University of Potsdam Institute of Physics and Astronomy
Lecture Stochastic Processes (SS 2018)
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# Problem Set 3

(discussion on may 17th)

**Diffusion in a harmonic potential**

A small particle of mass $m$ diffuses in a medium of viscosity $\eta$ at a temperature $T$. The particle is restricted to one dimensional movement and subjected to a linear restoring force $F(x) = -kx$, e.g. by optical tweezers or by tethering to an elastic polymer.

(a) Write down the Fokker-Planck-Smoluchovsky equation for the probability density function of the particle position. If possible, rescale space and time to reduce the number of parameters.

(b) Use partial integration to derive ordinary differential equations for the first moment $m = \langle x \rangle$ and the variance $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$ of the pdf. Solve the ODEs under the initial conditions $m(t = 0) = x_0$ and $\sigma^2(t = 0) = 0$.

(c) Show either that a Gaussian distribution $p(x, t) = G(x - m, \sigma^2)$ solves the Fokker-Planck-Smoluchovsky equation where $m$ and $\sigma^2$ evolve according to the ODEs derived above, or

(d) find the partial differential equation for the Fourier transform $p(k, t)$ of the pdf and solve it under the initial condition $p(x, t = 0) = \delta(x - x_0)$ by the method of characteristics [wikipedia]. Identify the solution as the Fourier transform of a Gaussian and determine its mean and variance.

(+) Solve the eigenvalue problem (in Fourier space) and find the expansion of the exact solution into these eigenfunctions. Find the real space representations of these eigenfunctions.