IX. Fractional Brownian motion & fractional Langevin equation motion.

(Kolmogorov 1940, Mandelbrot 1968)

$$\frac{dx(t)}{dt} = \xi(t) \quad \text{FBM}$$

The noise $\xi(t)$ is Gaussian distributed, $p(\xi) = \frac{1}{4\pi} \exp\left(-\frac{\xi^2}{4}\right)$

Its correlations decay in power-law fashion, $\langle \xi(t_1)\,\xi(t_2)\rangle \sim \alpha K_\alpha(\alpha-1)|t_1-t_2|^{\alpha-2}$

Here $x(t) = \int_0^t \xi(t')dt'$ is the position of the particle (formal solution).

To calculate the position autocorrelation function we use the autocorrelation of the noise & assume that $t_1 > t_2$:

$$\langle x(t_1)x(t_2)\rangle = \int_0^{t_1} dt' \int_0^{t_2} dt'' \,\langle \xi(t')\,\xi(t'')\rangle$$

$$= \alpha K_\alpha(\alpha-1) \int_0^{t_1} dt' \int_0^{t_2} dt'' \,|t'-t''|^{\alpha-2}$$

$$= \alpha K_\alpha(\alpha-1) \int_0^{t_1} dt' \left\{ \int_0^{t'} (t'-t'')^{\alpha-2} dt'' + \int_{t'}^{t_2} (t''-t')^{\alpha-2} dt'' \right\}$$

$$= \alpha K_\alpha(\alpha-1) \int_0^{t_1} dt' \left\{ \left[-\frac{(t'-t'')^{\alpha-1}}{\alpha-1}\right]_0^{t'} + \left[\frac{(t''-t')^{\alpha-1}}{\alpha-1}\right]_{t'}^{t_2} \right\}$$

$$= \alpha K_\alpha(\alpha-1) \int_0^{t_1} dt' \left\{ \frac{(t')^{\alpha-1}}{\alpha-1} + \frac{|t_2-t'|^{\alpha-1}}{\alpha-1} \right\}$$

$$= \alpha K_\alpha(\alpha-1) \left\{ \int_0^{t_1} \frac{(t')^{\alpha-1}}{\alpha-1} dt' + \int_0^{t_2} \frac{(t_2-t')^{\alpha-1}}{\alpha-1} dt' + \int_{t_2}^{t_1} \frac{(t'-t_2)^{\alpha-1}}{\alpha-1} dt' \right\}$$

$$= K_\alpha\, t_1^\alpha + t_2^\alpha + |t_1-t_2|^\alpha$$