DYNAMO IN ASYMMETRIC SQUARE CONVECTION

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Abstract

Linear and nonlinear dynamo action is investigated for square patterns in nonrotating and weakly rotating Boussinesq Rayleigh-Bénard convection in a plane horizontal layer. The square-pattern solutions may or may not be symmetric to up-down reflections. Vertically symmetric solutions correspond to checkerboard patterns. They do not possess a net kinetic helicity and are found to be incapable of kinematic dynamo action at least up to magnetic Reynolds numbers of $\approx 12000$. There also exist vertically asymmetric squares, characterized by rising (descending) motion in the centers and descending (rising) motion near the boundaries, among them such that possess full horizontal square symmetry and others lacking also this symmetry. The flows lacking both the vertical and horizontal symmetries possess kinetic helicity and show kinematic dynamo action even without rotation. The generated magnetic fields are concentrated in vertically oriented filamentary structures. Without rotation these dynamos are, however, always only kinematic, not nonlinear dynamos since the back-reaction of the magnetic field then forces the solution into the basin of attraction of a roll pattern incapable of dynamo action. But with rotation added parameter regions are found where stationary asymmetric squares are also nonlinear dynamos. These nonlinear dynamos are characterized by a subtle balance between the Coriolis and Lorentz forces. In some parameter regions also nonlinear dynamos with flows in the form of oscillating squares or stationary modulated rolls are found.

Keywords: Magnetohydrodynamics; Convection; Dynamo

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1 Introduction

The purpose of dynamo theory (Roberts and Soward, 1992), concerned with the generation of magnetic fields by the motion of electrically conducting fluids, is to explain the origin of the cosmical magnetic fields. In most cases,
like for instance those of the Earth and the Sun, the generating fluid motions are caused by heating, that is to say, the dynamos are driven by thermal convection. Studies of convection-driven dynamos have concentrated either on turbulent convection (Brandenburg et al., 1996) or on convection near onset, where simple steady flows can be obtained (Soward, 1974; Fautrelle and Childress, 1982; Zheligovsky and Galloway, 1998; Matthews, 1999). In this paper we report on the dynamo properties of convection in the simple form of squares. A preliminary account of the results presented here was published in Demircan and Seehafer (2001a).

The typical convective patterns are different for convection with up-down reflection symmetry and such lacking this symmetry, where symmetry of the convection means symmetry of the governing equations and boundary conditions for the deviations of the physical quantities from their values in the nonconvective state. Rayleigh-Bénard Boussinesq convection with symmetric top and bottom boundary conditions possesses the up-down reflection symmetry, and its preferred convection pattern near onset is rolls, i.e. the convective pattern is also up-down symmetric. However, recently it was found both experimentally and theoretically that other, asymmetric states, namely convection in the form of squares or hexagons, can coexist with the roll states in a parameter range where only rolls were previously known to be stable (Assenheimer and Steinberg, 1996; Clever and Busse, 1996; Busse and Clever, 1998; Demircan and Seehafer, 2001b). These asymmetric squares and hexagons, with rising or with descending motion in the center (and descending or rising motion near the boundary) are usually observed in convection lacking up-down reflection symmetry, namely in compressible convection (Matthews et al., 1995; Rucklidge et al., 2000), in fluids with strongly temperature dependent viscosity (Oliver and Booker, 1983) or in Bénard-Marangoni convection (Eckert et al., 1998; Schatz et al., 1999). To avoid confusion we note that by an asymmetric pattern we here simply mean one that lacks up-down reflection symmetry, following Busse and Clever (1998). In this sense an asymmetric square may or may not possess horizontal square ($D_4$) symmetry. Also, for instance, hexagon solutions formed by the superposition of three roll modes are asymmetric even if the roll amplitudes are equal [which is a necessary condition for the stability of the hexagon solutions in near-onset convection in systems lacking vertical symmetry (Soward, 1985)]. The asymmetric square pattern represents the dominating pattern over a wide range of the control parameters both in vertically symmetric and nonsymmetric convection. Details about this type of convection can be found in Proctor and Matthews (1996), Busse and Clever (1998) and Demircan and Seehafer (2001b). In the present paper we deal with squares in Boussinesq, i.e. vertically symmetric convection. Here besides the asymmetric square patterns also vertically symmetric ones are found, which appear in the form of checkerboard patterns. However, with respect to the dynamo effect the asymmetric squares turn out to be much more interesting than the symmetric ones.

It is well known that a nonvanishing kinetic helicity, for a given volume
defined by $H = \int_V \mathbf{v} \cdot \nabla \times \mathbf{v} \, d^3x$, where $\mathbf{v}$ denotes the fluid velocity and $h = \mathbf{v} \cdot \nabla \times \mathbf{v}$ is the helicity density, is favorable for the large-scale dynamo action of small-scale velocity fields. This is a major result of mean-field dynamo theory (Krause and Rädler, 1980). Here we consider small-scale dynamos, in which the magnetic field and the velocity field vary on comparable scales. Different from mean-field theory, no averaging is applied. The relation between small-scale dynamo action and the kinetic helicity of the underlying flow is not clear yet. Several strongly helical flows are known to be very dynamo effective. Intensively studied examples are the Arnold-Beltrami-Childress (ABC) flows (Arnold, 1965; Arnold and Korkina, 1983; Galloway and Frisch, 1986). These flows can be produced as steady solutions of the incompressible Navier-Stokes equations if an external force field of the ABC type is applied. Feudel et al. (1995) studied the full incompressible magnetohydrodynamic equations with a generalized ABC forcing for which the degree of helicity in the force field (and thus in the generated flow) can be varied by varying a parameter. It was found that, for increasing strength of the forcing, the primary bifurcation from the non-magnetic steady basic flow leads to a dynamo if the degree of helicity in the forcing exceeds a threshold value and to non-magnetic secondary flows for helicities below the threshold value. Thus at least for certain flow families, there is a correlation between helicity and small-scale dynamo action. On the other hand, helicity is certainly not necessary for a dynamo. So Hughes et al. (1996) found dynamo action for flows with vanishing net helicity $H$ or even identically vanishing helicity density $h$. An important question then is how convective flows behave in this respect.

We find different types of asymmetric square solutions, among them such with flows possessing a net helicity even in the absence of rotation and being kinematic dynamos (for which the back-reaction of the magnetic field on the velocity is neglected) in the absence as well as in the presence of rotation. Introducing rotation about a vertical axis increases the amount of helicity and the kinematic dynamo properties are slightly changed. Taking into account the full interaction of magnetic field and motion, the problem of self-extinguishing arises. Namely, the back-reaction of the magnetic perturbation can force the solution into the basin of attraction of another hydrodynamic state which is incapable of sustaining the magnetic field against Ohmic dissipation. This kind of behavior has recently been found by other authors (Brummell et al., 1998; Fuchs et al., 1999; Matthews, 1999). Our simulations show, however, that there is a range of rotation rates where the Coriolis force counteracts the Lorentz force in such a way that the dynamo action of the flow is preserved in the nonlinear regime. Thus it is found that asymmetric square patterns can act as convection-driven linear and nonlinear dynamos.
We consider buoyancy-driven rotating convection in an electrically conducting plane fluid layer of thickness $d$ heated from below. Using the Oberbeck-Boussinesq approximation, the governing system of partial differential equations reads as follows:

$$\nabla \cdot v = 0$$ (1)

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + P \nabla^2 v + PR \theta e_z + (\nabla \times B) \times B + P\sqrt{T} v \times e$$ (2)

$$\nabla \cdot B = 0$$ (3)

$$\frac{\partial B}{\partial t} + (v \cdot \nabla)B = PP_m^{-1} \nabla^2 B + (B \cdot \nabla)v$$ (4)

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = v_z + \nabla^2 \theta.$$ (5)

Here $B$ is the magnetic field and $p$ and $\theta$ represent the deviations of pressure and temperature from their values in the pure conduction state. We use Cartesian coordinates $x$, $y$ and $z$ with the $z$ axis in the vertical direction parallel to the gravitational force. $e_z$ is the unit vector in the vertical direction whereas the vector $e$ is the general notation for the unit vector in the direction of the rotation axis. For our special choice $e = e_z$ one has $v \times e = (v_y, -v_x, 0)$ in Eq. (2). Equations (1)–(5) are given in dimensionless form where the units of length and time are $d$ and $d^2/\kappa$, respectively, with $\kappa$ being the thermal diffusivity. $\theta$ is measured in units of the temperature difference $\delta T$ between the lower and upper boundaries of the fluid layer and $p$ and $B$ in units of $\rho\kappa^2/d^2$ and $\sqrt{\mu_0 \rho \kappa / d}$, respectively, where $\rho$ is the mass density and $\mu_0$ the vacuum magnetic permeability. There are four dimensionless parameters, the Prandtl number $P$, the magnetic Prandtl number $P_m$, the Rayleigh number $R$ and the Taylor number $T$, defined by

$$P = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{\eta}, \quad R = \frac{\alpha gd^3}{\nu\kappa} \delta T, \quad T = \left(\frac{2\Omega d^2}{\nu}\right)^2,$$ (6)

where $\nu$ is the kinematic viscosity, $\eta$ the magnetic diffusivity, $\alpha$ the volumetric expansion coefficient, $\Omega$ the angular velocity of the rotation and $g$ the gravitational acceleration. The Rayleigh number $R$ measures the strength of the buoyancy forces. We apply periodic boundary conditions with spatial period $L$ in the horizontal directions $x$ and $y$. The top and bottom planes are assumed to be stress-free, isothermal and impenetrable for matter and electromagnetic energy:

$$\frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = v_z = \theta = \frac{\partial B_x}{\partial z} = \frac{\partial B_y}{\partial z} = B_z = 0 \quad \text{at} \ z = 0, 1$$ (7)
For these boundary conditions the following Fourier expansions are appropriate:

\[ v_x = \sum_{n_x, n_y = -\infty}^{\infty} \sum_{n_z = 0}^{\infty} \tilde{v}_x(n) e^{i k_x x + i k_y y} \cos k_z z \]  

(8)

\[ v_y = \sum_{n_x, n_y = -\infty}^{\infty} \sum_{n_z = 0}^{\infty} \tilde{v}_y(n) e^{i k_x x + i k_y y} \cos k_z z \]  

(9)

\[ v_z = \sum_{n_x, n_y = -\infty}^{\infty} \sum_{n_z = 0}^{\infty} \tilde{v}_z(n) e^{i k_x x + i k_y y} \sin k_z z \]  

(10)

\[ \theta = \sum_{n_x, n_y = -\infty}^{\infty} \sum_{n_z = 0}^{\infty} \tilde{\theta}(n) e^{i k_x x + i k_y y} \sin k_z z \]  

(11)

\[ p = \sum_{n_x, n_y = -\infty}^{\infty} \sum_{n_z = 0}^{\infty} \tilde{p}(n) e^{i k_x x + i k_y y} \cos k_z z \]  

(12)

For the magnetic field components \( B_x, B_y \) and \( B_z \) expansions fully analogous to those for \( v_x, v_y \) and \( v_z \) are used. The wave number vectors \( \mathbf{k} = (k_x, k_y, k_z) \) are connected with the integer mode number vectors \( \mathbf{n} = (n_x, n_y, n_z) \) by

\[ k_x = n_x \frac{2\pi}{L}, \quad n_x = 0, \pm 1, \pm 2, \ldots \]  

(13)

\[ k_y = n_y \frac{2\pi}{L}, \quad n_y = 0, \pm 1, \pm 2, \ldots \]  

(14)

\[ k_z = n_z \pi, \quad n_z = 0, 1, 2, 3, \ldots \]  

(15)

As in Scheel and Seehafer (1997), Demircan et al. (2000) and Demircan and Seehafer (2001b) we restrict ourselves to the case of a vanishing mean horizontal flow since such a flow can be removed by a Galilean transformation (see also Sec. 4.2). In our numerics we used a pseudospectral method with a spatial resolution of \( 32^3 \) points for simulations and \( 16^3 \) points for non-simulative eigenvalue and eigenvector calculations. The main results were checked by additional simulations at a resolution of \( 64 \times 64 \times 16 \). Time integration was performed using an eighth-order Runge-Kutta scheme as described in Hairer et al. (1993). The aspect ratio is kept fixed at a value of \( L = 4 \) and the Prandtl number is 6.8. The Taylor number, measuring the rotation rate, is restricted to values below the critical one for the Küppers-Lortz (Küppers and Lortz, 1969) instability [the instability of convection rolls with respect to rolls rotated by a certain finite angle relative to the original rolls; the resulting dynamics is dominated by heteroclinic cycles formed by unstable roll states and connections between them (Busse and Heikes, 1980; Demircan et al., 2000)].
3 Convection in the Form of Asymmetric Squares

Without rotation, flows in the form of symmetric square or checkerboard patterns could be observed as transient phenomena but never as stable stationary patterns. By contrast, asymmetric-square solutions were found as stable stationary attractors. Examples of convection in the form of symmetric and asymmetric squares in the absence of rotation are shown in Fig. 1. The asymmetric squares [Fig. 1(b)] are a secondary convection pattern and appear via the skewed-varicose (Busse and Clever, 1979) instability of primary convection rolls; details are described in Demircan and Seehafer (2001b). Depending on the initial conditions, cells with rising or descending motion in the center appear. The spectrum of the excited Fourier modes shows that the asymmetric squares can be represented to lowest order by

\[
\left( A_1 e^{i k_1 x} + A_2 e^{i k_2 x} \right) + \left( B_1 e^{i (k_1 + k_2) x} + B_2 e^{i (k_1 - k_2) x} \right) + \text{c.c.} \tag{16}
\]

where \( k_1 \) and \( k_2 \) are horizontal wave vectors given by \( k_1 = (0, k) \) and \( k_2 = (k, 0) \) with \( k = 2\pi / L \), which is the fundamental wave number of the asymmetric squares (their side length is \( L \)). \( A_1 \exp(i k_1 x) \) and \( A_2 \exp(i k_2 x) \) represent two rolls with the same wave number \( k \), one parallel to the \( x \) axis and the other parallel to the \( y \) axis, while the two terms with coefficients \( B_1 \) and \( B_2 \), respectively,
correspond to rolls parallel to the diagonals of the periodicity square, perpendicular to each other and with the same wave number \( q = |k_1 + k_2| = \sqrt{2}k \), which is the wave number of the skewed-varicose unstable rolls (the instability thus leads to a pattern with a smaller wave number). For asymmetric squares as shown in Fig. 1(b) to appear it is essential that all four wave vectors \( k_1, k_2, k_1 + k_2 \) and \( k_1 - k_2 \) are excited [i.e. all four coefficients \( A_1, A_2, B_1 \) and \( B_2 \) in Eq. (16) must be different from zero]. The wave numbers \( k \) and \( q \) are in resonance through triadic interactions of these wave vectors (Silber and Skeldon, 1999). A representation like Eq. (16) was used in Proctor and Matthews (1996) to study square cells in non-Boussinesq convection near onset and is contained in a more general Galerkin ansatz used in Busse and Clever (1998) to study asymmetric squares in Boussinesq convection. Asymmetric squares were also found numerically in compressible magnetoconvection near onset (Matthews et al., 1995).

One can also think of the asymmetric squares as the superposition of two checkerboard patterns. Namely, each of the two parenthesized summands in Eq. (16) represents a checkerboard or symmetric square pattern as shown in Fig. 1(a). The sum of the rolls \( A_1 \exp(ik_1 x) \) and \( A_2 \exp(ik_2 x) \) represents a checkerboard pattern with wave number \( k \) [just as shown in Fig. 1(a)]. Similarly, the superposition of the two rolls with coefficients \( B_1 \) and \( B_2 \), respectively, is a checkerboard pattern with wave number \( q \) which is rotated by \( 45^\circ \) with respect to the checkerboard pattern with wave number \( k \). The sum of the two checkerboard patterns with wave numbers \( k \) and \( q \), respectively, gives the asymmetric square pattern shown in Fig. 1(b). A major difference between the checkerboard and asymmetric square solutions is that the latter ones require the excitation of two different wave numbers (\( k \) and \( q \)) and their nonlinear resonance, while the checkerboards are “linear” squares with only one wave number excited.

Without rotation, the checkerboard-pattern solutions are symmetric to reflections in vertical planes parallel to one of the sides or diagonals of a square. The symmetry to reflections in vertical planes implies zero net helicity (since helicity is a pseudoscalar and thus changes sign under reflections). We find the checkerboard-pattern solutions to be always unstable in the nonrotating case.

The asymmetric squares bifurcate from the thin rolls with wave number \( q \) if the Rayleigh number \( R \) is raised above a critical value. Following the path backwards towards lower values of \( R \), a back transition from stable asymmetric squares to stable rolls with wave number \( q \) is observed, but the stable squares can be traced to Rayleigh numbers below that at which they appear for increasing \( R \). There is a bistable region where both the thin rolls and the asymmetric squares are stable. The Rayleigh number of the back transition from squares to rolls is well above the critical one for the onset of convection \( (R = 657.5) \). Thus the asymmetric squares are not a primary convection branch. The hysteretic behavior, i.e. the coexistence of stable thin rolls and asymmetric squares, indicates that the bifurcation of the squares from the rolls is subcritical; for details see Demircan and Seehafer (2001b). Further-
more, stable thick rolls with wave number $k$ are found to coexist with stable asymmetric squares, in particular for Rayleigh numbers above that where the thin rolls with wave number $q$ become skewed-varicose unstable. Simulations for these supercritical Rayleigh numbers, starting from a superposition of the unstable rolls with a small perturbation, lead, depending on the perturbation added to the rolls, either to asymmetric squares or to rolls of wave number $k$ as final states.

In the case of $T = 0$, the square solutions obtained may or may not possess horizontal $D_4$ symmetry (the dihedral group $D_4$ contains all rotations and reflections which transform a square in a plane into itself). The square pattern shown in Fig. 1(b) corresponds to a solution lacking the $D_4$ symmetry; however, shadowgraph images of the two types of solutions are nearly indistinguishable. Differences are better seen in Fig. 2, showing for examples of the

![Figure 2: Vectors of the horizontal velocity in the midplane $z = 0.5$ for $T = 0$ and $R = 7500$. (a) $D_4$ symmetric solution, (b) solution lacking the $D_4$ symmetry.](image)

two cases vectors of the horizontal velocity in the horizontal midplane. For the solution depicted in Fig. 2(b), which corresponds to the nonsymmetric case, quite obviously the symmetries to rotations by $\pi/2$ and $3\pi/2$ about the vertical axis through the centers and to reflections in the lines joining the midpoints of opposite sides of the squares are broken. It is less obvious from this figure whether the symmetries to reflections in the diagonal lines and to rotations by $\pi$ are broken as well. This can be clarified by examining the Fourier coefficients of the velocity and temperature fields. The symmetry transformations that form the group $D_4$ and their consequences for the Fourier coefficients (in the case that the solution is symmetric to the respective transformation) are
the following:

\[ S_1 : (x, y, z) \mapsto (-x, y, z) \]
\[ (v_x, v_y, v_z, \theta) \mapsto (-v_x, v_y, v_z, \theta) \]
\[ \tilde{v}_x(i, j, k) = -\tilde{v}_x(-i, j, k) \]
\[ \tilde{v}_y(i, j, k) = \tilde{v}_y(-i, j, k) \]
\[ \tilde{v}_z(i, j, k) = \tilde{v}_z(-i, j, k) \]
\[ \tilde{\theta}(i, j, k) = \tilde{\theta}(-i, j, k) \] (17)

\[ S_2 : (x, y, z) \mapsto (x, -y, z) \]
\[ (v_x, v_y, v_z, \theta) \mapsto (v_x, -v_y, v_z, \theta) \]
\[ \tilde{v}_x(i, j, k) = \tilde{v}_x(i, -j, k) \]
\[ \tilde{v}_y(i, j, k) = -\tilde{v}_y(i, -j, k) \]
\[ \tilde{v}_z(i, j, k) = \tilde{v}_z(i, -j, k) \]
\[ \tilde{\theta}(i, j, k) = \tilde{\theta}(i, -j, k) \] (18)

\[ S_3 : (x, y, z) \mapsto (y, x, z) \]
\[ (v_x, v_y, v_z, \theta) \mapsto (v_y, v_x, v_z, \theta) \]
\[ \tilde{v}_x(i, j, k) = \tilde{v}_y(j, i, k) \]
\[ \tilde{v}_y(i, j, k) = \tilde{v}_x(j, i, k) \]
\[ \tilde{v}_z(i, j, k) = \tilde{v}_z(j, i, k) \]
\[ \tilde{\theta}(i, j, k) = \tilde{\theta}(j, i, k) \] (19)

\[ S_4 : (x, y, z) \mapsto (-y, -x, z) \]
\[ (v_x, v_y, v_z, \theta) \mapsto (-v_y, -v_x, v_z, \theta) \]
\[ \tilde{v}_x(i, j, k) = -\tilde{v}_y(-j, -i, k) \]
\[ \tilde{v}_y(i, j, k) = -\tilde{v}_x(-j, -i, k) \]
\[ \tilde{v}_z(i, j, k) = \tilde{v}_z(-j, -i, k) \]
\[ \tilde{\theta}(i, j, k) = \tilde{\theta}(-j, -i, k) \] (20)

\[ R_1 : (x, y, z) \mapsto (-y, x, z) \]
\[ (v_x, v_y, v_z, \theta) \mapsto (-v_y, v_x, v_z, \theta) \]
\[ \tilde{v}_x(i, j, k) = -\tilde{v}_y(-j, i, k) \]
\[ \tilde{v}_y(i, j, k) = \tilde{v}_x(-j, i, k) \]
\[ \tilde{v}_z(i, j, k) = \tilde{v}_z(-j, i, k) \]
\[ \tilde{\theta}(i, j, k) = \tilde{\theta}(-j, i, k) \] (21)
\[ R_2 : \quad (x, y, z) \mapsto (-x, -y, z) \]
\[ (v_x, v_y, v_z, \theta) \mapsto (-v_x, -v_y, v_z, \theta) \]
\[ \tilde{v}_x(i, j, k) = -\tilde{v}_x(-i, -j, k) \]
\[ \tilde{v}_y(i, j, k) = -\tilde{v}_y(-i, -j, k) \]
\[ \tilde{v}_z(i, j, k) = \tilde{v}_z(-i, -j, k) \]
\[ \tilde{\theta}(i, j, k) = \tilde{\theta}(-i, -j, k) \] \hfill (22)

\[ R_3 : \quad (x, y, z) \mapsto (y, -x, z) \]
\[ (v_x, v_y, v_z, \theta) \mapsto (v_y, -v_x, v_z, \theta) \]
\[ \tilde{v}_x(i, j, k) = \tilde{v}_y(j, -i, k) \]
\[ \tilde{v}_y(i, j, k) = -\tilde{v}_x(j, -i, k) \]
\[ \tilde{v}_z(i, j, k) = \tilde{v}_z(j, -i, k) \]
\[ \tilde{\theta}(i, j, k) = \tilde{\theta}(j, -i, k) \] \hfill (23)

\( S_1 \) and \( S_2 \) are reflections in the lines joining the midpoints of opposite sides of a square, \( S_3 \) and \( S_4 \) reflections in the diagonal lines and \( R_1 \), \( R_2 \) and \( R_3 \) rotations by \( \pi/2 \), \( \pi \) and \( 3\pi/2 \). We find that these symmetries are either all present (for the \( D_4 \) symmetric solutions) or all broken (for the solutions without the \( D_4 \) symmetry). For the \( D_4 \) symmetric solutions one has \( A_1 = A_2 \) and \( B_1 = B_2 \) in Eq. (16). Like for the checkerboard-pattern solutions, the symmetry to reflections in vertical planes then implies zero net helicity.

For the solutions lacking the horizontal \( D_4 \) symmetry at \( T = 0 \) there is no reflection symmetry that would prohibit a nonzero net helicity, and such a net helicity is indeed found: Fig. 3(a) shows the helicity of a nonrotating upflow square as a function of the Rayleigh number in the range where the flow is stationary. The solutions possess a net helicity even in the absence of rotation. If rotation at low rates about the vertical axis is added, the pattern is modified but still corresponds to asymmetric squares. Compared to the case without rotation, the stability boundary towards higher values of \( R \), where the pattern loses stability to oscillatory solutions (Demircan and Seehafer, 2001b), is shifted upwards. In Fig. 3(b) the helicity as a function of the Rayleigh number for a case with rotation (\( T = 100 \)) is shown. The helicity due to rotation is significantly larger than the “self-helicity” of the nonrotating squares already for very low rotation rates. In addition, stable rotating squares can be traced to higher values of the Rayleigh number where the helicity is by several orders of magnitude larger than for the nonrotating squares.
Figure 3: The helicity of an upflow square as a function of the Rayleigh number for (a) $T = 0$ and (b) $T = 100$.

4 Dynamo Action in Asymmetric Square Convection

4.1 Kinematic Dynamo

Our primary convection solutions are stationary and correspond either to rolls or to checkerboard patterns. Checkerboard patterns are observed for Taylor numbers around 225 (cf. Demircan et al., 2000). Through solving the magnetic induction equation, the primary roll and checkerboard-pattern solutions were checked for kinematic dynamo action in the (small) Rayleigh number interval close to the onset of convection where they are stable ($R \approx 1000 \ldots 2000$). The checkerboard-pattern solutions were additionally checked in the regions close to the onset of convection where they are unstable but the roll solutions are stable — the symmetric squares were then constructed as superpositions of two rolls with the same wave number and axes perpendicular to each other (cf. Sec. 3). The net helicity in the periodic box vanishes for both types of solutions, even if $T \neq 0$ [although for $T \neq 0$ each single roll or square (vertical upflow or downflow column with square cross section) has a nonvanishing helicity]. We always find the two primary convection states to be incapable of kinematic dynamo action. This partially results from our choice of the parameters, with in particular the rotation rate being restricted to values below the critical one for the Küppers-Lortz instability — for studies of dynamos in rapidly rotating plane layers see St. Pierre (1993) and Jones and Roberts (2000). The kinematic dynamo properties of the two flows were determined up to magnetic
Prandtl numbers $P_m = 5000$. The associated magnetic Reynolds numbers, according to Eq. (4) given by $R_m = v_{rms}L/(PP_m^{-1})$, where $v_{rms}$ is the root mean square velocity of the respective solution and $L = 4$, then increases to values $R_m \approx 12000$. No kinematic dynamo action was found. This does not exclude, of course, dynamo action at higher magnetic Reynolds numbers. We also note that a magnetic Reynolds number of 12000 is perhaps a little too high for our numerical resolution (though we did not see any indication of numerical errors).

Similarly, we find the asymmetric-square solutions with horizontal $D_4$ symmetry (existing only in the absence of rotation) to be incapable of kinematic dynamo action. This was tested for magnetic Reynolds numbers up to $R_m \approx 13000$ (at $R = 8000$ we have a flow for which $R_m \approx 13000$ if $P_m \approx 100$).

The convection flows in the form of asymmetric squares without horizontal $D_4$ symmetry, however, can act as kinematic dynamos even without rotation. In Fig. 4 results for the nonrotating case and for $T = 100$ are given. The

![Figure 4: Stability boundary for the kinematic dynamo instability in the $P_m-R$ plane. The dashed line corresponds to the nonrotating case and the continuous line to $T = 100$.](image)

two curves in the $P_m-R$ plane are stability boundaries where a single real eigenvalue becomes positive and the kinematic dynamo starts. The magnitude of the helicity does not seem to be the most crucial factor for the onset of the kinematic dynamo, though after onset the dynamo growth rates increase much faster with $R$ if rotation is present. For small Rayleigh numbers, $R < \approx 5000$, the dynamo sets in at lower values for the magnetic Prandtl number without rotation than with rotation. This can be explained by the fact that with rotation the convection is still very weak here since rotation increases the critical Rayleigh number for the onset of convection.

An example of the magnetic field generated by kinematic dynamo action is depicted in Figs. 5 and 6. The field is concentrated in filamentary struc-
Figure 5: Unstable magnetic eigenmode for the velocity field shown in Fig. 1(b) and $P_m = 5.5$. The isosurface of the magnetic field strength at 50% of the peak field is displayed.

Figure 6: Shadowgraph image of the vertical component $B_z$ of the unstable magnetic eigenmode for the velocity field shown in Fig. 1(b) and $P_m = 5.5$. The values in the horizontal midplane are displayed, bright areas indicating positive values. In addition the null line of the vertical velocity component $v_z$ in the midplane is shown.
tures which are aligned along the vertical axis (Fig. 5). Furthermore, the flux concentrations line out wavy stripes (bright and dark patches in Fig. 6) a distance $L$ apart and situated close to cell boundaries of the velocity field. The maximum magnetic energy density is found in the vicinity of the horizontal midplane. Remarkable in Fig. 6 is the left-right asymmetry of the magnetic field, seen e.g. by comparing the magnetic structure with the location of the null line of the vertical velocity component also shown in the figure. This is a further indication that broken reflection symmetry is essential for the underlying dynamo mechanism.

### 4.2 Nonlinear Dynamo

Fig. 7(a) shows the time evolutions of magnetic and kinetic energies starting from a square pattern velocity field and a small seed magnetic field for the case without rotation. Initially the magnetic field grows exponentially with a well defined growth rate. In this kinematic phase the Lorentz force is negligible and the square pattern remains undisturbed. Now one would in general expect a saturation of the exponential growth of the magnetic field and a modification of the velocity field in such a way that the magnetic field is maintained at a finite amplitude. However, after the magnetic perturbation has reached a strength sufficient to influence the flow, it forces the solution into the basin of attraction.
of the two-dimensional roll state with wave number $k$. The roll solution is incapable of dynamo action and the magnetic field decays to zero. There is no parameter region where this effect is suppressed. With higher growth rates (larger $R$) the nonlinear regime is reached earlier and the relaxation to a purely hydrodynamic state is accelerated. This effect of self-extinguishing of the dynamo by the back-reaction of the magnetic field was recently also observed for flow in triply periodic Cartesian geometry driven by an explicit forcing (Brummell et al., 1998), spherical dynamo models with rotation and explicit forcing (Fuchs et al., 1999) and two-dimensional convection rolls in a plane layer rotating about an oblique axis (Matthews, 1999).

That is to say, in the nonrotating case the asymmetric squares are only kinematic, not nonlinear dynamos. Nonlinear dynamo action is only possible if additional effects are included that counteract the self-extinguishing of the dynamo by the Lorentz force. We add background rotation at very low rates, namely $0 \leq T \leq 150$. For these small Taylor numbers the asymmetric square solutions are hydrodynamically stable, that is to say, the nonrotating squares can be continued on a stable solution branch towards higher Taylor numbers. Although the mechanism underlying the self-extinguishing is still acting, there are parameter ranges where a nonlinear dynamo is found. Time evolutions of kinetic and magnetic energies in such a case, with $T = 10$, $R = 7000$ and $P_m = 4.65$, are given in Fig. 7(b). After the initial kinematic phase, a back reaction of the magnetic field is clearly visible. But though the velocity field is modified, it still corresponds to an asymmetric square pattern. The magnetic field saturates and is maintained for all time. The corresponding time-asymptotic velocity and magnetic structures are shown in Figs. 8 and 9. Like in the kinematic case without rotation (Figs. 5 and 6), the magnetic field is concentrated in filaments aligned along the vertical axis, but now all these flux concentrations are located in the vicinity of the vertices where four velocity cells meet, while in the nonrotating kinematic case only half of the magnetic field concentrations are close to the vertices (see Fig. 6). The difference is obviously due to influence of rotation (and not due to nonlinear effects) since with rotation the kinematically generated magnetic fields (not shown) look similar to the nonlinear ones (Figs. 8 and 9). The whole phenomenon of field concentration is probably a result of flux expulsion from convective eddies (Weiss, 1966; Galloway et al., 1977; Moffatt, 1978).

Filamentary or ‘cigar-like’ structures were consequently also observed by Jones and Roberts (2000), who studied the plane-layer dynamo for the case of rapid rotation (and rigid-wall boundary conditions at the top and bottom). The dominating velocity modes are the same as in our case, namely rolls with horizontal wave vectors $(1, 0)$, $(0, 1)$, $(1, -1)$, $(1, 1) \times (2\pi/L)$, but the flow is in general chaotic due to irregular Kipppers-Lortz transitions between different roll states. The orientation of the flux ‘cigars’ observed by Jones and Roberts seems to be mainly horizontal (rather than vertical as in our case) and at a given instant of time to be correlated with a dominating roll mode of the velocity field (the magnetic field structures thus rotate irregularly like the rolls).
Figure 8: Isosurface of the magnetic field strength at 50% of the peak field for $R = 7000$, $T = 10$ and $P_m = 4.65$.

Figure 9: Shadowgraph images of the vertical velocity component $v_z$ (a) and the vertical magnetic field component $B_z$ (b) in the time-asymptotic state for $R = 7000$, $T = 10$ and $P_m = 4.65$. The values in the horizontal midplane are shown, bright areas indicating positive values. The line in (b) represents the null line of $v_z$. 
Also, the dynamo seems to act mainly during the transition periods between the (approximate) roll states. The detailed mode interactions responsible for the dynamo effect will thus be different in the asymmetric-square and Küppers-Lortz dynamos, which may explain e.g. different orientations of magnetic flux ropes.

On the other hand, a horizontal structure becomes also visible in Fig. 9(b) where the vertical magnetic field component for our asymmetric-square dynamo is shown. This structure corresponds to rolls which lie under 45° in the box. A tendency to enhance this roll mode is also seen for the velocity field [Fig. 9(a)]. Different from these oblique rolls with wavelength $L/\sqrt{2}$, the wavy stripe structures outlined by the magnetic flux in the nonrotating case (cf. Fig. 6) have wavelength $L$. The decrease from $L$ to $L/\sqrt{2}$ in wavelength is in accordance with the general observation that rotation tends to reduce the effective length scale of convective structures.

It seems that for our system and in the parameter range studied, non-linear dynamo action requires a subtle balance between the Coriolis and Lorentz forces. A similar balance between these two forces characterizes the weak-field limit of the Childress-Soward dynamo (Childress and Soward, 1972; Soward, 1974), which however works in a rapidly rotating convective layer. Plus symbols (+) in Fig. 10 mark the parameter range in the $T$-$P_m$ plane where we observe nonlinear dynamos with underlying stationary asymmetric square patterns. The Rayleigh number is fixed at $R = 7000$. By simultaneously varying $T$ and $P_m$ we also find magnetic attractors which differ from the stationary

![Figure 10: Parameter range of nonlinear dynamo action in the $T$-$P_m$ plane for $R = 7000$. Plus symbols (+) refer to dynamos with flows in the form of stationary asymmetric squares. Dynamos in oscillating squares are denoted by diamonds ($\diamond$) and such in stationary rolls by triangles ($\triangle$).]
squares. Varying $T$ corresponds to a variation of the Coriolis force, while varying $P_m$ in general amounts to a variation of the magnetic field strength and thus of the Lorentz force; however, the relation between $P_m$ and the Lorentz force is not simple because also the mechanism by which the dynamo saturates is important. The additional types of magnetic attractors are oscillating squares [indicated by diamonds (♦) in Fig. 10] and stationary rolls [indicated by triangles (△) in Fig. 10]. The stationary magnetic rolls are obtained in simulations starting from an asymmetric-square solution for the flow with a small seed magnetic field added. They show a modulation along the roll axis and disappear if the magnetic field is switched off. The solution then falls back on the simple roll state (without modulation) with wave number $k$, which is not capable of kinematic dynamo action.

In the non-marked regions of the $P_m$-$T$ plane self-extinguishing leads to nonmagnetic final states. However, long integration times may be needed to end in the magnetic-field-free purely hydrodynamic state (see Fig. 11). During

![Figure 11](image)

Figure 11: Time evolutions of kinetic and magnetic energies for $R = 7000$ and (a) $T = 16$, $P_m = 5.1$ and (b) $T = 15$, $P_m = 6$. Time is measured in units of the thermal diffusion time.

these long transient times a dynamo-like behavior can be observed. A further interesting effect is demonstrated in Fig. 11(b). Here the back-reaction of the kinematically amplified magnetic perturbation on an asymmetric-square pattern is leading to a modulated roll pattern which is also capable of kinematic dynamo action. The magnetic field grows again exponentially until it is reaching a sufficient strength for the self-extinguishing effect to act again, which brings the flow close to an asymmetric-square pattern. This switching between different states can take place several times until finally the system
settles into a solution which is not capable of dynamo action.

Our investigations were focused on small Taylor numbers $T < 150$. For higher Taylor numbers the convection is governed by alternating rolls (Demir- 
can et al., 2000) and we observe dynamo properties similar to those found by 
Jones and Roberts (2000).

An interesting question is whether mean flows or magnetic fields exist, 
defined by taking vertical or horizontal averages:

$$\mathbf{v}_v(x, y) = \int_0^1 \mathbf{v} \, dz, \quad \mathbf{v}_h(z) = \int_{\text{periodicity box}} \mathbf{v} \, dx \, dy$$

$$\mathbf{B}_v(x, y) = \int_0^1 \mathbf{B} \, dz, \quad \mathbf{B}_h(z) = \int_{\text{periodicity box}} \mathbf{B} \, dx \, dy$$

We first note that the total means of $\mathbf{v}$ and $\mathbf{B}$, obtained by averaging both 
vertically and horizontally (i.e. the modes with wave vector $k = 0$), vanish. 
For $\mathbf{v}_z$ and $B_z$ this follows immediately from the boundary conditions, Eq. (7). Furthermore, for our boundary conditions the total means of the horizontal 
parts of $\mathbf{v}$ and $\mathbf{B}$ are independent of time (see Seehafer et al., 1996). Consequently a magnetic field with a nonvanishing total mean cannot develop from 
a weak seed field [such a field can develop, however, for insulating top and bot-
tom boundaries (Jones and Roberts, 2000)]. Now all our magnetic attractors 
were found in simulations starting from nonmagnetic states with a small seed 
magnetic field added or from saturated states (obtained in this way) with de-
tuned parameters ($T$, $R$, $P_m$). Therefore only final states with vanishing total 
means of the magnetic field could be reached. In the numerical calculations 
the magnetic mode with wave vector $k = 0$ was set equal to zero explicitly.

The total mean of the horizontal flow, on the other hand, has been set equal 
to zero explicitly since it can be removed by a Galilean transformation. With 
respect to the more general mean fields defined in Eq. (24) we make the fol-
lowing observations:

(i) For all solutions studied (stationary or temporally periodic) $\mathbf{v}_v = 0$, no 
matter whether there is a magnetic field or not.

(ii) Similarly, always $\mathbf{B}_v = 0$ for the dynamo generated (stationary or tempo-
rally periodic) magnetic fields.

(iii) Always $\mathbf{B}_h \neq 0$ for the dynamo generated magnetic fields.

(iv) $\mathbf{v}_h = 0$ in general; $\mathbf{v}_h \neq 0$ only for flows that would not exist in the ab-

sence of (the dynamo generated) magnetic fields, among them the oscillating 
squares and the modulated rolls.

Here the observation (iii) appears most interesting, since $\mathbf{B}_h$ is a large-scale 
field. In an asymptotic analysis of dynamo action in rapidly rotating convec-
tion Soward (1974) found temporally periodic dynamos with $\mathbf{B}_h \neq 0$. Under 
the assumption of rapid rotation there are two length scales in the system 
(since the horizontal scale of the convective motions near onset is then much 
smaller than the thickness of the convective layer), and the dynamos found by 
Soward are large-scale rather than small-scale dynamos. Provided the flows
and the magnetic fields of our study remain the same if the aspect ratio $L$ is increased to large values (test calculations with aspect ratios up to 10 — see also Demircan and Seehafer (2001b) — indicate that this could be the case), our small-scale dynamos are mean-field or large-scale dynamos as well.

5 Conclusion

We have studied the dynamo properties of square patterns in Boussinesq Rayleigh-Bénard convection in a plane horizontal layer. Cases without rotation and with weak rotation about a vertical axis were considered, particular attention being paid to the relation between dynamo action and the kinetic helicity of the flows. While the fluid layer is symmetric with respect to up-down reflections, the square-pattern solutions may or may not possess this vertical symmetry. Vertically symmetric solutions appear in the form of checkerboard patterns. They do not possess a net kinetic helicity and we find them to be incapable of dynamo action (at least up to magnetic Reynolds numbers of $\approx 12000$). But there also exist square-pattern solutions lacking the vertical symmetry and being characterized by rising (descending) motion in the center and descending (rising) motion near the boundary of the squares. The vertically asymmetric solutions can in turn be divided into different types distinguished by their horizontal symmetry in the case of no rotation, namely such with the full horizontal $D_4$ symmetry of a square and others lacking this symmetry (with rotation added the horizontal $D_4$ symmetry is always broken). The solutions lacking the $D_4$ symmetry are particularly interesting in that they possess kinetic helicity and show kinematic dynamo action without rotation. The generated magnetic fields are concentrated in vertically oriented filamentary structures. The dynamos found in the nonrotating case are, however, always only kinematic, never nonlinear dynamos. Nonlinearly the back-reaction of the magnetic field then forces the flow into the basin of attraction of a roll-pattern solution incapable of dynamo action. But with rotation added parameter regions are found where stationary asymmetric squares are also nonlinear dynamos. These nonlinear dynamos are characterized by a subtle balance between the Coriolis and Lorentz forces. In some parameter regions this balance also leads to nonlinear dynamos with flows in the form of oscillating squares or stationary modulated rolls.

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References


