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Universal Pareto laws in agent-based exchange models: debt and varying initial-money distributions

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Abstract. We examine by Monte-Carlo simulations the behavior of a kinetic exchange-trading model for various initial distributions of money in the system of agents. Our goal is to analyze the characteristics of the Pareto laws for the long-time money distribution, in both closed and open systems. We consider three different initial distributions for these two situations. We first briefly summarize the concepts and results of some agent-based money-exchange models. Then, via employing the Monte-Carlo computer simulations, for both types of systems we obtain the long-time money distributions for the initially homogeneous or constant, for positive random, and finally, for both positive and negative random distributions of money among the agents. We conclude that the Pareto laws and their exponents remained nearly the same in all these situations showing little sensitivity to the initial conditions imposed.

1 Introduction

The distribution of income or wealth in a society or country often follows the Pareto law [1], that is used to quantify the degree of inequality [2–4]. As first observed empirically by Pareto in 1897, the higher end of the distribution of incomes $f(x)$ follows a power law [1,5–7]

$$f(x) \sim x^{-\alpha-1} \tag{1}$$

with the scaling exponent estimated to be $\alpha + 1 \approx 3/2$. Nearly the same scaling exponents were observed by Pareto for Italian cities, England, several German states, and Prussia as well as for ancient Peru and the Cherokee Indians [7]. For the last hundred years, this value of the Pareto exponent changed only slightly, both in time and across various capitalistic economies (see Ref. [8] for comparison to, e.g., Islamic economies). In 1931 Gibrat [9] clarified that the Pareto law is valid only for the high-income range. In some detailed empirical studies, it was demonstrated that the income distributions obey a log-normal law for $x \to 0$ and a power law for large x values [10,11].

The Pareto principle—as found by him for the distribution of land among Italian land-owners and stating that 80% of wealth is accumulated in the hands of 20% of the population—has since then been observed for a number of economic- and sales-related situations. Examples include, e.g., the splitting costs in the wealth-care spending, the distribution of cases in the criminal records, etc. The empirical and theoretical studies of inequality in different societies have become prolific in recent years [12–26].[1]

In addition to a flourishing diversity of recent data-based studies of inequality, some examinations in terms of various agent-based money-exchange statistical-mechanics models have also delivered a number of interesting results [6,26,31–43]. Importantly, for models of money-exchanging agents with random saving propensities the Pareto exponent $\alpha = 1$ was concluded in Ref.

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[1] Inequality is likely to be growing in recent decades due to, i.a., formation of huge corporate conglomerates, overall globalization of economies, rosy tax-evading schemes for "big global players", as well as due to other tricks invented by capitalistic magnates via, e.g., their political involvements and lobbying of appropriately "designed" laws. A 80/20 Pareto-type principle of wealth splitting and the steepness of the firm-size distribution [27,28] is likely to be even more drastic in modern times. Known as the Matthew effect from biblical times, a pronounced inequality *per se* is a rather negative factor for a long-term economic development and sustainable growth [29,30]: it can namely destabilize the "foundation" of the wealth-distribution pyramid and eradicate the middle-class, the core of many established capitalistic systems.

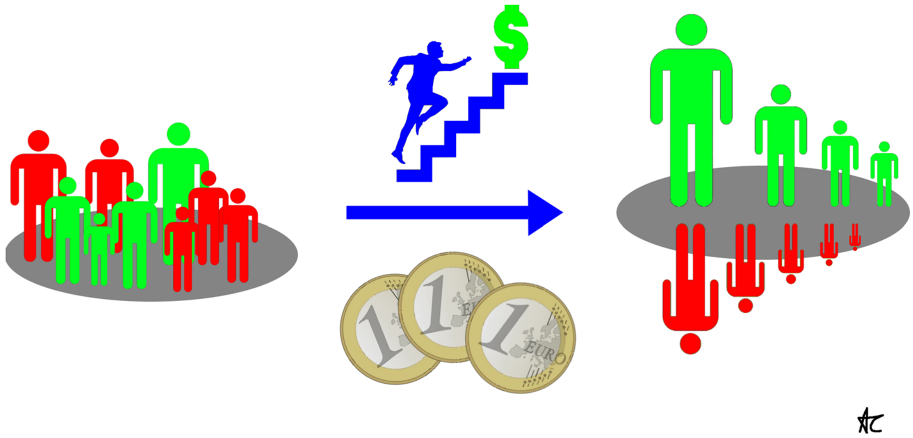

Fig. 1 Schematic illustration of money redistribution in a society of people (ensemble of traders) with initially nonequal amounts of money, as established via multiple money-exchange operations. The size of the agents/figures reflects the amount of their money. The red and green colors indicate, respectively, the presence and the absence of debt. Courtesy http://www.pixabay.com for the source images and to Alexey A. Cherstvy for preparing the artwork

[33]. In many economies, based on real data, the Pareto exponent in Eq. (1) is however varying and is located between $\alpha = 0$ and $\alpha = 2$ [13,20].

Previously, for example, in Refs. [44,45] we analyzed the Pareto law for an open system of agents corresponding to a non-conservative economic model. By adding "traps" into the model, such a closed conservative system is being transformed into an open one. In Ref. [44] we demonstrated that the Pareto exponents remained *unchanged* for different densities of traps in the system. Additionally, in Ref. [45] we have shown that *different* Pareto exponents can be generated by using position-exchange dynamics of agents in both closed and open systems.

Most agent-based money-exchange models assume that all agents start initially with a positive and equally distributed amount of money [46]. A model with "debt" was also developed [47], but its Pareto exponents were not discussed in detail there. The concepts of negative money and debt for money-exchange agent-based models were discussed in Refs. [48–50]. Unlike models, in real economic systems, everyone participates in the "game" with *different* amounts of money, and some agents may be in debt. In third-world countries, e.g., the majority of poor people are in debt of some kind. For this category of people, the initial amount of money is thus negative [51]. As an example, economic crashes [52–54] can cause some sort of wealth "resetting" [55–58], acting as a Pyrrhos war on the vitality of a community.

Here, we consider different distributions of money for both closed and open systems of agents in order to examine their Pareto distributions, see Fig. 1. Via simulating the kinetic agent-based exchange model, we obtain the Pareto distributions for different specific scenarios and conditions in both types of systems. The Pareto exponents are found to be universal.

The structure of the paper is as follows. In Sect. 2 we briefly recapitulate the main concepts of the agent-based money-exchange models. We provide the underlying money-exchange algorithms and the definition of the money distributions. In Sect. 3 we consider a closed conservative exchange system of agents and examine its Pareto distributions. In Sect. 4 we present the results for the Pareto distributions for an open non-conservative system for various initial money distributions, the main results of this study. Finally, we list and discuss the conclusions in Sect. 5.

2 Money-exchange models and distributions

A statistical model of a closed money-exchange system—that is analogous to the kinetic model of ideal gases—can be solved exactly or simulated numerically. In such a model, N agents exchange pairwise a measure x that can be denoted as wealth or money (we use below the term "money"). The state of the nth agent is characterized by its amount of money, x_n, where $n = \{1, 2, \ldots, N\}$, while the total money is conserved, $X = \sum_{n=1}^{N} x_n = \text{const}$, as the energy in a system of colliding gas molecules.

The evolution of the money distribution follows the prescribed trading rules between the agents. This evolution can be interpreted as a Monte-Carlo optimization procedure aimed at finding an equilibrium distribution (if it exists). At every time step, two agents i and j are chosen randomly from the system and an amount of money Δx_{ij} is exchanged between them,

$$x_i' = x_i - \Delta x_{ij}, \quad x_j' = x_j + \Delta x_{ij}. \tag{2}$$

Note that the quantity $x_i' + x_j' = x_i + x_j$ is conserved in every transaction. We examine an interconnected system in which only two neighboring agents interact at each time-step, but the pairs of these agents are chosen *randomly* on the lattice of agents for each exchanges of money. Therefore, effectively long-ranged interactions are present in this system.

Several exchange lattice-gas models exist in the literature, such as the basic models without saving [47,59–61], as well as with a constant global [10,62], with random [63–65], and fixed [66] saving factor propensities. Note that the models with diversified saving parameters were introduced in Refs. [65,67,68] to generalize the model of Refs. [63,64].

In the current study, we use a random saving factor λ_i distributed in the range $0 < \lambda_i < 1$. We use the following money-exchange rules

$$x_i' = \lambda_i x_i + \epsilon_{ij} \left[(1-\lambda_i)x_i + (1-\lambda_j)x_j\right], \\ x_j' = \lambda_j x_j + (1-\epsilon_{ij}) \left[(1-\lambda_i)x_i + (1-\lambda_j)x_j\right], \quad (3)$$

with

$$\Delta x_{ij} = (1-\epsilon_{ij})\,(1-\lambda_i)\,x_i - \epsilon_{ij}\,(1-\lambda_j)\,x_j, \quad (4)$$

where the stochastic parameter ϵ_{ij} is distributed uniformly on the interval [0,1]. The dynamics of money in *all* the scenarios (without and with traps, positive and negative money, etc.) follow these equations. In the presence of traps, no equilibrium can be reached because the money is not conserved: we then are mainly interested in the long-time stationary value of the Pareto exponent.

A debt represents former financial obligations of a given agent rendering the amount of its money negative (e.g. in real world, a credit taken in a bank to cover a bankruptcy, a property exchanged for a loan, etc.). The traps [44] perform here the very same way for positive and negative money of an agent interacting with it: in both cases the money is annihilated upon a contact with a trap. In a model with traps, the amount of money was shown to decay in time according to a stretched exponential function [44].

The values of λ_i are generated on each lattice site (or agent) for each realization of the system. These values stay constant in all the exchanges taking place within a given simulation run. For the next run, a new distribution of λ_i over all N lattice sites is created, yielding a statistical ensemble for later averaging of the results. In contrast, the values of ϵ_{ij}—which are also random in the model—are generated new for each money exchange taking place between a given pair of two agents on the lattice.

Note that as $\lambda \to 1$ the relaxation time in the system gets progressively longer. Thus, likely longer simulation times would be required to check if the tail of the money distribution has reached the stationarity. Note that the latter for a general non-conservative system is meant here as the constancy of the Pareto exponent in the tail of the distribution.

Note that, in addition to the stationary *money* distribution, a *class* distribution [69] can also be considered [44,45]. Specifically, the amount of money in the interval $[x, x+\Delta x]$ for a conservative system with the probability distribution $p(x)$,

$$f(x) = \int_x^{x+\Delta x} x' p(x') \mathrm{d}x', \quad (5)$$

corresponds to a certain amount of agents with this amount of money. The class distribution is the distribution of fractions of a population possessing money in the interval $[x, x+\Delta x]$.

3 Closed conservative systems

3.1 Parameters of computer simulations

We now consider a 1D-model to simulate money exchanges between the traders. To keep the simulation time manageable on a single PC, we set—if not specified otherwise—the lattice size to $L = L_0 = 100$, the number of realizations to $R = R_0 = 1000$, and the total simulation time to $t = t_0 = 1000$ steps, with each exchange taking place with the time-step of $\Delta t = 1$. In all the simulations, both for closed and open systems, we use the exchange model (3) with a random saving propensity $0 < \lambda < 1$. The computer simulations with different initial conditions are performed in Sects. 3 and 4.

3.2 Homogeneous or constant initial-money amounts

We start with considering the initially homogeneous money distribution in a closed system to demonstrate the Pareto distributions in the case of random saving propensity $0 < \lambda_i < 1$, with $i = \{1, \ldots, N\}$, based i.a. on the previous studies [63–65,67]. We set the initial money of each trader to $x = 10$ units. The results for the long-time money distribution for the *homogeneous* starting distribution of money among the agents is presented in the auxiliary Fig. 5. As it can be seen, a maximum of the money distribution $f(x)$ at small x values exists. The tail of this function for large x gives a Pareto inverse-power-law distribution, see the inset of Fig. 5, enabling a Pareto exponent to be extracted.

Often, in money-exchange models of similar type, at $x < x_c$ where x_c is the "critical" amount of money, the distribution of money is of Gibbs-Boltzmann type. That is, in this region it takes a nearly exponential form,

$$f(x) \sim e^{-x/x_c}. \quad (6)$$

At large x values, power-law dependence governs the money distribution,

$$f(x) \sim x^{-\alpha-1}. \quad (7)$$

In our case, the observed Pareto exponent is $\alpha \approx 1.02$, see the inset of Fig. 5. General interpolating functions between the forms (6) and (7) is an interesting issue that is, however, beyond the scope of this study, see Ref. [70]

The features of the observed money distribution are consistent with the previous well-known theoretical and empirical results [63,64,66,71] supporting the concept of agent-based exchange models as a powerful statistical-mechanics tool for modeling money distributions in a population of agents. We refer the reader to the study [72] where differences of the Gamma vs. exponential vs. log-normal distributions for the money distributions were examined. Note also that the size of a large system does not affect the scaling exponents of the money distributions we find in simulations (results not shown).

3.3 Random initial-money amounts

3.3.1 "Positive" money

Here, we examine a special case when all agents have random amounts of money initially, that is a random distribution for $0 < x < x_{\max} = 10$ is realized at $t = 0$. The results of our Monte-Carlo simulations are presented in the supplementary Fig. 6. The red circles in this figure show the results for the Pareto distribution at a homogeneous initial money (as obtained in Sect. 3.2 and shown in Fig. 5), while the green rhombuses are the results for the initial randomly-distributed amounts of money. The inset of Fig. 6 demonstrates the initial-money distribution along the lattice.

The tails of the stationary money distributions for both conditions yield the same Pareto exponent $\alpha = 1.02$ in Eq. (1). The maxima of the two distributions are shifted, in part because agents in these two scenarios have different amounts of money at $t = 0$. Similar "shift"-effects were found in our study of the agent-based models with traps [44,45]. Small-money holders thus act like temporary "traps".

The initial total amounts of money in the case of homogeneous and randomly distributed initial money are *not the same* for the simulation procedure we employed. This is the reason why the curves in some plots—which compare the results of the two scenarios such as, e.g., those in Fig. 6—do not have the same area underneath them.

3.3.2 "Negative" money

In Sects. 3.2 and 3.3 we considered the money-exchange model with homogeneous and randomly-distributed amounts of "positive" initial money. Now, within the same exchange model (3), we allow the initial money to be random with both "positive" and "negative" signs. The latter represents people in a society with a nonzero debt. Initial amounts of money vary uniformly on an interval $x_{\min} < x < x_{\max}$, where $-x_{\min} = x_{\max} = 10$. Other parameters are the same as in Sect. 3.1; we set below $L = L_0$, $R = R_0$, and $t = t_0$.

Via performing the Monte-Carlo simulations we obtained the long-time distributions of both positive

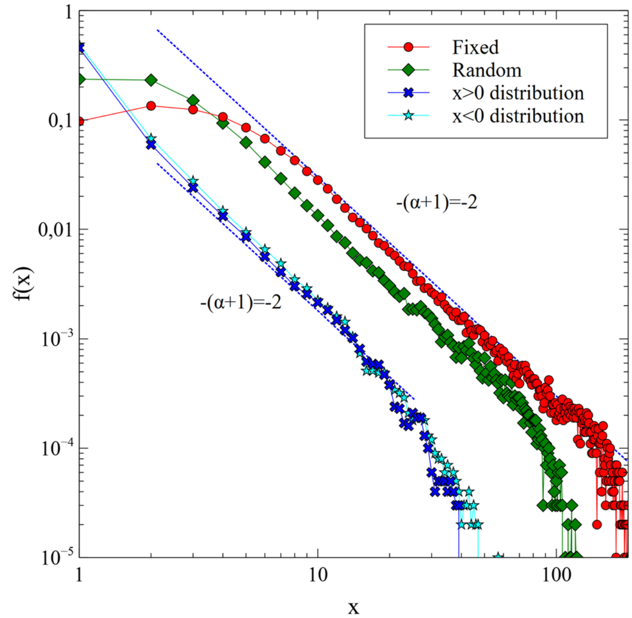

Fig. 2 The data on money distributions from Figs. 6 and 7 are shown in one plot (see the legend for notations). Parameters for $f_{\pm}(x)$ are the same as in Fig. 7. The asymptotes with the exponent-2 are shown, $f(x) \propto x^{-2}$

and negative money among the agents, see Fig. 7 in the Appendix. The blue crosses denote the agents with $x > 0$, while the cyan stars represent the traders with $x < 0$. Figure 7 shows that these two subclasses of traders yield similarly looking Paretian distributions $f_{\pm}(x)$, decaying with the same exponent. This indicates that—after numerous rounds of trader-trader interactions via exchanges—participants without and with debt follow the Paretian distributions with the same α values. For a system with a conserved amount of total money and with a finite number of agents it is not surprising that the tails of the final distributions scale virtually independently of the initial conditions. The behavior of the system with a finite number of agents in the long-time limit is expected to be ergodic [73].

For a homogeneous and random initial distribution of money, we conclude that the Pareto exponents for both subclasses of traders are the same. This means that the inverse power-law behavior of $f(x)$ given by (1) keeps its universality. Note that the agents with negative money can be considered as temporary trap-points in a process of reaching a stationary distribution in this system, as shown in Fig. 2.

To check whether the stationarity is reached —meant here only in terms of constant scaling exponent in the tail of the money distribution—we run the code for different simulation times. We observed a clear convergence of the long-time results, as shown in Fig. 8, indicating that a sufficient number of simulation steps was used in the current analysis.

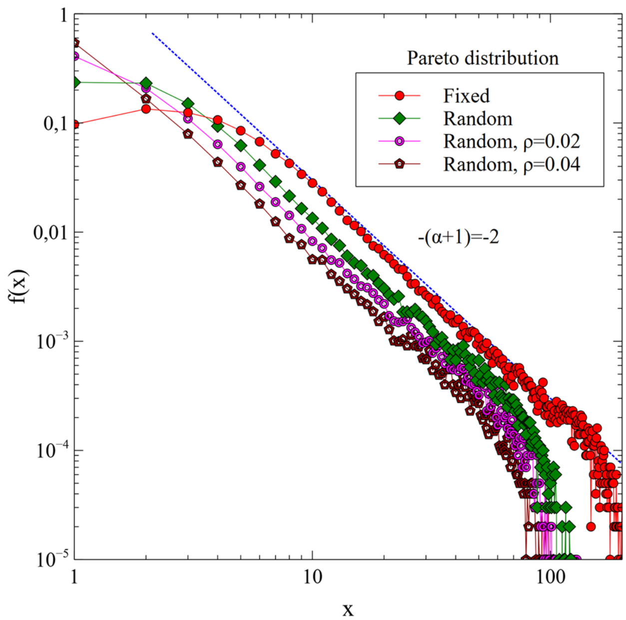

Fig. 3 Money distributions $f_+(x)$ for random initial amounts of positive money in the system with traps (see the legend), evaluated for the parameters of Fig. 2. The results of a trap-free model of Fig. 6 are shown for comparison as well

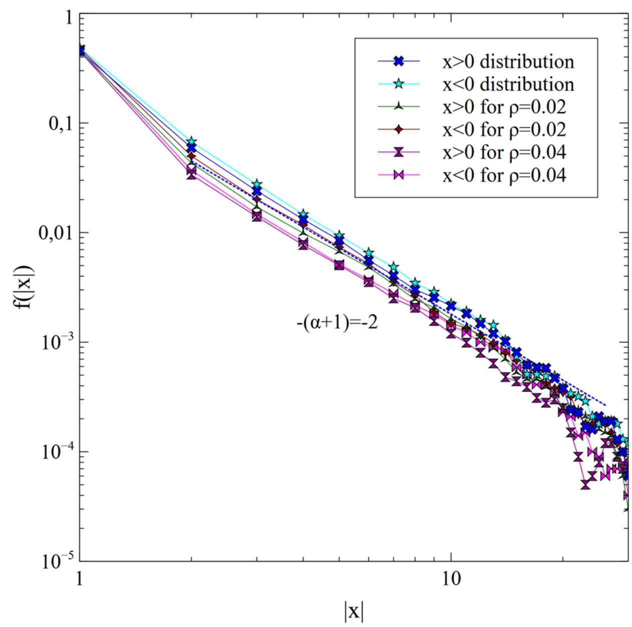

Fig. 4 Money distribution $f_\pm(x)$ for random initial money (both positive and negative) and in the presence of traps. Trap densities and notations for the curves are provided in the legend

4 Open non-conservative systems

Here, we examine the Pareto distributions in an open, non-conservative system, as discussed also in Refs. [44, 45]. One can turn a closed system into an open one by adding so-called "traps" which "absorb" the money when a given site is involved in a money exchange. Such traps, therefore, diminish the overall amount of money in the system with time acting as "sinks"[2] capable of changing the distribution of money in such systems. Previously, we have shown that traps *do not alter* the Pareto distribution and its exponent [44].

Here, we consider three different scenarios of the initial money distributions for open systems in Sects. 4.1, 4.2.1, and 4.2.2 within the 1D exchange-based model of Eq. (3). In the simulation procedure, we set the lattice size to $L = L_0$ and the realization number to $R = R_0$. The time required to properly conduct the Monte-Carlo simulations varies due to different numbers of traps in the lattice: the overall times are thus chosen from the requirement of getting the stationary final results.

[2] In a real economic system, the total amount of money is not preserved: a country can lose some of its wealth as a consequence of e.g. inflation, unexpected debt [to cover the costs of war, for example], a big natural catastrophe [drought, etc.], etc. Such a model could, thus, explain an effective decline of the wealth *per capita*.

4.1 Initially homogeneous distribution of money

We first revisit the results for the effects of traps for initially homogeneous amounts of money. We set $x_{\max} = 10$, $L = L_0$, $R = R_0$, and $t = t_0$. For two different trap densities with the fraction of trapping sites $\rho = \{0.02, 0.04\}$ [44,45] the results of simulations are shown in Fig. 9. We find that the long-tail decay of the final money distributions is still a power-law with nearly the same exponent. As, after a given number of simulation steps, the amount of "liquid" money remained in the system with a higher density of traps is smaller, the respective distribution has smallest area under its profile, see the curve for $\rho = 0.04$ in Fig. 9. Perpetual leakage of money into traps prohibits an open system from reaching an equilibrium, as compared to a closed system in Sect. 3. To better understand this issue, we conducted the Monte-Carlo simulations for varying simulation times, with $t = \{1 \times t_0, 5 \times t_0, 10 \times t_0, 15 \times t_0\}$, and found nearly the same values for the Pareto exponent, as illustrated in Fig. 10.

4.2 Initially random money distributions

4.2.1 "Positive" money

The results of the analysis of the money distribution in the system of agents with traps and randomly distributed initial money the most general situation in the current analysis are presented in Fig. 3. For comparison, the results of a trap-free model are also shown here. We find that the final distributions have similar shapes and reveal similar Pareto exponents in their tails.

4.2.2 "Negative" money

We finally analyze the effects of traps in the system of agents with randomly distributed initial money of *both* signs, see Fig. 4. The most general system with positive- and negative-money agents and in the presence of traps is thus considered here. To facilitate the comparison, the results of the previous models and the same starting conditions are shown in this plot. We find that the decay exponents in the tails of the distributions stay remarkably similar in these situations as well.

Similarly to Fig. 8 performed for a conservative system, for a nonconservative system here we also run simulations for varying total times. The results demonstrate the stationarity of the value of the Pareto scaling exponent in the tails of the money distributions, Fig. 10. The total amount of money is changing with time (see Fig. 11) and thus no universal normalization of these distributions is possible. The amount of money in a system with traps is not conserved: the higher the density of traps, the quicker both positive and negative money decreases with time, see Fig. 11.

5 Discussion and conclusions

We considered various initial-money distributions for closed and open systems of money-exchanging agents in order to test the universality of the realizable Pareto distributions. Via simulating the 1D agent-based money-exchange models, we demonstrated that the Pareto exponents remain the same, independent of the initial conditions, consistent with some previous observations [44,45,67,74]. We also showed that the resulting distributions of positive and negative money in the system can be treated separately yielding again the same characteristic Pareto exponents in their tails. Neither different money-exchange rules nor the presence of traps in the system of agents had a significant effect onto the final Pareto exponent.

The obtained universality is clearly the finding from computer simulations (no analytics) and will not stay universal if other distributions of parameter λ are chosen (see, e.g., Refs. [67,68]). While for a conservative system [49] and positive-only money these conclusions are rather expected (see Ref. [33]), for a general non-conservative system—in the presence of debt (negative money acting as traps) and of money traps—these statements are much less obvious. We demonstrated the universality of the Pareto exponent in all these situations within the current model: this is the main novel result of the current study.

This work can shed new light onto the principles of money distribution in different societies where numerous "initial conditions" can realize. In order to arrive at different exponents in the tails of the money distributions, as observed from the data in different economies, however, other features than traps and debt considered here are to be included in agent-based models of this type. This is the matter for future investigations.

In the current model, as shown in Fig. 7, each agent or site on the lattice has an equal probability to belong to a cohort with or without a debt. In a more general model—with the initial condition of unbalanced proportions of agents with positive and negative money in the system—it would be interesting to investigate the dynamics of the percentage of "bankrupt" agents with time in such a "society" of interacting agents. This can be another subject of future studies.

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Author contributions

EA performed Monte-Carlo Simulations and analyzed the data; EA, AGC, RM and IMS discussed the results and wrote the manuscript.

Data availibility statement No data is associated with the current manuscript.

Appendix A: Auxiliary figures

Here, we present some supplementary plots supporting the claims of the main text (see Figs. 5, 6, 7, 8, 9, 10, 11).

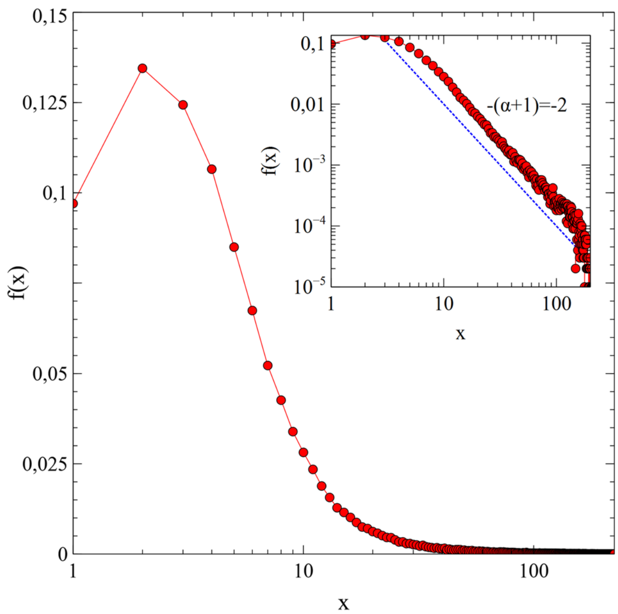

Fig. 5 Money distribution $f(x)$ for the simulated 1D traps-free exchange model. Inset: a log–log plot of the same $f(x)$ yielding the Pareto exponent $-(1+\alpha) = -2.0$

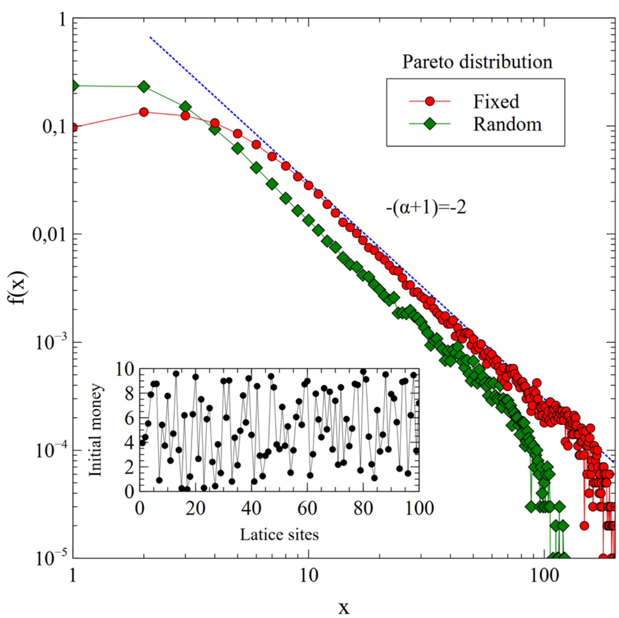

Fig. 6 Money distributions $f(x)$ for randomly distributed and homogeneous initial amounts of positive money (green squares/rhombuses and red circles, correspondingly). The inset shows the initial money amounts for each site of the lattice. In our simulations, each agent positioned at each site of the 1D lattice acquires a certain random amount of money at the start of the simulation procedure. Each simulation run these numbers are chosen anew so that a statistical ensemble for later averaging of the results is being created

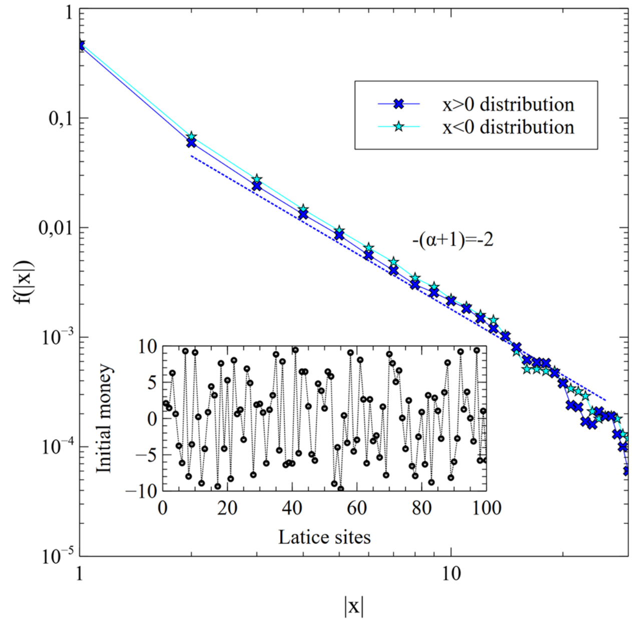

Fig. 7 Money distribution $f_{\pm}(x)$ for randomly distributed initial amounts of positive and negative money (see the legend for notations), used as the starting conditions in simulations. Parameters: $L = L_0$, $R = R_0$, and $t = t_0$

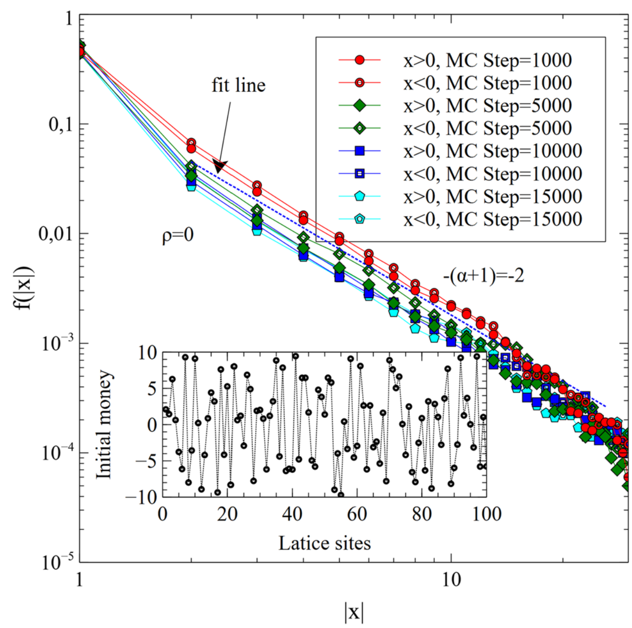

Fig. 8 Long-time money distribution for a closed system with debt shown for different lengths of simulations, see the legend. Other parameters are the same as in Fig. 2

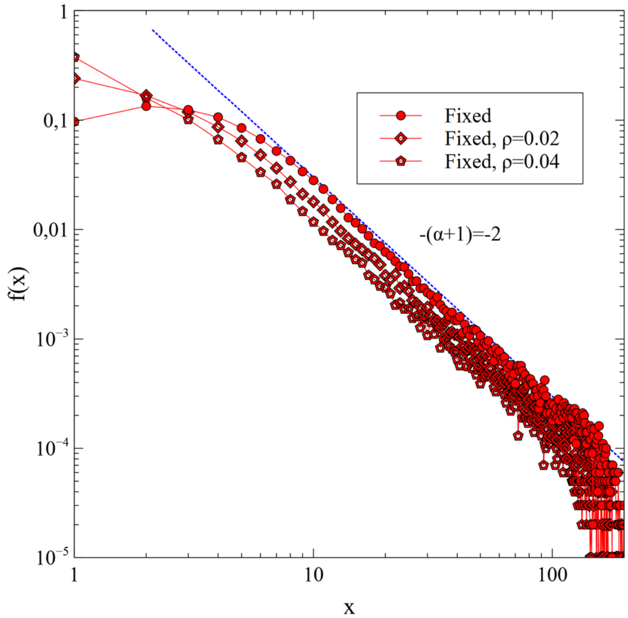

Fig. 9 Money distributions $f_+(x)$ for random initial conditions, shown for two densities of the trapping sites ρ (see the legend). As a reference, the results for a homogeneous money distribution from Fig. 6 are also shown

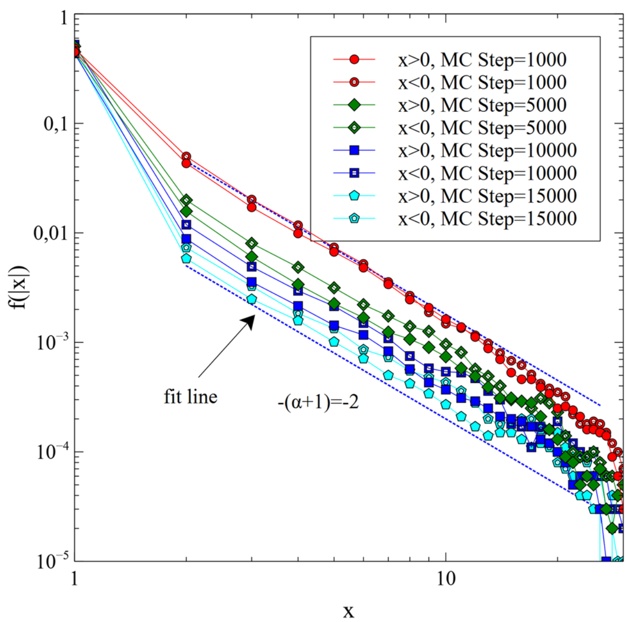

Fig. 10 The same quantities as in Fig. 8 but for an open system with positive and negative money and in the presence of traps, computed for different trajectory lengths (see the legend for the values of parameters)

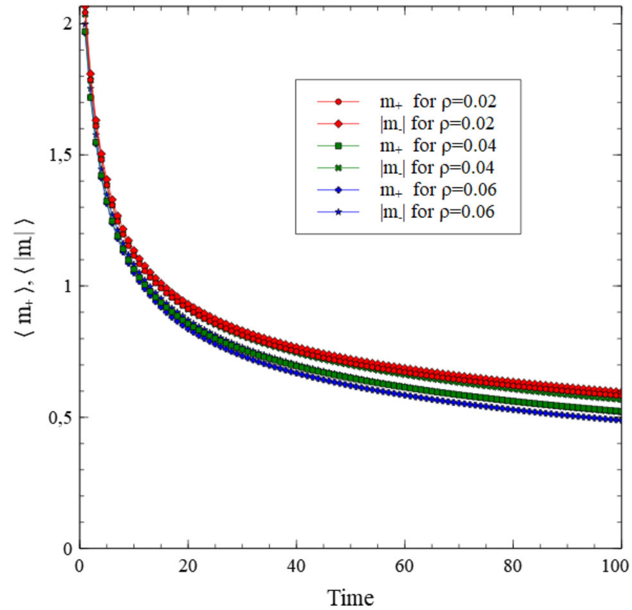

Fig. 11 Evolution of money in a system with variable density of traps and with both positive- and negative-money agents. Other parameters are the same as in Fig. 4

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