Linear Filters and Wavelet Analysis
– lecture script –

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Chapter 1

Filters

Filter are a fundamental tool in data analysis. They are not used to give a specific result, such as, e.g., the fractal dimension, but rather for preprocessing the data to clean it from certain undesired structures, which might distort the results of following algorithms. A typical example of a filter is noise reduction or limiting the signal to a certain frequency bandwidth. In this chapter we give an basic introduction to linear filter theory.

A general approach of filtering is computing the Fourier transform. In Fourier space a signal is represented by fundamental frequencies (sinusoids), where each frequency has an amplitude and a phase. The filtering process is then simply a multiplication of the Fourier transform with some function, which characterizes the filter, and which thus alters the frequency spectrum as desired.

Typically one chooses filters of one of the following categories: lowpass filters, which reduce all frequencies above some cutoff frequency, highpass filters, which reduce all frequencies below some cutoff frequency, bandpass filters, which reduce all frequencies outside a certain frequency band, band reject or band stop filters, which reduce all frequencies within a certain frequency band, and allpass filters, which don't change the magnitude of the spectrum, but only alter the phases (used for example for computing the Hilbert transform).

In quick the Fourier filter approach can be represented like this:

1. given some time series x_n, compute its Fourier transform

$$X(\omega) = \langle e^{i\omega n}, x_n \rangle = \sum_{n=-\infty}^{\infty} e^{-i\omega n} x_n,$$

2. choose a filter function $F(\omega)$ and multiply it with $X(\omega)$ to obtain the filtered signal Fourier transform $Y(\omega) = X(\omega)F(\omega)$,

3. compute the inverse Fourier transform of $Y(\omega)$ to get the filtered time series

$$y_n = \langle e^{-i\omega n}, Y(\omega) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega n} Y(\omega) d\omega$$

Two examples for the filter function $F(\omega)$ are

- lopass filter: $F(\omega) = \begin{cases} 1 & , |\omega| < \omega_{\text{cutoff}} \\ 0 & , |\omega| > \omega_{\text{cutoff}} \end{cases}$

- Hilbert transformer (creates a 90 degree phase shift):

$$F(\omega) = -i\,\mathrm{sign}\,\omega = \begin{cases} i & ,\omega < 0 \\ -i & ,\omega > 0 \end{cases}$$

The approach of computing the Fourier transform has the advantage that any desired filter response can be realized. However, the computation amount is high and the implementaion is not straight forward. Further, this type of filtering is not instantanious, but which might be desired for real-time applications (e.g., in medical applications) or for adaptive filter schemes (e.g. for chaos control). Therefore it is good to know some alternatives of realizing a filtering process.

Let's begin with a simple example. We want to lowpass filter a time series in order to reduce high frequency fluctuations (again: lowpass filtering means keeping low fequencies and eliminating high frequencies). An intuative way of doing so is a gliding mean:

$$y_n = \frac{1}{2K+1}\sum_{k=-K}^{K} x_{k-n}, \tag{1.1}$$

where K is the order of the mean. Here each output sample y_n is the mean of the corresponding input sample x_n and its K neighboring samples to the left and to right in the series. This approach is very simply, but it the disadvantage, that the number of additions needed to get one output sample become very large if the cutoff of this filter is low.

A general discrete linear filter is defined as follows

$$y_n = \sum_{k=0}^{K} b_k x_{k-n} - \sum_{l=1}^{L} a_l y_{l-n}. \tag{1.2}$$

This means a filter can be eralized as a linear combination of the previous filter outputs and the current and previous filter inputs. From a physicist or engineer point of view equation (1.2) resembles a discrete linear dynamical system. Such systems can better be analyzed and quantified by their z-transform. The z-transform of a series x_n is defined as

$$\mathcal{Z}x_n \;=\; X(z) \;:=\; \sum_{k=-\infty}^{\infty} x_n z^{-n}. \tag{1.3}$$

Note that the z-transform along the unit circle $z = e^{i\omega}$ is the discrete Fourier transform

$$\mathcal{F}x_n \;=\; \sum_{k=-\infty}^{\infty} x_n e^{-i\omega n}. \tag{1.4}$$

The z-transform of equation (1.2) is then (skipping the mathematical details of the transformation)

$$Y(z)\sum_{l=0}^{L} a_l z^{-l} \;=\; X(z)\sum_{k=0}^{K} b_k z^{-k}, \tag{1.5}$$

where $Y(z)$ is the z-transform of the filter output, and $X(z)$ is the z-transform of the filter input. The quotient of both is the so called *filter transfer function*

$$H(z) := \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{K} b_k z^{-k}}{\sum_{l=0}^{L} a_l z^{-l}}. \tag{1.6}$$

The inverse z-transform of of $H(z)$ is the so called filter impulse response h_n, i.e. the output of the filter to a single input impulse $\delta_n = \{...0, 0, 0, 1, 0, 0, 0...\}$. The transfer function limited to the unit circle is the filter Fourier transform $H(e^{i\omega})$. This one is used to analyze the filter characteristics. The filter magnitude $A(\omega)$ is the absolute value of the filter Fourier transform. It shows the attenuation of the filter at a certain frequency. The filter phase response $\phi(\omega)$ is the phase of the filter Fourier transform. It specifies by what amount a specific input frequency is delayed by the filter.

$$\begin{aligned} A(\omega) &= |H(\omega)| \\ \phi(\omega) &= \arg H(\omega) = \arctan \frac{\mathtt{im}(H(e^{i\omega}))}{\mathtt{re}(H(e^{i\omega}))} \end{aligned}$$

A filter is designed by choosing a filter order, i.e. choose an K and L in (1.2) and putting constraints on $A(\omega)$ and $\phi(\omega)$ in order to meet the desired filter response, and then calculating the coefficients a_l and b_k in (1.2). This is mainly an engineering research field. Many classical filters and algorithms for filter design have been developed. Some classical filter are for example Butterworth filters and Chebyshev filters which have originally been developed for realization in analogue circuits and have a wide spread application. But their digital counterparts can be very useful, too, in real time signal processing and data analysis.

For more detail on the topic see for example the free online book [1].

Chapter 2

Wavelet Transform

The wavelet transform is just one of several existing time-frequency transforms. All of them have been developed to overcome the drawback of the Fourier transform, namely the lack of time information of the frequency components. There are the windowed Fourier transform, the Gabor transform and the Wigner-Ville-transform. All these different transforms have different advantages and disadvantages. However, the wavelet transform has become the most successful of those, due to its strong theoretical backgound and connections with the theory of frames and filterbanks in the engineering field.

The term *wavelet transform* actually describes two in some aspects different transforms, namely the continous wavelet transform (CWT) and the discrete wavelet transform (DWT). In contrary to the the discrete Fourier transform, where the term *discrete* is usually related to the discrete signal, the DWT is a discrete transform of a continuous signal. The DWT is precisely a special case of the CWT, but with a lot of theoretical consequences, which will also be discussed briefly.

2.1 Continous Wavelet Transform

The Fourier transform is a projection of a signal onto a basis of pure frequencies $e^{i\omega t}$ with an infinit support, i.e. which are spread over the whole time axis. The Basis of the wavelet transform is constructed from frequencies, which are localized in time. This for example can be accomplished by taking a pure frequency and multiply it with some envelope function (e.g. the Morlet wavelet is constructed by multiplying with a gauss bell). That basis wavelet is then shifted in time and rescaled in order to obtain the basis for all time- and frequency-positions.

The continous wavelet transform $\mathcal{W}_\Psi x$ of a signal $x(t)$ is given by the inner product

$$\mathcal{W}_\Psi x\,(s,\tau) \;:=\; \langle \psi_{s\tau}, s\rangle \;=\; \int\limits_{-\infty}^{\infty} \psi^*_{s\tau}(t)\,x(t)\;dt \tag{2.1}$$

with a wavelet $\psi_{s\tau}$ from a wavelet family

$$\Psi \;=\; \left\{\psi_{s\tau}(t) = \tfrac{1}{s}\psi\left(\tfrac{t-\tau}{s}\right) \;\mid\; s \in \mathbb{R}_+, \tau \in \mathbb{R}\right\} \tag{2.2}$$

of a by s dilated τ translated mother wavelet $\psi(t)$.

Now, what does this mean in detail. Equation (2.1) together with (2.2) can be rewritten as a convolution

$$\mathcal{W}_{\Psi}x\,(s,\tau) \;=\; \left(\bar{\psi} * x\right)(s,\tau) \;=\; \frac{1}{s}\int\limits_{-\infty}^{\infty} \bar{\psi}^{*}\left(\frac{\tau - t}{s}\right)\, x(t)\; dt, \tag{2.3}$$

where $\bar{\psi}(t) = \psi(-t)$ is the reversed wavelet. In this sence it is clear that the wavelet transform can be seen as a filter bank, where the reversed wavelet is the filter impulse response. But what kind of filter is the wavelet? To answer this question we have to take a closer look on the wavelet ψ itself.

In order for $\psi(t)$ to be a wavelet its Fourier transform $\hat{\psi}(t)$ has to satisfy the admissibility condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega \;<\; \infty, \tag{2.4}$$

from which follows that the wavelet has a bandpass filter characteristic, i.e.

$$\hat{\psi}(\omega = 0) \;=\; 0 \quad \text{and} \quad \hat{\psi}(\omega = \infty) \;=\; 0.$$

Further follows from that, that at least the first moment of the wavelet vanishes:

$$\int_{-\infty}^{\infty} \psi(t)\; dt = 0,$$

which also assures that the wavelet has at least one oscillation and therefore deserves being called a *wavelet*.

Beside of (2.4) there are no other constraints to the wavelet $\psi(t)$. Thus, it can be a real valued or complex valued function. However, complex valued wavelets, precisely analytic wavelets, i.e., wavelets $\psi(t)$ which Fourier transform has only positive frequencies, have the nice feature of yielding an instantanious amplitude and phase.

2.1.1 Wavelet transform of some exemplary signals

Here are some CWTs of some examplary time series. The frequency is the inverse of the scale s in the CWT. All plots use a logarithmic frequency/scale axis.

- **Morlet/Gabor Wavelet**:

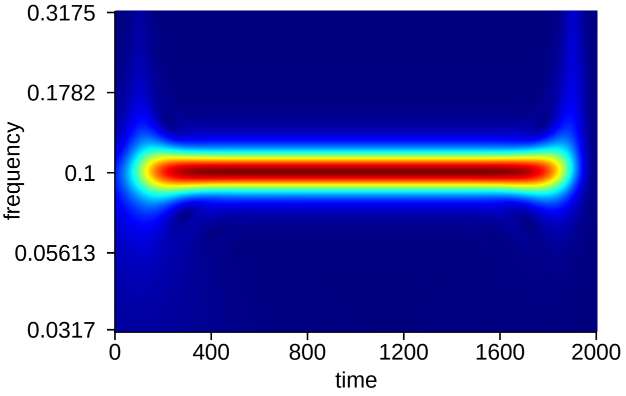

Figure 2.1: CWT of a pure sinusoidal: $\sin(0.1t)$

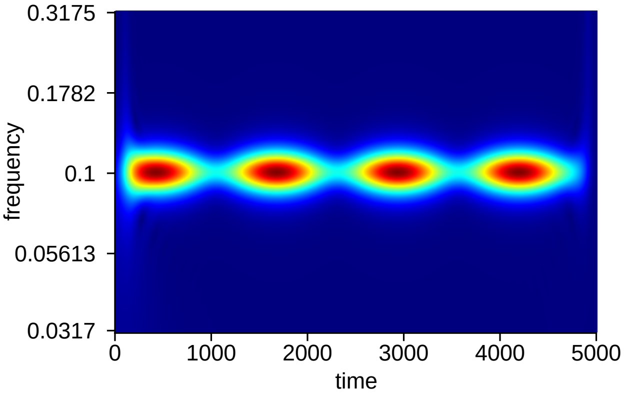

Figure 2.2: CWT of an amplitude modulated sinusoidal: $\sin(0.1t)(1 + 0.5\sin(0.005t))$

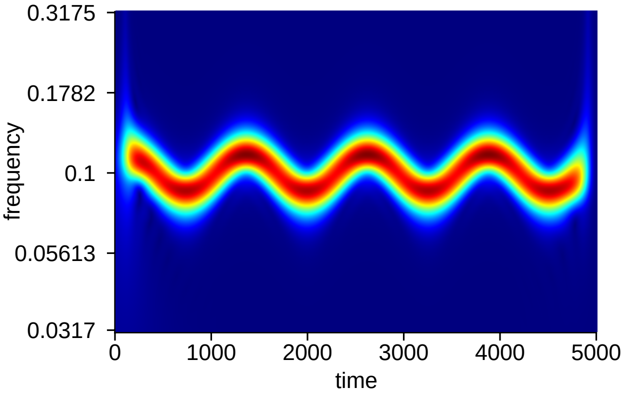

Figure 2.3: CWT of an frequency modulated sinusoidal: $\sin(0.1t + 3\sin(0.005t))$

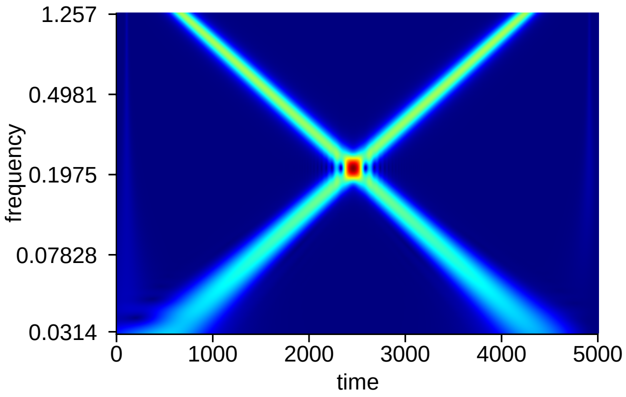

Figure 2.4: CWT of two logarithmic chirps (sinusoidals with raising or falling frequency): $\sin(\exp(7.7-0.001t))+\sin(\exp(3+0.001t))$

2.1.2 List of Wavelets

The following list of wavelets can in some sence be seen as a list of classical wavelets, which also appear a lot in literature. They all have the nice feature of being expressed by an closed mathematical expression and a therefore well suited for theoretical analysis, too.

- **Morlet/Gabor Wavelet**:

$$\psi(t) = e^{it}e^{-t^2/a}$$

The parameter a controls the width of the wavelet and thus the compromise between good time and good frequency localization.

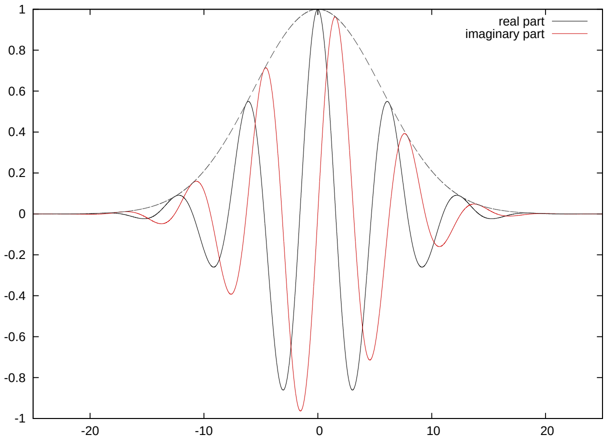

Figure 2.5: Morlet Wavelet for $a = 64$.

- The **Mexican Hat Wavelet** is the normalized second deriavative of the

Gaussian function:

$$\psi(t) = \frac{1}{\sqrt{2\pi}s^3}\left(1 - \frac{t^2}{s^2}\right) e^{\frac{-t^2}{2s^2}}.$$

It is just a special case of a family of wavelets which are also called Gaussain wavelets, and which are the n'th derivative of the Gaussian function.

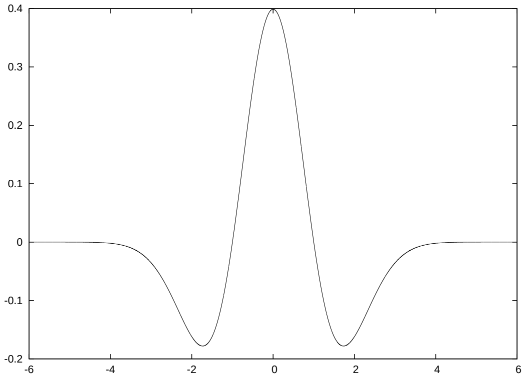

Figure 2.6: Maxican Hat Wavelet for $s = 1$.

- The **Cauch-Paul Wavelet** is a nice analytic wavelet and is usually defined by its Fourier transform

$$\hat{\psi}(t) = \begin{cases} \omega^\alpha e^{-\omega} & ,\omega > 0 \\ 0 & ,\omega < 0 \end{cases}$$

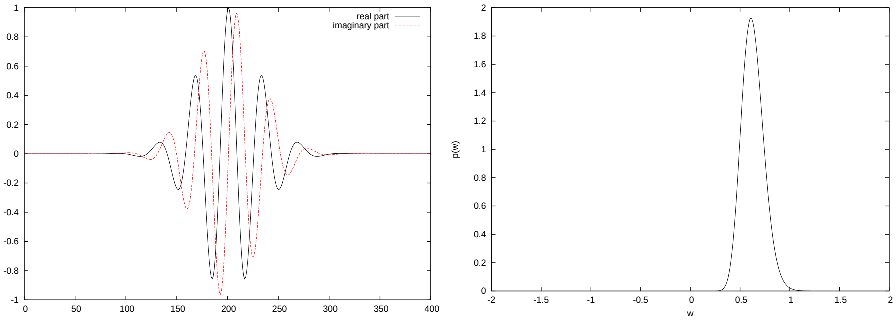

Figure 2.7: The Cauchy Paul Wavelet (left) and its Fourier transform (right) for $\alpha = 30$.

2.1.3 Inversion of the CWT

The CWT is invertible and the inversion formula is

$$\mathcal{M}_\Phi w(s,\tau) = \frac{1}{C_{\psi\phi}} \int_0^\infty \frac{ds}{s} \int_{-\infty}^\infty d\tau \, \frac{1}{s}\phi\left(\frac{\tau - t}{s}\right) w(s,\tau). \tag{2.5}$$

The reconstruction wavelet $\phi(t)$ must not be the same as the analysis wavelet $\psi(t)$. The only constraints are that the constant $C^{+}_{\psi\phi}$ and $C^{-}_{\psi\phi}$ are equal and finit. They are defined as

$$C^{\pm}_{\psi\phi} = \int\limits_{0}^{\infty} \frac{\hat{\bar{\phi}}(\pm\omega)\hat{\psi}(\pm\omega)}{\omega} d\omega.$$

For practical use the reconstruction wavelet $\phi(t)$ can be the dirac delta function $\delta(t)$, even though it is not a wavelet in a strict sence. In that case the reconstruction formula (2.5) reduces to

$$\mathcal{M}_{\Phi} w(s,\tau) \;=\; \frac{1}{C_{\psi}} \int\limits_{0}^{\infty} \frac{ds}{s} w(s,\tau). \tag{2.6}$$

This is intuitive quite clear. Since the wavelet transform can be interpreted as a bandpass filterbank, the above reconstruction formula represents the mere summing up of the indivdual channels of the filterbank.

2.1.4 The reproducing kernel

An interesting question that comes up is, if every possible function $w(s,\tau)$ can be a wavelet transform. To make the answer short: No! This can be seen if we take the inverse transform and then again the wavelet transform. If the result is the same function w again, it is a wavelet transform. So, a function $w(s,\tau)$ is a wavelet transform if

$$w(s,\tau) = \mathcal{W}_{\Psi} \mathcal{M}_{\Psi} w(s,\tau).$$

The expression $\mathcal{W}_{\Psi} \mathcal{M}_{\Psi} w$ can be seen as a special kind of convolution with the so called reproducing kernel $\mathcal{K}_{\psi}(s,\tau)$:

$$\mathcal{W}_{\Psi} \mathcal{M}_{\Psi} w(s,\tau) \;=\; \int\limits_{0}^{\infty} \frac{ds'}{s'} \int\limits_{-\infty}^{\infty} d\tau' \frac{1}{s'} \mathcal{K}_{\psi}\!\left(\frac{s}{s'}, \frac{\tau-\tau'}{s'}\right) w(s,\tau), \tag{2.7}$$

where the reproducing kernel $\mathcal{K}_{\psi}(s,\tau)$ is the wavelet transform of the wavelet itself:

$$\mathcal{K}_{\psi}(s,\tau) \;=\; \int\limits_{-\infty}^{\infty} \frac{1}{s} \psi\left(\frac{t-\tau}{s}\right) \psi(t) dt. \tag{2.8}$$

The reproducing kernel also symbolizes the intrinsic uncertainty of the wavelet transform. The variances in time σ_{τ} and in scale σ_{s} of the reproducing kernel obey the well known Heisenberg principle of uncertainty, thus they cannot be both arbitrary small at the same time. Their product $\sigma_{\tau}\sigma_{s}$ is bound by a positive constant.

2.2 Discrete Wavelet Transform

The DWT has been developed in the 1980ies mainly by the Belgian Phyisicist Ingrid Daubieches. The CWT has the disadvantage is the huge amount of redundancy. This becomes quite clear if one consider a time series of, say, N samples, then the CWT has N times S samples, where S is the number of scales. Obviously the amount of information in the SN samples of the CWT cannot be more then the amount of information of the N original samples. Therefore there must be a way to reduce the data amount of the wavelet transform. The solution of the DWT is to sample the whole CWT only on discrete points. Usually those point lie on the so called dyadic grid, that are the points $(s, \tau) = (2^m, n)$. Daubieches showed that sampling the CWT only on that dyadic grid, all information to reconstruct the original signal is preserved, if the analysis and synthesis wavelets form an orthogonal or biorthogonal basis system.

This discovery started the wavelet boom in the 80ies. Many techniques have been devoloped by now, such as the fast wavelet transform and many algorithms for signal compression, noise reduction, and feature detection of signals and images, as well.

Bibliography

[1] Steve Smith *The Scientist and Engineer's Guide to Digital Signal Processsing*, `http://www.dspguide.com`.

[2] M. Holschneider, *Wavelets: an analysis tool*, Oxford University Press, 1995.

[3] S. Mallat, *a Wavelet Tour of signal processing*, Academic Press, 1998.