptotic limit in both space and time, such that $|x|/t^H\gg 1$, gives rise to an exponential distribution function. This result is inconsistent with the Lévy distribution obtained earlier, since the latter has an inverse power-law form asymptotically. How can these two results be resolved?

The answer lies in the choice of the correlation function used in Sec. IV. If the two-point correlation function is to have a unit value initially then

$$\Phi_\xi(t)=\frac{A}{(B+t)^\beta}, \tag{45}$$

which is unity at $t=0$ if $A=B^\beta$. The asymptotic form for the correlation function (45) was assumed to be given by Eq. (15) and was used in the fractional diffusion equation (9). Equation (15) does not properly describe the dynamics of the two-state process, however. Note that

$$\Phi_\xi(t-t')=\frac{A}{(B+t-t')^\beta} \tag{46}$$

and t' approaches t at the upper limit of the integral (18) so that B cannot be neglected relative to $(t-t')$ even at very long times. To properly account for the nonzero value of B we change the non-Markovian equation (9) into a Markovian equation using the constraint

$$\sigma_0(x,t-t')=\frac{1}{2W}\int_{-\infty}^{\infty}dx'\,\delta\left(t'-\frac{|x-x'|}{W}\right)\sigma_0(x',t). \tag{47}$$

This constraint implies that the transition time t to the time $(t-t')$ is obtained by assuming that the velocity is kept constant for the whole interval of time t'. This constraint would be violated by ordinary Brownian motion, but is certainly fulfilled by dynamical systems with the correlation function (46) for time intervals of the order of B. For longer time intervals the constraint (47) is violated and its introduction into Eq. (9) turns out to be an approximation. This approximation, however, serves the important purpose of preventing us from overestimating the short-time region of the correlation function and in so doing from introducing a fictitious ''microscopic'' times scale, which is responsible for the results illustrated in Figs. 2–4. Thus the results of Sec. IV are a consequence of overemphasizing the short–time behavior of the approximate correlation function, even in the ''asymptotic'' regime.

Substituting the distribution function (47) into Eq. (9) yields

$$\frac{\partial\sigma_0(x,t-t')}{\partial t}=\frac{1}{2W}\int_0^t dt'\,\Phi_\xi(t')\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\times dx'\,\delta\left(t'-\frac{|x-x'|}{W}\right)\sigma_0(x',t), \tag{48}$$

so that changing the x derivative on the Dirac δ function to a t' derivative and integrating the resulting equation by parts give rise to

$$\frac{\partial\sigma_0(x,t)}{\partial t}=\int_{-\infty}^{\infty}dx'\,\psi\left(\frac{|x-x'|}{W}\right)\sigma_0(x',t), \tag{49}$$

where the kernel is given by

$$\psi(t)=\frac{\langle\xi^2\rangle}{2W^3}\frac{\partial^2\Phi_\xi(t)}{\partial t^2}. \tag{50}$$

Note that here we do not need to assume the relation between the correlation function and the waiting-time distribution function from CTRWs given by Eq. (12). If we now use the inverse power law (45) in Eq. (50), taking cognizance of the fact that $\langle\xi^2\rangle=W^2$, we can rewrite Eq. (49) as

$$\frac{\partial\sigma_0(x,t)}{\partial t}=C\int_{-\infty}^{\infty}dx'\,\frac{\sigma_0(x',t)}{(B+|x-x'|)^{\beta+2}}. \tag{51}$$

Here is where we take advantage of the fact that the time constraint has already been accounted for through Eq. (47), so we can neglect B in Eq. (51). Introducing the parameter $\alpha=\beta+1$ and evaluating the coefficient C to be

$$C=\frac{b}{\pi}\sin\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha+1)=\beta(\beta+1)AW^{\beta+1}, \tag{52}$$

we obtain from Eq. (51)

$$\frac{\partial\sigma_0(x,t)}{\partial t}=\frac{b}{\pi}\sin\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha+1)\int_{-\infty}^{\infty}dx'\,\frac{\sigma_0(x',t)}{|x-x'|^{\alpha+1}}. \tag{53}$$

Seshadri and West [9] established that Eq. (53) is the integro-differential equation describing the evolution of an α-stable Lévy process for $0<\alpha\leqslant 2$.

The Fourier transform of Eq. (53) is given by

$$\frac{\partial\hat{\sigma}_0(k,t)}{\partial t}+b|k|^\alpha\hat{\sigma}_0(k,t)=0, \tag{54}$$

which immediately integrates to an exponential with the initial condition $\hat{\sigma}_0(k,0)=1$. Therefore, in terms of Fox functions we can write

$$\hat{\sigma}_0(k,t)=\exp(-bt|k|^\alpha)=\frac{1}{\alpha}H_{0,1}^{1,0}\left((bt)^{1/\alpha}|k|\,\middle|\,\left(0,\frac{1}{\alpha}\right)\right) \tag{55}$$

and subsequently the inverse Fourier transform of Eq. (55) yields the probability density

$$\sigma_0(x,t)=\frac{\pi}{\alpha|x|}H_{2,2}^{1,1}\left(\frac{|x|}{(bt)^{1/\alpha}}\,\middle|\,\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right)=\sum_{l=1}^{\infty}(-1)^{l+1}\frac{\Gamma(1+l\alpha)}{l!}\sin\left(\frac{l\pi\alpha}{2}\right)\frac{(bt)^l}{|x|^{l\alpha+1}}, \tag{56}$$

the expansion being valid for large argument $|x|/(bt)^{1/\alpha}\gg 1$. A generalization of this series expansion for the Fox function is given by Glöckle and Nonnenmacher [6] and the identical series is given by Montroll and West [2] for α-stable Lévy distributions. It appears that Schneider [10] was the first to realize that Lévy α-stable processes can be expressed in terms of Fox functions.

Note that the asymptotic form of the Lévy α-stable process is an inverse power law

$$\sigma_0(x,t) \approx \frac{bt}{|x|^{\alpha+1}}. \tag{57}$$

This asymptotic behavior is distinct from the exponential from observed in Eq. (39) and is a consequence of maintaining the effect of $B>0$ in the asymptotic limit through the constraint (47).

VI. CONCLUSION

A simple two-state random process with an inverse power-law correlation function was shown to produce a random walk described by an exact fractional diffusion equation. Such equations describe anomalous transport and were shown to have exact solutions in terms of Fox functions. The fractional calculus was shown to present a powerful mathematical method for deriving and solving fractional diffusion equations. Exact analytic solutions to such equations are obtained in terms of Fox functions by using Fourier, Laplace, and Mellin transforms. The property that the Laplace transform of a Fox function is still a Fox function with altered indices enables us to obtain exact solutions to the fractional diffusion equations and ultimately to express the solutions in terms of more familiar special functions.

ACKNOWLEDGMENTS

B.J.W. thanks the U.S. Office of Naval Research for partial support of this work. R.M. and T.F.N. thank the Deutsche Forschungsgemeinschaft (SFB 239, Project No. B10) for financial support.

APPENDIX: FOX FUNCTIONS

Fox's H function is defined by the Mellin-Barnes-type integral

$$H_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} (\underline{a},\underline{\alpha}) \\ (\underline{b},\underline{\beta}) \end{matrix} \right) = \frac{1}{2\pi i}\int_{\mathcal{C}} h(s) z^s ds, \tag{A1}$$

where $(\underline{a},\underline{\alpha}) = (a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)$; $(\underline{b},\underline{\beta}) = (b_1,\beta_1),(b_2,\beta_2),\ldots,(b_q,\beta_q)$; and $h(s)$ is given by

$$h(s) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)\prod_{k=1}^{m}\Gamma(b_k-\beta_k s)}{\prod_{l=m+1}^{q}\Gamma(1-b_l+\beta_l s)\prod_{r=n+1}^{p}\Gamma(a_r-\alpha_r s)}, \tag{A2}$$

where p, q, m, and n are integers satisfying $0\leqslant n\leqslant p$, $1\leqslant m\leqslant q$, and empty products are interpreted as unity; see [18] and [19]. The parameters α_j $(j=1,\ldots,p)$ and β_j $(j=1,\ldots,q)$ are positive numbers and a_j $(j=1,\ldots,p)$ and b_j $(j=1,\ldots,q)$ are complex numbers satisfying

$$\alpha_j(b_l+\nu) \neq \beta_l(a_j-1-\lambda) \tag{A3}$$

for $\nu,\lambda=0,1,\ldots$; $l=1,\ldots,m$; and $j=1,\ldots,n$. Here $\mathcal{C}$ is a contour in the complex s plane separating the poles in such a way that the poles of $\Gamma(b_j-\beta_j s)$ $(j=1,\ldots,m)$ lie to the right and the poles of $\Gamma(1-a_j+\alpha_j s)$ $(j=1,\ldots,n)$ lie to the left of the contour $\mathcal{C}$. The Fox function is an analytic function of z (i) for every $z\neq 0$ if $\mu>0$ and (ii) for $0<|z|<\beta^{-1}$ is $\mu=0$, where

$$\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j \tag{A4}$$

and

$$\beta = \prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j}. \tag{A5}$$

The H function is, in general, multiple valued due to the factor z^s in the integral (A1), but it is single valued on the Riemann surface of ln$z$.

The theorem of residues enables us to express the Fox function as the infinite series

$$H_{p,q}^{m,n}(z) = \sum_{l=1}^{m}\sum_{k=0}^{\infty}\frac{\prod_{j=1,j\neq l}^{m}\Gamma(b_j-\beta_j s_{lk})\prod_{r=1}^{n}\Gamma(1-a_r+\alpha_r s_{lk})}{\prod_{u=m+1}^{q}\Gamma(1-b_u+\beta_u s_{lk})\prod_{v=n+1}^{p}\Gamma(a_v-\alpha_v s_{lk})} \times \frac{(-1)^k}{k!}\frac{z^{s_{lk}}}{\beta_l}, \tag{A6}$$

where $s_{lk}=(b_l+k)/\beta_l$. The prime indicates the product without the factor $j=l$. The formula (A6) can be used for the calculation of special values of Fox functions and to derive the asymptotic behavior for $z\to 0$.

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