Infinite density and relaxation for Lévy walks in an external potential: Hermite polynomial approach

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(Received 7 January 2022; accepted 22 March 2022; published 14 April 2022)

Lévy walks are continuous-time random-walk processes with a spatiotemporal coupling of jump lengths and waiting times. We here apply the Hermite polynomial method to study the behavior of LWs with power-law walking time density for four different cases. First we show that the known result for the infinite density of an unconfined, unbiased LW is consistently recovered. We then derive the asymptotic behavior of the probability density function (PDF) for LWs in a constant force field, and we obtain the corresponding qth-order moments. In a harmonic external potential we derive the relaxation dynamic of the LW. For the case of a Poissonian walking time an exponential relaxation behavior is shown to emerge. Conversely, a power-law decay is obtained when the mean walking time diverges. Finally, we consider the case of an unconfined, unbiased LW with decaying speed $v(\tau) = v_0/\sqrt{\tau}$. When the mean walking time is finite, a universal Gaussian law for the position-PDF of the walker is obtained explicitly.

DOI: 10.1103/PhysRevE.105.044118

I. INTRODUCTION

Brownian motion is characterized by the linear time dependence of the mean-squared displacement (MSD) defined as $\langle x^2(t)\rangle = \int_{\infty}^{\infty} x^2 p(x,t)dx$, where $p(x,t)$ is the probability density function (PDF) of finding the particle at position x at time t [1,2]. However, many processes observed in nature do not satisfy this property [3]. Instead, often so-called anomalous diffusion with power-law MSD $\langle x^2(t)\rangle \simeq K_\alpha t^\alpha$ with $\alpha \neq 1$ is observed. Here the anomalous diffusion coefficient K_α has physical dimension of length^2/time$^\alpha$ [2]. Respectively, when $\alpha > 1$ or $0 < \alpha < 1$ the corresponding process is called superdiffusive or subdiffusive. Among a variety of other models [4–6], a highly successful approach to describe anomalous diffusion is the Montroll-Weiss-Scher continuous-time random walk (CTRW) [7–9] consisting of two independent series of independently and identically distributed random variables, the waiting time and the jump length. When the PDF of waiting times is exponential and the PDF of the jump lengths l is given by a Lévy stable law with asymptotic form $\simeq |l|^{-1-\mu}$ and $\mu \in (0,2)$ the corresponding CTRW process is a Lévy flight [2,9,10]. Although Lévy flights have found many applications in describing the paths of foraging animals [11–13] the drawback of Lévy flights are their instantaneous long jumps to a new position, causing the MSD to diverge, a property sometimes referred to as pathologic. There exist only few cases, in particular, the effective Lévy flight motion on looping polymer chains [14,15], for which the divergence occurs only in the chemical coordinate but not in the physically relevant embedding space. Moreover, in recent work, long-lived, effective Lévy flights indeed arise from some well-defined microscopic dynamics due to the coupling of the tracer motion to the nonlocal hydrodynamic interactions with active swimmers [16].

To solve this divergence problem in direct geometric space, the Lévy walk (LW) process with spatiotemporal coupling of jump lengths l and waiting times τ and resulting finite propagation speed was formulated [17]. Specifically LWs can either have a fixed propagation speed or a scaling form $\simeq l^{-\mu}\delta(l-\tau^\nu)$ in the coupling [9,18]. When the propagation speed is a constant, the LW can assume superdiffusion or normal diffusion depending on the exact choice for the density of waiting times. Note that in the context of LWs we will refer to the distribution of τ as the walking time density in the following. LWs have widespread use, e.g., to optimize [19,20] search processes of higher animals [21] or bacteria [22]. Note, however, that in the field of movement ecology recent analyses point at persistent motion as an alternative to LWs [23]. LWs are identified in human movement patterns [24,25], spreading of the specific SARS-CoV-2 outbreak [26], pedestrian movement [27], human hunter-gatherer foraging [28], and optimized robotic search [29,30]. LWs are also shown to emerge in many microscopic phenomena, e.g., molecular-motor motion in living biological cells [31], motor-driven transport in dendritic cells [32], spreading of cancer cells [33], human memory retrieval [34] and cognition processes [35]. Moreover LWs are also used to describe anomalous heat transport [36,37], the dynamics of blinking quantum dots [38], Lévy-Lorentz gas dynamics [39], or the intermittent motion in weakly chaotic systems [40–42]. We also mention that LWs emerge near critical points in nonlinear systems [43].

The "ultraweak" nonergodicity (in the sense that time- and ensemble-averaged MSDs only differ by a constant) and the generalized fluctuation-dissipation theorem for LWs are discussed in Refs. [44–48]. These properties are intimately related to the infinite density of LWs [49]. An infinite density is a special density that cannot be normalized, i.e., the

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total area beneath the density function is divergent. Infinite densities can, e.g., also be found in cold atom systems and thus have an actual physical meaning [50,51]. One of the most important applications of infinite densities is to calculate the relation between the qth-order moment of a process $x(t)$ and time t, which can be generally given as $\langle |x|^q\rangle(t) \sim t^{q\nu(q)}$. Obviously for Brownian motion, $\nu(q) = 1/2$. When $\nu(q)$ is not a constant the process is referred to as strong anomalous diffusion [52]. As shown in Ref. [49] an LW whose velocity is a constant or uniformly distributed in a symmetric finite domain, and whose walking time follows a power-law density with scaling exponent $\alpha \in (1, 2)$ and thus finite mean walking time $\langle\tau\rangle$, is characterized by some critical value $q_c = \alpha$: If $q > q_c$, then $q\nu(q) = q + 1 - \alpha$ while if $q < q_c$, $\nu(q) = q/\alpha$.

In the following, we establish a useful alternative method to calculate the infinite density of LWs in terms of a Hermite polynomial expansion. This new method particularly allows us to solve some problems of LWs, such as LWs driven by a constant or Hookean external force, that are not amenable to the more conventional integral transform methods. We also consider the case of an LW with a walk time-dependent speed.

Through a fractional material derivative, LWs in a constant external force field can be described by a deterministic equation [53,54]. For arbitrary external forces, the corresponding LW follows a generalized Kramers-Fokker-Planck equation [55,56]. However, this dynamic equation is hard to solve for the relevant statistical properties. Moreover, the traditional Fourier-Laplace transform cannot be directly applied to solve LWs in external potentials because of the spatiotemporal coupling. To deal with the constant force field or an external harmonic potential, the Hermite orthogonal polynomial approach was introduced in Refs. [57,58]. These analyses showed that LWs always become localized in an harmonic potential, i.e., the corresponding MSD converges to a plateau at long times. We will derive here the decay rate of this localization, providing additional insight into the influence of the external harmonic potential.

For the case of a constant external force, the corresponding asymptotic behavior of the MSD and the qth-order moments were discussed in Ref. [58]: Based on simulations the above-mentioned strong anomalous diffusion behavior of the scaling exponent $\nu(1)$ was recovered. Here we demonstrate this result explicitly in the range $q < \alpha/2$ for $1 < \alpha < 2$. Due to the intricate dependence on the LW parameters, this is a significant progress.

We here also consider another important generalization of LWs, namely, those with walking time-dependent speed [59,60], concretely, for the form $v(\tau) = v_0/\sqrt{\tau}$. In this case the MSD always has the same form $\langle x^2(t)\rangle \sim t$, independent on the exact form of the walking time density. We further show that when the walking time density has a finite mean, the PDF of the corresponding LW converges to a Gaussian law, again fully independent of the exact shape of the walking time density.

The paper is organized as follows. In Sec. II, we introduce the Hermite polynomial method to obtain the infinite density for unconfined and unbiased LWs. In Sec. III we discuss the infinite density for LWs in a constant force field via the Hermite polynomial approach and derive the qth-order moment. Section IV is devoted to the study of the relaxation dynamics to the stationary state of LWs in an harmonic potential. Finally, in Sec. V, we consider an unconfined, unbiased LW with walking time-dependent speed $v(\tau) = v_0/\sqrt{\tau}$, and we derive the corresponding asymptotic behavior of the PDF. We draw our conclusions in Sec VI, while mathematical details are deferred to the Appendices.

II. INFINITE DENSITY FOR UNCONFINED, SYMMETRIC LÉVY WALKS

To establish our approach used in the presence of external forces below we first consider unconfined, symmetric LWs for power-law forms of the walking time PDF $\phi(\tau)$ with finite mean walking time $\langle\tau\rangle = \int_0^\infty \tau\phi(\tau)d\tau$,

$$\phi(\tau) = \alpha/(1+\tau)^{1+\alpha}, \quad \text{for } \alpha \in (1, 2), \tag{1}$$

so that the Laplace transform $\widehat{\phi}(s) = \mathscr{L}_\tau\{\phi(\tau)\}(s) = \int_0^\infty e^{-s\tau} f(\tau)d\tau$ has the asymptotic form [18,59]

$$\widehat{\phi}(s) \sim 1 - \frac{1}{\alpha - 1}s - \Gamma(1-\alpha)s^\alpha. \tag{2}$$

Here we consider LWs with a constant speed v_0, and we assume that the walking time for each renewal is drawn identically from the PDF $\phi(\tau)$. Then the PDF $q(x, t)$ that the renewal event finishes at time t and the walker arrives at position x satisfies the relation [18,59]

$$q(x, t) = \frac{1}{2}\int_0^t \sum_\pm q(x \pm v_0\tau, t-\tau)\phi(\tau)d\tau + p_0(x)\delta(t), \tag{3}$$

where we use the notation $\sum_\pm q(a \pm b) = q(a+b) + q(a-b)$, and where $p_0(x)$ represents the density of the initial position. Finally, $\delta(\cdot)$ denotes the Dirac δ function. Then the PDF $p(x, t)$ to find the walker at position x at time t satisfies

$$p(x, t) = \frac{1}{2}\int_0^t \sum_\pm q(x \pm v_0\tau, t-\tau)\Psi(\tau)d\tau, \tag{4}$$

where $\Psi(\tau) = \int_\tau^\infty \phi(\tau')d\tau'$ represents the survival probability of $\phi(\tau)$, i.e.,

$$\widehat{\Psi}(s) = \frac{1 - \widehat{\phi}(s)}{s}. \tag{5}$$

We now assume that the densities $p(x, t)$ and $q(x, t)$ can be explicitly expressed as series of Hermite orthogonal polynomials $H_n(x)$ for $n = 0, 1, \ldots,$

$$p(x, t) = \sum_{n=0}^\infty R_n(t)H_n(x)e^{-x^q},$$

$$2(x, t) = \sum_{n=0}^\infty T_n(t)H_n(x)e^{-x^2}. \tag{6}$$

If we choose the initial condition $p_0(x) = \delta(x)$, then the relations between the coefficients $\widehat{T}_n(s)$ and $\widehat{R}_n(s)$ can be obtained from substituting the series (6) into relations (3) and (4), respectively, yielding [59]

$$\sqrt{\pi}2^n n!\widehat{T}_n(s) = \frac{1}{2}\sum_{m=0}^{n}\frac{\sqrt{\pi}2^n n!}{(n-m)!}\left[\sum_{\pm}(\pm v_0)^{n-m}\right] \times \mathscr{L}_\tau\{\tau^{n-m}\phi(\tau)\}(s)\widehat{T}_m(s) + H_n(0) \quad (7)$$

and

$$\widehat{R}_n(s) = \frac{1}{2}\sum_{m=0}^{n}\frac{\sum_{\pm}(\pm v_0)^{n-m}}{(n-m)!}\mathscr{L}_\tau\{\tau^{n-m}\Psi(\tau)\}(s)\widehat{T}_m(s). \quad (8)$$

Next we derive the asymptotic form of $\widehat{R}_n(s)$ for each n at sufficiently long times t (i.e., the variable s in Laplace space is small). As illustrated in Ref. [59], for symmetric LWs the odd terms of $\widehat{R}_n(s)$ and $\widehat{T}_n(s)$ equal zero; therefore, in the following we only consider the asymptotic forms of even terms. Then, according to the derivations in Appendix B, it can be obtained from (7) and the form of $\widehat{\phi}(s)$ in (2) that

$$\widehat{\mathcal{T}}_{2n}(s) \sim \begin{cases} \frac{\alpha-1}{\sqrt{\pi}(2n)!}v_0^{2n}(\alpha)_{2n}\Gamma(1-\alpha)s^{2\alpha-3} & \text{for } n \geqslant 1; \\ (\alpha-1)s^{\alpha-2}/\sqrt{\pi} & \text{for } n = 0, \end{cases} \quad (9)$$

where $\widehat{\mathcal{T}}_{2m}(s) = s^{\alpha-1+2m}\widehat{T}_{2m}(s)$ and $(z)_n = \prod_{m=1}^{n}(z-m+1)$ is the falling factorial. Further from (5), (8), and (9), for $n \geqslant 1$ we see that

$$\widehat{R}_{2n}(s) \sim \frac{v_0^{2n}\Gamma(2-\alpha)}{(2n)!\sqrt{\pi}}[(\alpha)_{2n} - (\alpha-1)_{2n}]s^{\alpha-2n-2}, \quad (10)$$

and the corresponding inverse Laplace transform is given by

$$R_{2n}(t) \sim -\frac{2n(\alpha-1)\sqrt{2\pi}v_0^{2n}\csc(\alpha\pi)}{(2n)!\Gamma(1+\alpha-2n)\Gamma(2-\alpha+2n)}t^{2-\alpha+2n}. \quad (11)$$

Particularly, when $n=0$, it can be easily found that $\widehat{R}_0(s) \sim 1/(\sqrt{\pi}s)$, so that $R_0(t) \sim 1/\sqrt{\pi}$. Next applying a Fourier transformation, defined as $\tilde{f}(k) = \mathscr{F}_x\{f(x)\}(k) = \int_{-\infty}^{\infty} e^{-ikx}f(x)dx$, to (6) and substituting the asymptotic forms (11), we obtain

$$\begin{aligned}\tilde{p}(k,t) &\sim \sum_{n=1}^{\infty}\sqrt{\pi}(-1)^n k^{2n}R_{2n}(t) + 1 \\ &\sim 1 - \frac{(\alpha-1)\pi t^{3-\alpha}v_0^2\csc(\alpha\pi)k^2}{\Gamma(4-\alpha)\Gamma(\alpha-1)} \\ &\quad\times\left\{(\alpha-3)\,{}_1F_2\left(1-\frac{\alpha}{2};\frac{3}{2},2-\frac{\alpha}{2};-\frac{1}{4}k^2t^2v_0^2\right)\right. \\ &\quad\left.-(\alpha-2)\,{}_1F_2\left(\frac{3-\alpha}{2};\frac{3}{2},\frac{5-\alpha}{2};-\frac{1}{4}k^2t^2v_0^2\right)\right\},\end{aligned} \quad (12)$$

where the hyper-geometric function is defined as [61]

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty}\frac{(a_1)^n\cdots(a_p)^n}{(b_1)^n\cdots(b_q)^n}\frac{z^n}{n!}. \quad (13)$$

Here $(z)^n = \prod_{k=1}^{n}(z+k-1)$ is the rising factorial. Taking the inverse Laplace transform in expression (12) with respect to k, we finally obtain

$$p(x,t) \sim \begin{cases}\frac{\alpha-1}{2}t^{-\alpha}\left[\alpha v_0^{\alpha}(t/x)^{1+\alpha} - (\alpha-1)v_0^{-1+\alpha}(t/x)^{\alpha}\right] & \text{if } |x| \leqslant v_0 t; \\ 0 & \text{if } |x| > v_0 t.\end{cases} \quad (14)$$

This is exactly the same result as the one obtained in Ref. [49] using different methods. As we have assumed that both s and k are sufficiently small (the "diffusion limit"), at sufficiently long times t and large x, we can conclude the asymptotic form (14). Moreover, in order to ensure the existence of the series $\widehat{\tilde{p}}(k,s) \sim \sum_{n=0}^{\infty}\sqrt{\pi}(-1)^n k^{2n}\widehat{R}_{2n}(s)$, where $\widehat{R}_{2n}(s)$ is given asymptotically in (10), $v_0|k/s|$ should be small enough, i.e., (14) is valid for x around $v_0 t$, given time t to be sufficiently long. This analysis is in accordance with Ref. [49]. One of the main advantages of the Hermite polynomials approach pursued here is that it allows solutions of more complicated LW processes, particularly in the presence of an external potential, for which the traditional integral transform approaches cannot be applied. In the next section we consider the infinite density for LWs in a constant external force field, which can be solved smoothly in terms of Hermite polynomials, and in section IV we consider the case of an external harmonic potential.

III. INFINITE DENSITY FOR LÉVY WALKS IN A CONSTANT EXTERNAL FORCE FIELD

We follow the approach introduced in Ref. [58], according to which we assume that the LW has the same initial speed v_0 for each renewal. The LW dynamic under the influence of a constant external force F_0 can then be established along with the deterministic equation for the PDF. In contrast, we note that the standard method based on Fourier and Laplace transforms is not straightforward, as it involves integral transforms of the types $\mathscr{L}_\tau\{\exp(-ikF_0\tau^2)\cos(kv_0\tau)\phi(\tau)\}(s)$ and $\mathscr{L}_\tau\{\exp(-ikF_0\tau^2)\cos(kv_0\tau)\Psi(\tau)\}(s)$. These are hard to solve explicitly for power-law PDFs $\phi(\tau)$. With this ordinary method given in Ref. [49] it is then difficult to calculate the associated infinite density, a task that we here demonstrate can be achieved by utilizing the formulation in terms of Hermite polynomials. Without loss of generality we assume that $F_0 > 0$. Again we denote $q(x,t)$ to find the particle at position x when the renewal event finishes at time t. We obtain [58]

$$q(x,t) = \frac{1}{2}\sum_{\pm}\int_0^t q\left(x - \frac{1}{2}F_0\tau^2 \pm v_0\tau, t-\tau\right)\phi(\tau)d\tau + p_0(x)\delta(t). \quad (15)$$

Then the PDF $p(x,t)$ to find the walker at position x at time t satisfies

$$p(x,t) = \frac{1}{2}\sum_{\pm}\int_0^t q\left(x-\frac{1}{2}F_0\tau^2 \pm v_0\tau, t-\tau\right)\Psi(\tau)d\tau. \quad (16)$$

We now assume that $p(x,t)$ and $q(x,t)$ can be expressed as series of Hermite polynomials according to relation (6), and then from (16) we find

$$\widehat{R}_n(s) = \frac{1}{2}\sum_{m=0}^{n}\frac{2^{m-n}}{(n-m)!}\widehat{T}_m(s) \times \mathscr{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2 \pm 2v_0\tau)^{n-m}\right]\Psi(\tau)\right\}(s). \tag{17}$$

Moreover, from (15), the recurrence relation between the $\{\widehat{T}_n(s)\}$ is given by

$$\sqrt{\pi}2^n n!\widehat{T}_n(s) = \frac{1}{2}\sum_{m=0}^{n}\frac{2^m\sqrt{\pi}n!}{(n-m)!}\widehat{T}_m(s) \times \mathscr{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2 \pm 2v_0\tau)^{n-m}\right]\phi(\tau)\right\}(s) + H_n(0). \tag{18}$$

Combining (2) and (18), the asymptotic form of $\widehat{\mathcal{T}}_m(s) = s^{\alpha-1+2m}\widehat{T}_m(s)$ for small s is given by

$$\widehat{\mathcal{T}}_n(s) \sim \begin{cases} -\frac{\Gamma(1-\alpha)}{n!\sqrt{\pi}}\left(\frac{F_0}{2}\right)^n(\alpha)_{2n}(\alpha-1)^2 s^{2\alpha-3} & \text{for } n \geqslant 1; \\ (\alpha-1)s^{\alpha-2}/\sqrt{\pi} & \text{for } n \geqslant 0, \end{cases} \tag{19}$$

The detailed derivations can be found in Appendix C. Next we conclude the asymptotic forms of $\widehat{R}_n(s)$ from (17), (5), and (19), for $n \geqslant 1$,

$$\widehat{R}_n(s) \sim \frac{\alpha-1}{\sqrt{\pi}n!}\left(\frac{F_0}{2}\right)^n[(\alpha-1)_{2n}-(\alpha)_{2n}] \times \Gamma(1-\alpha)s^{-2n+\alpha-2}. \tag{20}$$

After inverse Laplace transform of $\widehat{R}_n(s)$ with respect to the variable s we have, for $n \geqslant 1$,

$$R_n(t) \sim -\frac{2\sqrt{\pi}nF_0^n(\alpha-1)\csc(\alpha\pi)}{2^n n!\Gamma(1+\alpha-2n)\Gamma(2-\alpha+2n)}t^{1+2n-\alpha}. \tag{21}$$

Now, according to the Fourier transform of the assumed expression of $p(x,t)$ in (6) and utilizing the property (A7) together with (21), we have

$$\widehat{p}(k,t) \sim \sum_{n=0}^{\infty}\sqrt{\pi}(-ik)^n R_n(t) \sim -2^{-\frac{\alpha}{2}}(\alpha-1)(iF_0k)^{\frac{\alpha-1}{2}}\left[\sqrt{ikF_0}t\gamma\left(1-\frac{\alpha}{2},\frac{1}{2}ikF_0t^2\right) - \sqrt{2}\gamma\left(\frac{3-\alpha}{2},\frac{1}{2}ikF_0t^2\right)\right], \tag{22}$$

where γ is the lower incomplete gamma function defined as

$$\gamma(\beta,x) = \int_0^x u^{\beta-1}e^{-u}du, \qquad \text{for } \mathrm{Re}\{\beta\} > 0. \tag{23}$$

Finally after inverse Fourier transform, we have that

$$p(x,t) = -\frac{1}{\pi}2^{-(\frac{3+\alpha}{2})}(\alpha-1)t^{1-\alpha}|x|^{-\frac{\alpha}{2}}\Bigg(-\frac{i\alpha\pi}{\sqrt{2}|x|}\csc\left(\frac{\alpha\pi}{2}\right)t^\alpha\left\{(-iF_0)^{\frac{\alpha}{2}}\left[\cos\left(\frac{\alpha\pi}{4}\right)\mathrm{sign}(x) - i\sin\left(\frac{\alpha\pi}{4}\right)\right] - (iF_0)^{\frac{\alpha}{2}}\left[\cos\left(\frac{\alpha\pi}{4}\right)\mathrm{sgn}(x) + i\sin\left(\frac{\alpha\pi}{4}\right)\right]\right\} + \frac{(\alpha-1)\pi\sec\left(\frac{\alpha\pi}{2}\right)}{F_0\sqrt{|x|}}t^{\alpha-1}\{\cos[(1+\alpha)\pi/4][(-iF_0)^{\frac{\alpha}{2}}(iF_0)^{\frac{1}{2}} + (iF_0)^{\frac{\alpha}{2}}(-iF_0)^{\frac{1}{2}}] - \mathrm{sgn}(x)\sin[(1+\alpha)\pi/4]i[(-iF_0)^{\frac{\alpha}{2}}(iF_0)^{\frac{1}{2}} - (iF_0)^{\frac{\alpha}{2}}(-iF_0)^{\frac{1}{2}}]\}\Bigg). \tag{24}$$

According to the fact that

$$(\pm iF_0)^{\frac{\alpha}{2}} = F_0^{\frac{\alpha}{2}}\left[\exp\left(\pm\frac{\pi}{2}i\right)\right]^{\frac{\alpha}{2}} = F_0^{\frac{\alpha}{2}}\exp\left(\pm\frac{\pi\alpha}{4}i\right) = F_0^{\frac{\alpha}{2}}\left[\cos\left(\frac{\pi\alpha}{4}i\right) \pm \sin\left(\frac{\pi\alpha}{4}i\right)\right], \tag{25}$$

we can simplify $p(x,t)$ and find the form

$$p(x,t) \sim \begin{cases} 2^{-1-\frac{\alpha}{2}}(\alpha-1)F_0^{\frac{\alpha-1}{2}}x^{-1-\frac{\alpha}{2}}[\alpha\sqrt{F_0}t - \sqrt{2}(\alpha-1)\sqrt{x}] & \text{for } x > 0; \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

It should be noted that although directly applying the inverse Fourier transform to (22) produces a positive PDF for each point of the interval $0 < x < \alpha^2F_0t^2/[2(\alpha-1)^2]$, we should also keep in mind that at given time t the LW in the constant force field $F_0 > 0$ can at most reach the point $x = F_0t^2/2 + v_0t$. Combining both conditions guarantees that in the asymptotic limit $p(x,t)$ in (26) is always positive. Explicitly, we need to adjust the asymptotic behavior for the first case in (26) under the condition $x \in (0, \alpha^2F_0t^2/[2(\alpha-1)^2]) \cap (0, F_0t^2/2 + v_0t)$. The agreement between the asymptotic theoretical result in Eq. (26) with stochastic simulations is very good, as verified in Fig. 1.[1]

Looking more closely at Fig. 1 we observe a systematic deviation between theory and simulations around $x = 10^5$ and, less severely, at the edge $x = F_0t^2/2$ ($= 10^9$ with the parameters used in Fig. 1). The former deviation at smaller

[1]The simulations code is available at URL: https://github.com/Peter-Bloomberg/LW_F0_Infinite_density

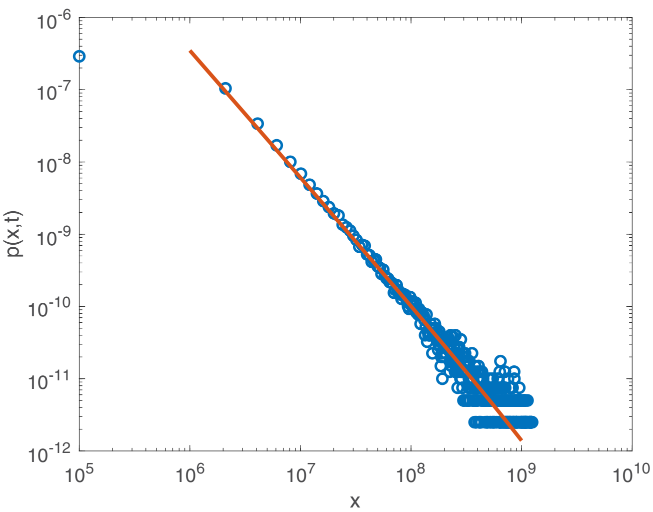


FIG. 1. Infinite density for an LW process in the presence of a constant external force $F_0 = 1$ at dimensionless time $t = 5 \times 10^4$. The walking time PDF for the LW process follows the power-law form (1) with $\alpha = 1.5$. The circles are obtained from taking the average over 2×10^5 samples, the initial velocity for each step is $v_0 = 1$. The full red line corresponds to the theoretical result in (26).

x arises since the asymptotic form of $p(x,t)$ in (26) is only valid for sufficiently large $x > x_c$, for reasons analogous to the arguments below Eq. (14), in order to ensure the existence of the series $\sum_{n=0}^{\infty}(-ik)^n\widehat{R}_n(s)$, i.e., the condition that $|k/s^2|$ is small enough. From the simulations here we see that $x_c \approx 10^6$. Since the probability of finding the particle around the maximum position it can travel is quite small, the deviation around the leading edge $F_0t^2/2$ is expected to be larger than in the well-sampled central part. Since the exact form of this central part of the PDF $p(x,t)$ for LWs in a constant external force field is unknown, we cannot give a specific analytical estimate for x_c, compare the case of free LWs discussed in Ref. [49]. Roughly speaking, for LWs in a constant external force field, the critical point x_c should be related with F_0 and α, while the exact role of v_0 remains unclear. From our simulations, we also estimate that $t \geqslant 10^4$ can be considered as sufficiently long.

We finally note that the result of (26) when $x > 0$ can be rearranged in the form

$$\begin{aligned} p(x,t) &\sim 2^{-1-\frac{\alpha}{2}}(\alpha-1)F_0^{\frac{\alpha-1}{2}}x^{-1-\frac{\alpha}{2}} \\ &\quad\times[\alpha\sqrt{F_0}t-\sqrt{2}(\alpha-1)\sqrt{x}] \\ &= \frac{(\alpha-1)}{t^\alpha\sqrt{2xF_0}}\left[\alpha\left(\frac{L}{t}\right)^{-\alpha-1}-(\alpha-1)\left(\frac{L}{t}\right)^{-\alpha}\right], \end{aligned} \tag{27}$$

where $L = \sqrt{2x/F_0}$. Therefore we obtain the same result as the one presented in Ref. [62]. It should be noted that although Eq. (27) can be found in Ref. [62], the process considered in Ref. [62] is fundamentally different from our case in the present paper. Specifically, Ref. [62] discusses a CTRW whose space and time steps are independent, while we here consider LWs with spatiotemporal coupling.

We also conclude from (26) that the integration of the asymptotic behavior of $p(x,t)$ over the whole space $(-\infty,\infty)$ is always divergent at $x = 0$, a key property of an infinite density. Next we utilize the asymptotic behavior of $p(x,t)$ in (26) to theoretically calculate the absolute qth-order moment $\langle|x|^q\rangle$, that was considered in Ref. [58] only from numerical simulations.

Before proceeding with the calculation of the qth order, we briefly detail how we implement the LW simulations in this work. To simulate a large number of LW particle trajectories in a constant external force field over the time period t, we first generate a series of independently and identically distributed random variables τ_i, $i = 1,\ldots,N$ from the waiting time PDF $\phi(\tau)$, Eq. (1), and where $N = \max\{n; \sum_{i=1}^{n}\tau_i \leqslant t\}$. The method of generating the random variables for a given PDF is given in Ref. [63]. Since an LW particle does not rest from time $T_N = \sum_{i=1}^{N}\tau_i$ to t but continues to move, the last step should also be considered, and we define $\tau_{N+1} = t - T_N$. It should be noted that τ_{N+1} does not follow the PDF $\phi(\tau)$ [64]. Then, according to the specific dynamics of LWs in an external constant force field, we can simulate the trajectory with the algorithm sketched in Appendix D. From a large number of independent trajectories we can then obtain the corresponding PDF, the moments, or other quantities.

Derivation of qth-order moments

We show that our Hermite polynomial approach allows us to explicitly calculate the qth-order moment

$$\langle|x|^q\rangle = \int_{-\infty}^{\infty}|x|^q p(x,t)dx. \tag{28}$$

Since $p(x,t) > 0$ only for $x \in (0, \alpha^2F_0t^2/[2(\alpha-1)^2]) \cap (0, F_0t^2/2 + v_0t)$, when time t is sufficiently long we can approximate

$$F_0t^2/2 + v_0t \approx F_0t^2/2, \tag{29}$$

and for $\alpha \in (1,2)$ the following inequity is always valid:

$$\frac{\alpha^2F_0t^2}{2(\alpha-1)^2} > F_0t^2/2. \tag{30}$$

Therefore, together with (26) we have

$$\begin{aligned} \langle|x|^q\rangle &\approx \int_0^{\frac{1}{2}F_0t^2}|x|^q p(x,t)dx \\ &\sim \left(\frac{F_0}{2}\right)^q\frac{2(\alpha-1)q}{(\alpha-2q)(\alpha-2q-1)}t^{-\alpha+2q+1} \quad \text{if } q > \frac{\alpha}{2}. \end{aligned} \tag{31}$$

The result (31), which is verified in Fig. 2, corresponds to the one in Ref. [58] concluded directly from numerical simulations. However, for $0 < q < \alpha/2$, the corresponding $\langle|x|^q\rangle$ may not be obtained directly from the infinite density, since the information of the central part of the PDF $p(x,t)$ is required, which is beyond our current analysis.

IV. TYPICAL RELAXATION DYNAMICS OF LÉVY WALKS IN AN HARMONIC POTENTIAL

The PDF of spatiotemporally independent Lévy flights in an harmonic confinement can be obtained explicitly in Fourier space using the method of characteristics, and the stationary

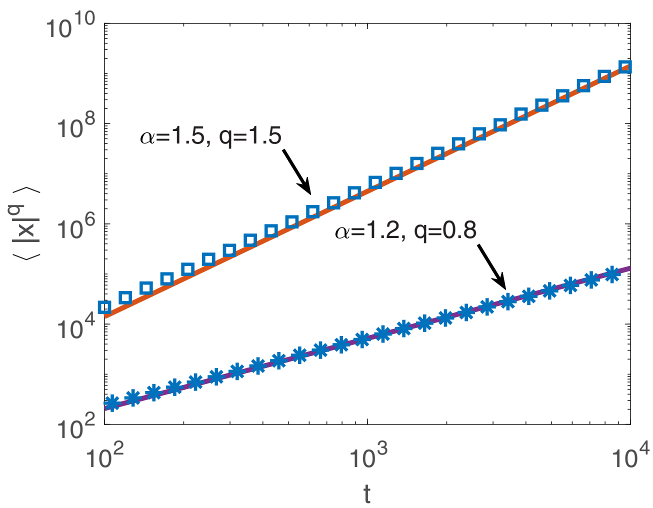


FIG. 2. Absolute qth-order moment $\langle |x|^q\rangle$ for an LW in the presence of the constant force $F_0 = 1$. The walking time PDF follows the power law (1) with different pairs of α and q under the condition $q > \alpha/2$. The stars and squares represent simulations results obtained from averaging over 5×10^4 samples. The initial speed for each step is $v_0 = 1$. The solid lines correspond to the theoretical results in Eq. (31).

solution is a Lévy stable law with the same Lévy index as the scaling exponent of the underlying PDF of jump lengths [65]. The situation is more intricate for the LW case with its spatiotemporal coupling.

As shown in Ref. [57], an LW in the harmonic potential $V(x) = \gamma x^2/2$ with "spring" constant $\gamma > 0$ and initial speed v_0 at the start of each renewal step, has the PDF

$$q(x,t) = \sum_{\pm} \frac{1}{2}\int_0^t q(x^{\pm}, t-\tau)\frac{\phi(\tau)}{|\cos(\omega\tau)|}d\tau + p_0(x)\delta(t) \tag{32}$$

to find a particle at position x at time t at the end of the renewal event. Here $x^{\pm} = [x/\cos(\omega\tau)] \pm (v_0/\omega)\tan(\omega\tau)$ and $\omega = \sqrt{\gamma/M}$ and where M is the mass of the LW particle. The PDF $p(x,t)$ in this case can be expressed as

$$p(x,t) = \sum_{\pm} \frac{1}{2}\int_0^t q(x^{\pm}, t-\tau)\frac{\Psi(\tau)}{|\cos(\omega\tau)|}d\tau. \tag{33}$$

It can be found that the traditional integral transform methods cannot solve this problem, and we expand $p(x,t)$ and $q(x,t)$ in Hermite polynomials (6). For the initial condition $p_0(x) = \delta(x)$, $\{T_n(t)\}$ and $\{R_n(t)\}$ satisfy the recurrence relations in Laplace space

$$\begin{aligned}\widehat{T}_n(s) - \frac{H_n(0)}{\sqrt{\pi}2^n n!} &= \sum_{m=0}^{n}\sum_{r=0}^{\lfloor \frac{m}{2}\rfloor}\frac{2^{-1-2r}}{(n-m)!r!}\left(\frac{v_0}{\omega}\right)^{n-m}(-1)^i(1+(-1)^{n-m})\\ &\quad\times \mathscr{L}_\tau\big\{\sin^{n-m+2r}(\omega\tau)\cos^{k-2i}(\omega\tau)\phi(\tau)\big\}(s),\end{aligned} \tag{34}$$

and

$$\begin{aligned}\widehat{R}_n(s) &= \sum_{m=0}^{n}\sum_{r=0}^{\lfloor \frac{m}{2}\rfloor}\frac{2^{-1-2r}}{(n-m)!r!}\left(\frac{v_0}{\omega}\right)^{n-m}(-1)^i(1+(-1)^{n-m})\\ &\quad\times \mathscr{L}_\tau\{\sin^{n-m+2r}(\omega\tau)\cos^{k-2i}(\omega\tau)\Psi(\tau)\}(s).\end{aligned} \tag{35}$$

It was shown in Ref. [57] that there exists a stationary PDF when $\phi(\tau)$ is of exponential or power-law form. However, the explicit form of this stationary PDF is hard to calculate explicitly, and it is already hard to discuss directly the relaxation dynamics of the LW PDF to the stationary state. It can, however, be shown that the asymptotic behavior of the MSD for an LW in the harmonic potential reaches the plateau $\langle x^2(t)\rangle \sim v_0^2/\omega^2$ [57]. In this sense the relaxation dynamics of the LW in the harmonic potential corresponds to the crossover to the stationary value of the MSD. We note that in Ref. [57] the theoretical result $\langle x^2(t)\rangle \sim v_0^2/\omega^2$ is only derived for the case of exponential $\phi(\tau)$, here we fill the gap when $\phi(\tau)$ is power law. According to the property (A7) of Hermite polynomials and the fact that $\langle x^m(t)\rangle = i^m \frac{d^m}{dk^m}p(k,t)|_{k=0}$, the MSD in Laplace space generally reads

$$\langle \widehat{x}^2(s)\rangle = \mathscr{L}_t\{\langle x^2(t)\rangle\}(s) = \frac{\sqrt{\pi}}{2}\widehat{R}_0(s) + 2\sqrt{\pi}\widehat{R}_2(s). \tag{36}$$

Further when $n = 0, 2$ from (34) and (35), we have $\widehat{R}_0(s) = 1/(\sqrt{\pi}s)$,

$$\begin{aligned}\widehat{T}_2(s) &= \left[\frac{1}{2}\left(\frac{v_0}{\omega}\right)^2 - \frac{1}{4}\right]\mathscr{L}_\tau\{\sin^2(\omega\tau)\phi(\tau)\}(s)\widehat{T}_0(s)\\ &\quad+ \mathscr{L}_\tau\{\cos^2(\omega\tau)\phi(\tau)\}(s)\widehat{T}_2(s) + \frac{H_2(0)}{8\sqrt{\pi}},\end{aligned} \tag{37}$$

and

$$\begin{aligned}\widehat{R}_2(s) &= \left[\frac{1}{2}\left(\frac{v_0}{\omega}\right)^2 - \frac{1}{4}\right]\mathscr{L}_\tau\{\sin^2(\omega\tau)\Psi(\tau)\}(s)\widehat{T}_0(s)\\ &\quad+ \mathscr{L}_\tau\{\cos^2(\omega\tau)\Psi(\tau)\}(s)\widehat{T}_2(s).\end{aligned} \tag{38}$$

First we choose the exponential form $\phi(\tau) = \beta e^{-\beta\tau}$ with $\beta > 0$. According to (37), (38), and

$$\widehat{T}_0(s) = \frac{1}{\sqrt{\pi}[1-\widehat{\phi}(s)]} = \frac{\beta+s}{\sqrt{\pi}s}, \tag{39}$$

we have

$$\widehat{T}_2(s) = \frac{\beta+s}{4\sqrt{\pi}s}\left[\frac{4\beta v_0^2}{s(\beta+s)^2 + 2\omega^2(\beta+2s)} - 1\right] \tag{40}$$

and

$$\widehat{R}_2(s) = \frac{-\beta^2 s - 2\beta\left(s^2 - 2v_0^2 + \omega^2\right) - s\left(s^2 - 4v_0^2 + 4\omega^2\right)}{4\sqrt{\pi}s[\beta^2 s + s^3 + 4s\omega^2 + 2\beta(s^2+\omega^2)]}. \tag{41}$$

It can then be obtained from relation (36) and neglecting small higher orders of s that

$$\langle \widehat{x}^2(s)\rangle \sim \frac{2(\beta+s)v_0^2}{s[\beta^2 s + 4\omega^2 s + 2\beta(s^2+\omega^2)]}, \tag{42}$$

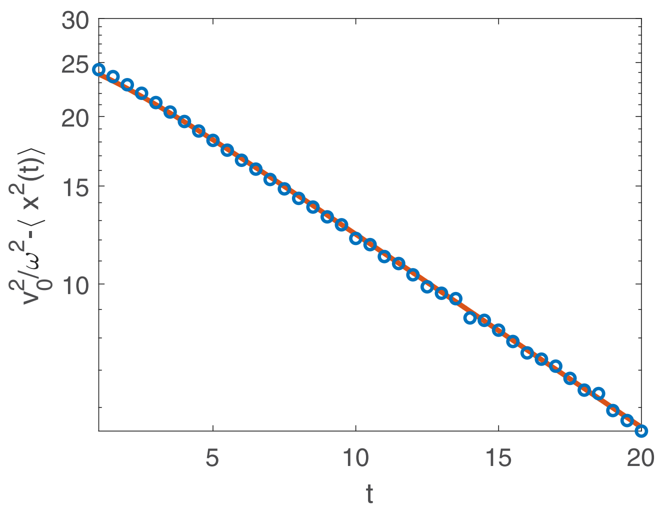


FIG. 3. Relaxation dynamics for an LW in an harmonic potential with $\omega = 0.2$. The walking time PDF is exponential with unit average ($\beta = 1$), and the initial speed for each step is $v_0 = 1$. The circles are obtained from averaging over 5×10^4 samples. The full red line corresponds to the theoretical result (43).

which corresponds to

$$\langle x^2(t)\rangle \sim \frac{v_0^2}{\omega^2} + \frac{2v_0^2}{\beta^2 - 4\omega^2} e^{-\beta t/2} + \frac{v_0^2(\beta^2 - 2\omega^2)}{\omega^2(-\beta^2 + 4\omega^2)} e^{-2\omega^2 t/\beta}. \tag{43}$$

Therefore when $\phi(\tau)$ is an exponential density, the corresponding relaxation of an LW in the harmonic potential to the stationary state is exponential with decay rate $\beta/2$. The result (43) is verified in Fig. 3.

For the power-law shape $\phi(\tau) = \alpha(1+\tau)^{-1-\alpha}$ with $\alpha \in (0, 1)$, the corresponding $\widehat{T}_0(s) \sim s^{-\alpha}/[\Gamma(1-\alpha)]$, and then from (37), (105), and (106) we have

$$\widehat{T}_2(s) \sim \frac{1}{4\sqrt{\pi}\omega^2}\left[\frac{2v_0^2 + \omega^2}{\alpha\mathcal{I}_{\alpha+1}(\omega)} + \frac{2v_0^2 - \omega^2}{\Gamma(1-\alpha)} s^{-\alpha}\right], \tag{44}$$

where

$$\mathcal{I}_\theta(\omega) = \int_1^\infty \frac{\cos[2\omega(t+1)]}{t^\theta} dt. \tag{45}$$

The asymptotic behavior of $\widehat{R}_2(s)$ can now be obtained through (38), (F8), and (F9), resulting in

$$\widehat{R}_2(s) \sim \frac{1}{8\sqrt{\pi}\omega^2}\left[\frac{4v_0^2 - 2\omega^2}{s} + \frac{(2v_0^2 + \omega^2)\Gamma(1-\alpha)}{-1 + \alpha\mathcal{I}_{\alpha+1}(\omega)} s^{\alpha-1}\right]. \tag{46}$$

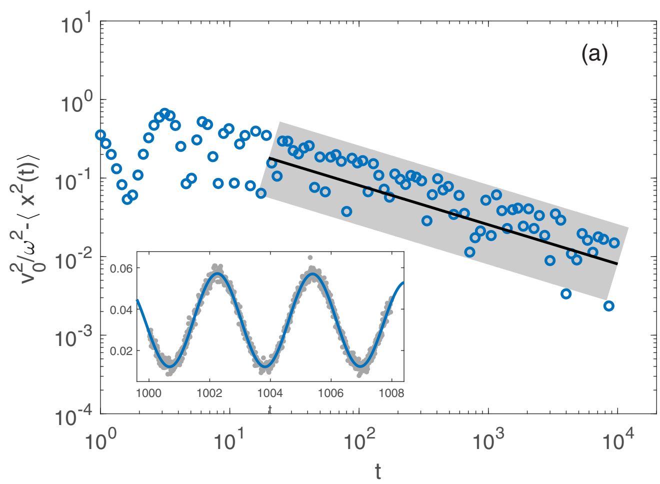
Finally, the corresponding asymptotic behavior of the MSD in Laplace space is

$$\langle \widehat{x}^2(s)\rangle \sim \frac{v_0^2}{\omega^2 s} + \frac{s^{-1+\alpha}(2v_0^2 + \omega^2)\Gamma(1-\alpha)}{4\omega^2[-1 + \alpha\mathcal{I}_{\alpha+1}(\omega)]}, \tag{47}$$

corresponding to

$$\langle x^2(t)\rangle \sim \frac{v_0^2}{\omega^2} - \frac{2v_0^2 + \omega^2}{4\omega^2[1 - \alpha\mathcal{I}_{\alpha+1}(\omega)]} t^{-\alpha}. \tag{48}$$

Therefore, we conclude that when $\phi(\tau)$ is a power law with infinite average the typical relaxation dynamics has a power-law decay. This result is nicely verified in Fig. 4(a). It can observed from the numerical simulations in Fig. 4(a) that the decay dynamics of the difference $v_0^2/\omega^2 - \langle x^2(t)\rangle$ from the stationary plateau value exhibits some oscillations. The reason for these is that each step of an LW particle in the harmonic potential performs harmonic oscillations [57] (see also Ref. [66]). These are still present even in the long time limit of $\langle x^2(t)\rangle$ close to the plateau v_0^2/ω^2. For a power-law walking time PDF $\phi(\tau)$ with $\alpha \in (0, 2)$ the duration between two renewal events is typically longer than in the exponential case. Therefore, intuitively speaking, the oscillations will be more pronounced for the power-law case. We further note that when $\alpha \in (1, 2)$, the corresponding relaxation dynamics



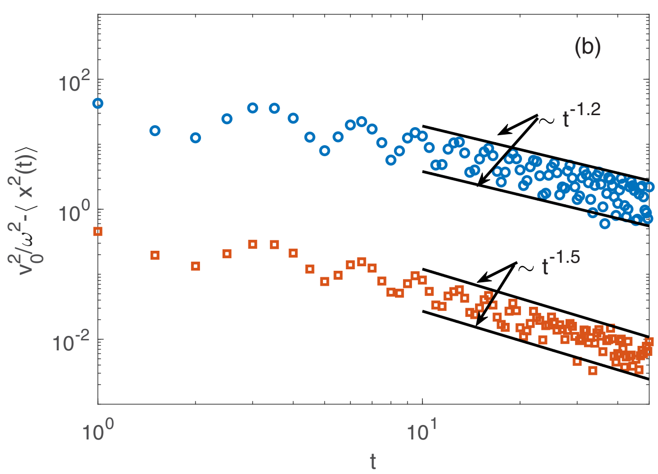


FIG. 4. Relaxation dynamics of the MSD for LWs in an harmonic potential with $\omega = 1$ and power-law walking time PDF (1), on log-log scale. For (a), the value of the walking time scaling exponent is $\alpha = 0.5$ [i.e., the corresponding $\mathcal{I}_{\alpha+1}(\omega) = 0.14026$]. The starting speed for each step is $v_0 = 1$. The circles are obtained from averaging over 5×10^4 samples. The small oscillations, resolved in the inset using linear axes, are bounded in the gray band around the theoretical result (48) represented by the solid black line. In panel (b), the simulations results represented by the red squares and blue circles are averaged over 10^5 samples with $\alpha = 1.5$ and 1.2, and with $v_0 = 1$ and 100, respectively. Again, the oscillations are observed, and they lie in a band bounded by power laws decaying as (49), with scaling exponents $-\alpha$.

is hard to derive, but from numerical simulations we conclude an power-law decay of the form

$$\langle x^2(t)\rangle \sim \frac{v_0^2}{\omega^2} - C_1 t^{-\alpha}, \tag{49}$$

where C_1 is a positive constant, see the results shown in Fig. 4(b). Again, the persistent oscillations cause the relaxation dynamics to the stationary value to lie in a band delimited by two power laws with slope $-\alpha$.

V. FREE LÉVY WALK WITH SPEED $v(\tau) = v_0/\sqrt{\tau}$

We finally consider a less standard, unconfined LW process, for which the speed is a decaying function of the walking time in the form $v(\tau) = v_0/\sqrt{\tau}$ [9]. As concluded in Ref. [59] for this choice, the MSD is always $\langle x^2(t)\rangle \sim \sqrt{v_0}t$, independent of the walking time PDF $\phi(\tau)$. In this section we consider $\phi(\tau)$ to assume the power-law form (1) with $0 < \alpha < 2$. We find that when the walking time has a finite mean, the PDF $p(x,t)$ asymptotically always has a Gaussian shape at long times, while the asymptotic behavior of the PDF is different when the average walking time is infinite.

We start again with the assumption that the $q(x,t)$ and $p(x,t)$ can be expressed as series of Hermite polynomials, corresponding to the form of Eq. (6). According to Ref. [59], we then have

$$\widehat{R}_n(s) = \frac{1}{2}\sum_{m=0}^{n}\frac{v_0^{n-m}}{(n-m)!}\widehat{T}_m(s) \times \mathscr{L}_\tau\{[(-\sqrt{\tau})^{n-m} + (\sqrt{\tau})^{n-m}]\Psi(\tau)\}(s). \tag{50}$$

Since the process is symmetric, we only need to consider the even terms. With (B1) we have

$$\begin{aligned}\widehat{R}_{2n}(s) &= \sum_{m=0}^{n}\frac{v_0^{2n-2m}}{(2n-2m)!}\widehat{T}_{2m}(s)\mathscr{L}_\tau\{\tau^{n-m}\Psi(\tau)\}(s)\\ &= \sum_{m=0}^{n-1}(-1)^{n-m}\frac{v_0^{2n-2m}}{(2n-2m)!}(\alpha-1)_{n-m}\Gamma(1-\alpha)\\ &\quad \times s^{-n}\widehat{\mathcal{T}}_{2m}(s) + \frac{1}{\alpha-1}s^{-n-\alpha+1}\widehat{\mathcal{T}}_{2n}(s),\end{aligned} \tag{51}$$

where $\widehat{\mathcal{T}}_{2m}(s) = s^{-1+\alpha+m}\widehat{T}_{2m}(s)$. Additionally, from Ref. [59] we have the recurrence relations

$$\begin{aligned}&\sqrt{\pi}2^{2n}(2n)!\widehat{T}_{2n}(s)\\ &= \frac{1}{2}\sum_{m=0}^{n}\frac{\sqrt{\pi}2^{2n}(2n)!}{(2n-2m)!}\widehat{T}_{2m}(s)\\ &\quad\times \mathscr{L}_\tau\left\{\sum_{\pm}(\pm v_0\sqrt{\tau})^{2n-2m}\phi(\tau)\right\}(s) + H_{2n}(0)\\ &= \sum_{m=0}^{n}\frac{\sqrt{\pi}2^{2n}(2n)!}{(2n-2m)!}v_0^{2n-2m}\mathscr{L}_\tau\{\tau^{n-m}\phi(\tau)\}(s)\widehat{T}_{2m}(s)\\ &\quad + H_{2n}(0).\end{aligned} \tag{52}$$

Then from (B4), for $n \geqslant 2$ we see that

$$\begin{aligned}\widehat{\mathcal{T}}_{2n}(s) &\sim \left(1 - \frac{s}{\alpha-1}\right)\widehat{\mathcal{T}}_{2n}(s) + \frac{v_0^2}{2(\alpha-1)}\widehat{\mathcal{T}}_{2n-2}(s)\\ &\quad + \sum_{m=0}^{n-2}\frac{(-1)^{n-m+1}}{(2n-2m)!}v_0^{2n-2m}(\alpha)_{n-m}\Gamma(1-\alpha)\\ &\quad \times s^{\alpha-(n-m)}\widehat{\mathcal{T}}_{2m}(s) + \frac{H_{2n}(0)}{\sqrt{\pi}2^{2n}(2n)!}.\end{aligned} \tag{53}$$

Finally we obtain

$$\begin{aligned}\frac{1}{\alpha-1}\widehat{\mathcal{T}}_{2n}(s) &\sim \frac{v_0^2}{2(\alpha-1)}\widehat{\mathcal{T}}_{2n-2}(s) + s^{\alpha+n-2}\frac{H_{2n}(0)}{\sqrt{\pi}2^{2n}(2n)!}\\ &\quad + \sum_{m=0}^{n-2}\frac{(-1)^{n-m+1}}{(2n-2m)!}v_0^{2n-2m}(\alpha)_{n-m}\Gamma(1-\alpha)\\ &\quad \times s^{\alpha-1}\widehat{\mathcal{T}}_{2m}(s),\end{aligned} \tag{54}$$

which indicates that $\widehat{\mathcal{T}}_{2n}(s) = \widehat{\mathcal{T}}_{2n-2}(s)/2$ when only $1 < \alpha < 2$. For $n = 0, 1$, we can directly obtain the corresponding asymptotic behaviors from (52)

$$\widehat{\mathcal{T}}_0(s) = s^{-1+\alpha}\widehat{T}_0(s) = \frac{s^{-1+\alpha}}{\sqrt{\pi}[1-\widehat{\phi}(s)]} \sim \frac{\alpha-1}{\sqrt{\pi}}s^{\alpha-2}, \tag{55}$$

and

$$\begin{aligned}\widehat{\mathcal{T}}_2(s) = s^{\alpha}\widehat{T}_2(s) &= \frac{v_0^2 s^{\alpha}\mathscr{L}_\tau\{\tau\phi(\tau)\}(s)}{2\sqrt{\pi}[1-\widehat{\phi}(s)]^2} - \frac{s^{\alpha}}{4\sqrt{\pi}[1-\widehat{\phi}(s)]}\\ &\sim \frac{\alpha-1}{2\sqrt{\pi}}v_0^2 s^{\alpha-2}.\end{aligned} \tag{56}$$

We then conclude that

$$\widehat{\mathcal{T}}_{2n}(s) \sim \frac{\alpha-1}{\sqrt{\pi}2^n}v_0^{2n}s^{\alpha-2} \text{ for } n \geqslant 0. \tag{57}$$

Therefore, for $n \geqslant 1$ the asymptotic behaviors of the $\widehat{R}_{2n}(s)$ can be obtained from (51),

$$\begin{aligned}\widehat{R}_{2n}(s) &\sim \sum_{m=0}^{n-1}\frac{(-1)^n(\alpha-1)_{n-m}\Gamma(1-\alpha)(\alpha-1)}{2^m v_0^{-2n}\sqrt{\pi}(2n-2m)!}s^{\alpha-2-n}\\ &\quad + \frac{1}{\sqrt{\pi}}\left(\frac{v_0^2}{2}\right)^n s^{-n-1}\\ &\sim \frac{1}{\sqrt{\pi}}\left(\frac{v_0^2}{2}\right)^n s^{-n-1}.\end{aligned} \tag{58}$$

Additionally, $\widehat{R}_0(s) = 1/(\sqrt{\pi}s)$, and thus

$$\widehat{\widehat{p}}(k,s) \sim \sum_{n=0}^{\infty}(-1)^n k^{2n}\left(\frac{v_0^2}{2}\right)^n s^{-n-1} = \frac{2}{2s + k^2 v_0^2}, \tag{59}$$

which leads to the Gaussian law

$$p(x,t) \sim \frac{1}{v_0\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2tv_0^2}\right). \tag{60}$$

The result (60) does not contain the exponent α explicitly, and we thus conclude that LWs with speed $v(\tau) = v_0/\sqrt{\tau}$ and power-law walking time PDF $\phi(\tau)$ with finite mean walking time, Eq. (1) with $1 < \alpha < 2$, always converges to the same

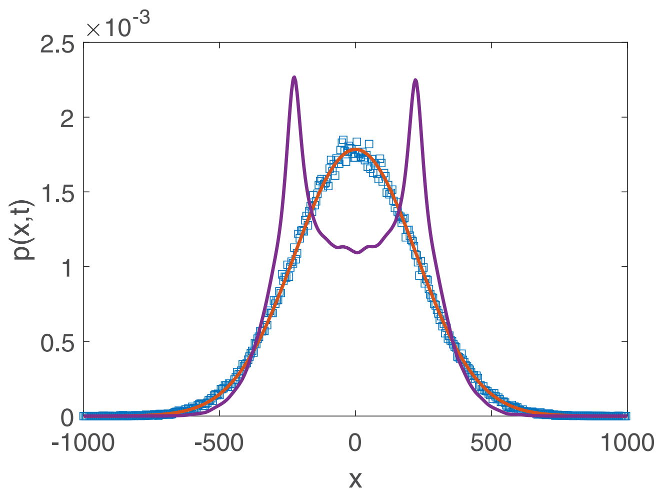


FIG. 5. PDF of a free LW with decaying speed $v(\tau)=1/\sqrt{\tau}$ as function of the walking time τ, at process time $t=5\times10^4$. The squares are obtained from sampling over 2×10^5 realization with power-law walking time (1) with $\alpha=1.5$, the (red) full line corresponds to the theoretical result (60). The (purple) bimodal curve is obtained from numerical simulations of the same number of realizations for the case when the mean walking time is infinite, with $\alpha=0.5$.

Gaussian (60). This result is reasonable since the average of τ with PDF (1) is always finite, and given the speed $v(\tau)=v_0/\sqrt{\tau}$, the length of each step is typically $\rho=v_0\sqrt{\tau}$. Roughly speaking, the second moment for each step is $\langle\rho^2\rangle$, and according to the central limit theorem such a random walk should have a Gaussian limit form of width ρ.

Although the MSD for a free LW with speed $v(\tau)=v_0/\sqrt{\tau}$ is always the same for different kinds of $\phi(\tau)$, including the case of divergent mean [59], the PDF is distinctly different in the case when the mean of the PDF $\phi(\tau)$ diverges. While we did not find an explicit derivation, this fact can be concluded from the numerical simulations presented in Fig. 5. Indeed, when the mean walking time $\langle\tau\rangle$ diverges ($0<\alpha<1$) the stationary PDF becomes bimodal, in contrast to the Gaussian shape (60) for $1<\alpha<2$.

We note that similar bimodality was found for time-fractional wave or generalized Cattaneo equations [67–69]. The two humps will spread as function of time and are thus different from multimodal stationary solutions for LWs in harmonic confinement [57] and with soft resetting events [70], as well as Lévy flights [71–73] and superdiffusive (persistent) fractional Brownian motion [74] in steeper than harmonic potentials.

VI. CONCLUSION

Based on the Hermite orthogonal polynomial approach to LWs we considered four different realizations based on a power-law walking time PDF. For unconfined and unbiased LWs with constant speed we showed that our method recovers the infinite density for the PDF to find the walker at position x at time t determined in Ref. [49]. In the presence of a constant external force we demonstrated that the Hermite polynomial approach can be successfully employed to explicitly calculate the PDF in the diffusion limit. We showed that the asymptotic form is not integrable over the whole space, since the integration is divergent at the origin, as expected for an infinite density. From the asymptotic behavior the qth-order moment $\langle|x|^q\rangle$ for $q>\alpha/2$ could be obtained explicitly, filling the gap from [59] where the asymptotic behavior of $\langle|x|^q\rangle$ was only deduced from numerical simulations. However, when $q<\alpha/2$, the information contained in the central part of the PDF is required, and in this range the current method cannot be applied. How to calculate the qth-order moment for the case of general q along with the asymptotic form of the central part of the PDF remains to be discussed in the future. Moreover, for the case when $\phi(\tau)$ has a divergent average, such as a power-law form with $\alpha\in(0,1)$, the asymptotic behaviors of the PDF, even for the free LW process, so far remains elusive with our method. How to treat this case will be a question of future studies.

The strength of the Hermite polynomial approach was demonstrated in the case of LWs in an external harmonic potential, in which we showed that localization always occurs, i.e., the corresponding MSD asymptotically converges to a plateau, and there exists a stationary state for such a process. From the crossover of the MSD we define the typical relaxation dynamic for this process, finding that when $\phi(\tau)$ is exponential, an exponential decay to the stationary MSD can be found. In contrast, when $\phi(\tau)$ is a power law with $\alpha\in(0,2)$, a power-law decay of the form $\langle x^2(t)\rangle\sim v_0^2/\omega^2+\mathcal{O}(t^{-\alpha})$ is found. We showed that the relaxation dynamics of the MSD to its plateau value exhibits clear oscillations reflecting the classical motion of the LW in the harmonic confinement.

We finally discuss an LW with decaying speed $v(\tau)=v_0/\sqrt{\tau}$, for which the MSD was shown to be independent of the specific walking time PDF $\phi(\tau)$. From the Hermite polynomial approach we calculated the asymptotic form of the PDF when $\phi(\tau)$ has a finite average, finding a universal Gaussian law, independent of the power-law exponent α. This result was rationalized from the finite walk time, that should invoke the validity of the central limit theorem. A rigorous mathematical proof of this result will be provided in future work. From simulations we showed that when the mean walking time is infinite, $0<\alpha<1$, the stationary PDF is no longer Gaussian, but assumes a distinct bimodal stationary PDF.

We believe that the results presented here further demonstrate the usefulness of the Hermite polynomial method in the analysis of LW processes, and we are confident that more results can be obtained that remain inaccessible by integral transformation methods. Specifically, in the future it may be interesting to combine the motion in the external harmonic potential with damping of the LW, or with a walking time dependent speed.

ACKNOWLEDGMENTS

P.X. gratefully acknowledges financial support from China Postdoctoral Science Foundation funded project No. 8206300491. R.M. acknowledges financial support from the German Science Foundation (DFG, Grant No. ME 1535/12-1) and from the Foundation for Polish Science (Fundacja na rzecz Nauki Polskiej, Humboldt Polish Honorary Re-search Scholarship). W.W. is supported by the National Natural Science Foundation of China under Grant No. 12105243 and the Zhejiang Province Natural Science Foundation LQ22A050002.

APPENDIX A: A BRIEF INTRODUCTION OF HERMITE ORTHOGONAL POLYNOMIALS

Hermite polynomials $\{H_n(x)\}$ are orthogonal over $(-\infty,\infty)$ with respect to the weight function $w(x)=e^{-x^2}$ [59],

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)w(x)dx=\sqrt{\pi}2^n n!\delta_{n,m}, \tag{A1}$$

where $\delta_{n,m}$ is the Kronecker δ symbol defined as

$$\delta_{n,m}=\begin{cases}1 & \text{for } m=n\\ 0 & \text{for } m\neq n\end{cases}. \tag{A2}$$

The Hermite polynomials are given by

$$H_n(x)=\begin{cases} n!\sum_{i=0}^{n/2}\frac{(-1)^{\frac{n}{2}-i}}{(2i)!(\frac{n}{2}-i)!}(2x)^{2i} & \text{for even } n;\\ n!\sum_{i=0}^{\frac{n-1}{2}}\frac{(-1)^{\frac{n-1}{2}-i}}{(2i+1)!(\frac{n-1}{2}-i)!}(2x)^{2i+1} & \text{for odd } n.\end{cases} \tag{A3}$$

The values of the Hermite polynomials at $x=0$ are

$$H_n(0)=\begin{cases}\frac{(-1)^{n/2}n!}{(n/2)!}, & \text{for even } n;\\ 0, & \text{for odd } n.\end{cases} \tag{A4}$$

Another two important properties for Hermite polynomials are

$$H_n(x+y)=\sum_{l=0}^{n}\binom{n}{l}H_l(x)(2y)^{n-l}, \tag{A5}$$

and

$$H_n(cx)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}c^{n-2k}(c^2-1)^k\binom{n}{2k}\frac{(2k)!}{k!}H_{n-2k}(x), \tag{A6}$$

where $\binom{n}{l}=\frac{n!}{l!(n-l)!}$ and $\lfloor\frac{n}{2}\rfloor$ is the floor function representing the biggest integer less than $n/2$. Further there exists the following Fourier transform [59]

$$\mathscr{F}_x\{H_n(x)e^{-x^2}\}(k)=\sqrt{\pi}(-ik)^n\exp\left(-\frac{k^2}{4}\right). \tag{A7}$$

APPENDIX B: DERIVATIONS OF (9) AND (10)

In this part we derive the asymptotic form of $\widehat{R}_n(s)$ for each n for small s. From the asymptotic behavior of $\widehat{\phi}(s)$ in (2) and relation (5) there exists

$$\begin{aligned}&\mathscr{L}_\tau\{\tau^{n-m}\Psi(\tau)\}(s)\\ &=(-1)^{n-m}\widehat{\Psi}^{(n-m)}(s)\\ &\sim\begin{cases}\frac{1}{\alpha-1} & \text{if } m=n;\\ (-1)^{n-m}(\alpha-1)_{n-m}\Gamma(1-\alpha)s^{\alpha-1-(n-m)} & \text{if } n>m,\end{cases}\end{aligned} \tag{B1}$$

where $(z)_n=\prod_{m=1}^{n}(z-m+1)$ is the falling factorial. Therefore, substituting (B1) into (8) gives the following asymptotic form of $\widehat{R}_n(s)$,

$$\begin{aligned}\widehat{R}_n(s)\sim&\frac{1}{2}\sum_{m=0}^{n-1}\frac{\sum_{\pm}(\pm v_0)^{n-m}}{(n-m)!}(-1)^{n-m}(\alpha-1)_{n-m}\\ &\times\Gamma(1-\alpha)s^{\alpha-1-(n-m)}\widehat{T}_m(s)+\frac{1}{\alpha-1}\widehat{T}_n(s).\end{aligned} \tag{B2}$$

As illustrated in Ref. [59], for symmetric LWs the odd terms of $\widehat{R}_n(s)$ and $\widehat{T}_n(s)$ equal zero. After further simplification, we obtain the asymptotic behaviors of the even terms $\widehat{R}_{2n}(s)$ in the form

$$\begin{aligned}\widehat{R}_{2n}(s)\sim&\frac{\Gamma(1-\alpha)}{s^{2n}}\sum_{m=0}^{n-1}\frac{v_0^{2n-2m}}{(2n-2m)!}(\alpha-1)_{2n-2m}\widehat{\mathcal{T}}_{2m}(s)\\ &+\frac{s^{1-\alpha-2n}}{\alpha-1}\widehat{\mathcal{T}}_{2n}(s),\end{aligned} \tag{B3}$$

where $\widehat{\mathcal{T}}_{2m}(s)=s^{\alpha-1+2m}\widehat{T}_{2m}(s)$. Next we consider the asymptotic form of $\widehat{T}_{2m}(s)$ from the recurrence relation in (7), and similarly we find the following asymptotic behaviors

$$\begin{aligned}&\mathscr{L}_\tau\{\tau^{n-m}\phi(\tau)\}(s)\\ &=(-1)^{n-m}\widehat{\phi}^{(n-m)}(s)\\ &\sim\begin{cases}1-\frac{s}{\alpha-1} & \text{if } m=n;\\ \frac{1}{\alpha-1} & \text{if } m=n-1;\\ (-1)^{n-m+1}(\alpha)_{n-m}\Gamma(1-\alpha)s^{\alpha-(n-m)} & \text{if } n>m+2.\end{cases}\end{aligned} \tag{B4}$$

Then from (7) and (B4), the even terms can be asymptotically obtained as

$$\begin{aligned}&\sqrt{\pi}2^{2n}(2n)!s^{\alpha-1+2n}\widehat{T}_{2n}(s)\\ &\sim-\sum_{m=0}^{n-1}\frac{\sqrt{\pi}2^{2n}(2n)!}{(2n-2m)!}v_0^{2n-2m}(\alpha)_{2n-2m}\Gamma(1-\alpha)s^{2\alpha+2m-1}\\ &\quad\times\widehat{T}_{2m}(s)+\sqrt{\pi}2^{2n}(2n)!\widehat{T}_{2n}(s)s^{-1+\alpha+2n}-\sqrt{\pi}2^{2n}(2n)!\\ &\quad\times\frac{s^{\alpha+2n}}{\alpha-1}\widehat{T}_{2n}(s)+s^{-1+\alpha+2n}H_{2n}(0),\end{aligned} \tag{B5}$$

that can be further simplified as

$$\begin{aligned}&\sqrt{\pi}2^{2n}(2n)!\frac{s}{\alpha-1}\widehat{\mathcal{T}}_{2n}(s)\\ &\sim-\sum_{m=0}^{n-1}\frac{\sqrt{\pi}2^{2n}(2n)!}{(2n-2m)!}v_0^{2n-2m}(\alpha)_{2n-2m}\Gamma(1-\alpha)s^{\alpha}\widehat{\mathcal{T}}_{2m}(s)\\ &\quad+s^{\alpha-1+2n}H_{2n}(0).\end{aligned} \tag{B6}$$

It should be noted that (B6) is only valid for $n\geqslant 1$; if $n=0$ according to (7) we have

$$\widehat{T}_0(s)=\frac{1}{\sqrt{\pi}[1-\phi(s)]}\sim\frac{\alpha-1}{\sqrt{\pi}s}, \tag{B7}$$

i.e., $\widehat{\mathcal{T}}_0(s)\sim(\alpha-1)s^{\alpha-2}/\sqrt{\pi}$. Further, it can be concluded from (B6) that

$$(2n)!\frac{s}{\alpha-1}\widehat{\mathcal{T}}_{2n}(s)\sim-v_0^{2n}(\alpha)_{2n}\Gamma(1-\alpha)s^{\alpha}\widehat{\mathcal{T}}_0(s). \tag{B8}$$

Then, substituting the asymptotic form of $\widehat{\mathcal{T}}_0(s)$ into (B8) yields

$$\widehat{\mathcal{T}}_{2n}(s) \sim \frac{\alpha-1}{\sqrt{\pi}(2n)!}v_0^{2n}(\alpha)_{2n}\Gamma(1-\alpha)s^{2\alpha-3}. \tag{B9}$$

Since $(2\alpha-3)$ is larger than $(\alpha-2)$ when $1<\alpha<2$, the first term on the right-hand side of (B3) can be further simplified to

$$\begin{aligned}&\frac{\Gamma(1-\alpha)}{s^{2n}}\sum_{m=0}^{n-1}\frac{v_0^{2n-2m}}{(2n-2m)!}(\alpha-1)_{2n-2m}\widehat{\mathcal{T}}_{2m}(s)\\&\sim\frac{1}{(2n)!}v_0^{2n}\Gamma(1-\alpha)(\alpha-1)_{2n}s^{-2n}\widehat{\mathcal{T}}_0(s)\\&\sim\frac{\alpha-1}{\sqrt{\pi}(2n)!}v_0^{2n}(\alpha-1)_{2n}\Gamma(1-\alpha)s^{\alpha-2-2n}.\end{aligned} \tag{B10}$$

Finally, from (B3) and (B9), for $n\geqslant 1$ we see that

$$\widehat{R}_{2n}(s) \sim \frac{v_0^{2n}\Gamma(2-\alpha)}{(2n)!\sqrt{\pi}}[(\alpha)_{2n}-(\alpha-1)_{2n}]s^{\alpha-2n-2}.$$

APPENDIX C: DERIVATIONS OF (19) AND (20)

We first consider the asymptotic form of the Laplace transform in result (17),

$$\begin{aligned}&\mathcal{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2\pm 2v_0\tau)^{n-m}\right]\Psi(\tau)\right\}(s)\\&=\sum_{r=0}^{n-m}\binom{n-m}{r}F_0^{n-m-r}(2v_0)^r(1+(-1)^r)\\&\quad\times\mathcal{L}_\tau\{\tau^{2n-2m-r}\Psi(\tau)\}(s).\end{aligned} \tag{C1}$$

When $n=m$ (such that $r=0$ is the sole remaining term), we conclude from (B1) that

$$\mathcal{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2\pm 2v_0\tau)^{n-m}\right]\Psi(\tau)\right\}(s)\sim\frac{2}{\alpha-1}. \tag{C2}$$

Otherwise, when $n>m$ the dominant term is the one with index $r=0$, i.e.,

$$\begin{aligned}&\mathcal{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2\pm 2v_0\tau)^{n-m}\right]\Psi(\tau)\right\}(s)\\&\sim 2F_0^{n-m}(\alpha-1)_{2n-2m}\Gamma(1-\alpha)s^{\alpha-1-(2n-2m)}.\end{aligned} \tag{C3}$$

Finally after combining (17) with (C2) and (C3), we conclude that for $n\geqslant 1$

$$\begin{aligned}\widehat{R}_n(s)\sim&\sum_{m=0}^{n-1}\frac{(\alpha-1)_{2n-2m}}{(m-n)!}\left(\frac{F_0}{2}\right)^{n-m}\Gamma(1-\alpha)\\&\times s^{-2n}\widehat{\mathcal{T}}_m(s)+\frac{s^{1-\alpha-2n}}{\alpha-1}\widehat{\mathcal{T}}_n(s),\end{aligned} \tag{C4}$$

where $\widehat{\mathcal{T}}_m(s)=s^{\alpha-1+2m}\widehat{T}_m(s)$. Moreover, when $n=0$, $\widehat{R}_0(s)=\widehat{\Psi}(s)\widehat{T}_0(s)=1/(\sqrt{\pi}s)$, since $\widehat{T}_0(s)=1/\{\sqrt{\pi}[1-\widehat{\phi}(s)]\}$.

Now we consider the Laplace transform in (18), and similarly we have

$$\begin{aligned}&\mathcal{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2\pm 2v_0\tau)^{n-m}\right]\phi(\tau)\right\}(s)\\&=\sum_{r=0}^{n-m}\binom{n-m}{r}F_0^{n-m-r}(2v_0)^r(1+(-1)^r)\\&\quad\times\mathcal{L}_\tau\{\tau^{2n-2m-r}\phi(\tau)\}(s).\end{aligned} \tag{C5}$$

Therefore, r must be zero when $m=n$, and according to (B4) we find

$$\mathcal{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2\pm 2v_0\tau)^{n-m}\right]\phi(\tau)\right\}(s)\sim 2\left(1-\frac{s}{\alpha-1}\right). \tag{C6}$$

For $n\geqslant m+1$, the term with index $r=0$ dominates the right-hand side of Eq. (C5), such that

$$\begin{aligned}&\mathcal{L}_\tau\left\{\left[\sum_{\pm}(F_0\tau^2\pm 2v_0\tau)^{n-m}\right]\phi(\tau)\right\}(s)\\&\sim -2F_0^{n-m}(\alpha)_{2n-2m}\Gamma(1-\alpha)s^{\alpha-(2n-2m)}.\end{aligned} \tag{C7}$$

Then for $n\geqslant 1$, according to (18) we see that

$$\begin{aligned}&\sqrt{\pi}2^n n!\widehat{T}_n(s)\\&=-\sum_{m=0}^{n-1}\frac{2^m\sqrt{\pi}n!}{(n-m)!}F_0^{n-m}(\alpha)_{2n-2m}\Gamma(1-\alpha)\\&\quad\times s^{\alpha-(2n-2m)}\widehat{T}_m(s)+2^n\sqrt{\pi}n!\left(1-\frac{s}{\alpha-1}\right)\widehat{T}_n(s)\\&\quad+H_n(0),\end{aligned} \tag{C8}$$

which can be further simplified to

$$\begin{aligned}&\frac{\sqrt{\pi}2^n n!}{\alpha-1}\widehat{\mathcal{T}}_n(s)\\&=-\sum_{m=0}^{n-1}\frac{2^m\sqrt{\pi}n!}{(n-m)!}F_0^{n-m}(\alpha)_{2n-2m}\Gamma(1-\alpha)\\&\quad\times s^{\alpha-1}\widehat{\mathcal{T}}_m(s)+H_n(0)s^{\alpha+2n-2}.\end{aligned} \tag{C9}$$

Before further analyzing the asymptotic behavior of $\widehat{\mathcal{T}}_n(s)$ from (C9), we need to consider $\widehat{\mathcal{T}}_0(s)$. In fact when $n=0$, the asymptotic behavior of $\widehat{T}_0(s)$ can be immediately obtained through

$$\widehat{T}_0(s)=\frac{1}{\sqrt{\pi}[1-\widehat{\phi}(s)]}\sim\frac{\alpha-1}{\sqrt{\pi}s}, \tag{C10}$$

so that $\widehat{\mathcal{T}}_0(s)=s^{\alpha-1}\widehat{T}_0(s)\sim(\alpha-1)s^{\alpha-2}/\sqrt{\pi}$. Therefore when $n\geqslant 1$, the dominant term on the right of (C9) is the one with index $m=0$, i.e.,

$$\widehat{\mathcal{T}}_n(s)\sim-\frac{\Gamma(1-\alpha)}{n!\sqrt{\pi}}\left(\frac{F_0}{2}\right)^n(\alpha)_{2n}(\alpha-1)^2s^{2\alpha-3}. \tag{C11}$$

Combining the asymptotic forms of $\{\widehat{\mathcal{T}}_n(s)\}$ with (C4), the leading term in the series on the right-hand side of (C4) is the term with $m = 0$, consequently

$$\sum_{m=0}^{n-1}\frac{(\alpha-1)_{2n-2m}}{(m-n)!}\left(\frac{F_0}{2}\right)^{n-m}\Gamma(1-\alpha)s^{-2n}\widehat{\mathcal{T}}_m(s) \sim \frac{\alpha-1}{\sqrt{\pi}n!}\left(\frac{F_0}{2}\right)^{n}(\alpha-1)_{2n}\Gamma(1-\alpha)s^{-2n+\alpha-2}. \tag{C12}$$

Further combining with (C11), we obtain for $n \geqslant 1$

$$\begin{aligned}\widehat{R}_n(s) \sim{}& \frac{\alpha-1}{\sqrt{\pi}n!}\left(\frac{F_0}{2}\right)^{n}(\alpha-1)_{2n}\Gamma(1-\alpha)s^{-2n+\alpha-2} \\ &- \frac{\Gamma(1-\alpha)}{n!\sqrt{\pi}}\left(\frac{F_0}{2}\right)^{n}(\alpha)_{2n}(\alpha-1)s^{-2n+\alpha-2} \\ \sim{}& \frac{\alpha-1}{\sqrt{\pi}n!}\left(\frac{F_0}{2}\right)^{n}[(\alpha-1)_{2n}-(\alpha)_{2n}] \\ &\times\Gamma(1-\alpha)s^{-2n+\alpha-2}.\end{aligned} \tag{C13}$$

APPENDIX D: SPECIFIC ALGORITHM FOR LW IN CONSTANT EXTERNAL FORCE FIELD

The numerical pseudo code for simulation of LW in constant external force field F_0 is illustrated in Algorithm 1.

APPENDIX E: SOME RESULTS WHEN $\alpha \in (0, 1)$

In this part, we consider ordinary LWs with constant speed v_0. When the power-law density $\phi(\tau)$ in (1) has infinite average, i.e., $\alpha \in (0, 1)$, then $\widehat{\phi}(s) \sim 1 - \Gamma(1-\alpha)^{\alpha}$ and $\widehat{\Psi}(s) \sim \Gamma(1-\alpha)s^{\alpha-1}$. It follows that

$$\mathscr{L}_\tau\{\tau^{n-m}\Psi(\tau)\}(s) \sim \begin{cases}(-1)^{n-m}(\alpha-1)_{n-m}\Gamma(1-\alpha)s^{-1+\alpha-(n-m)} & \text{if } n>m;\\ \Gamma(1-\alpha)s^{\alpha-1} & \text{if } n=m.\end{cases} \tag{E1}$$

Then we have

$$\begin{aligned}\widehat{R}_n(s) ={}& \frac{1}{2}\sum_{m=0}^{n-1}\frac{(-v_0)^{n-m}}{(n-m)!}[1+(-1)^{(n-m)}](\alpha)_{n-m} \\ &\times\Gamma(1-\alpha)s^{-1+\alpha-(n-m)}\widehat{\mathcal{T}}_m(s) \\ &+\Gamma(1-\alpha)s^{\alpha-1}\widehat{T}_n(s).\end{aligned} \tag{E2}$$

Similarly, according to the symmetry of the process we only need to consider the even terms,

$$\begin{aligned}\widehat{R}_{2n}(s) \sim{}& \sum_{m=0}^{n-1}\frac{v_0^{2n-2m}}{(2n-2m)!}(\alpha-1)_{2n-2m}\Gamma(1-\alpha)s^{-2n}\widehat{\mathcal{T}}_{2m}(s) \\ &+\Gamma(1-\alpha)s^{-2n}\widehat{\mathcal{T}}_{2n}(s),\end{aligned} \tag{E3}$$

where in this part $\widehat{\mathcal{T}}_{2n}(s) = s^{-1+\alpha+2n}\widehat{T}_{2n}(s)$. Conversely,

$$\mathscr{L}_\tau\{\tau^{2n-2m}\phi(\tau)\}(s) \sim \begin{cases}-(\alpha-1)_{2n-2m}\Gamma(1-\alpha)s^{\alpha-(2n-2m)} & \text{if } n>m;\\ \Gamma(1-\alpha)s^{\alpha} & \text{if } n=m.\end{cases} \tag{E4}$$

Then for $n \geqslant 1$

$$\begin{aligned}&\sqrt{\pi}2^{2n}(2n)!\widehat{T}_{2n}(s)s^{\alpha-1+2n} \\ &\sim -\sum_{m=0}^{n-1}\frac{\sqrt{\pi}2^{2m}(2n)!}{(2n-2m)!}(2v_0)^{2n-2m}(\alpha)_{2n-2m} \\ &\quad\times\Gamma(1-\alpha)s^{-1+2\alpha+2m}\widehat{\mathcal{T}}_{2m}(s) \\ &\quad+\sqrt{\pi}2^{2k}(2k)!s^{-1+\alpha+2k}\widehat{T}_{2k}(s) \\ &\quad-\sqrt{\pi}2^{2k}(2k)!\Gamma(1-\alpha)s^{-1+2\alpha+2k}\widehat{\mathcal{T}}_{2k}(s) \\ &\quad+s^{-1+\alpha+2n}H_{2n}(0),\end{aligned} \tag{E5}$$

which leads to

$$\begin{aligned}&\sqrt{\pi}2^{2n}(2n)!\Gamma(1-\alpha)\widehat{\mathcal{T}}_{2n}(s) \\ &\sim -\sum_{m=0}^{n-1}\frac{\sqrt{\pi}2^{2n}(2m)!}{(2n-2m)!}v_0^{2n-2m} \\ &\quad\times(\alpha)_{2n-2m}\Gamma(1-\alpha)\widehat{\mathcal{T}}_{2m}(s)+s^{2n-1}H_{2n}(0).\end{aligned} \tag{E6}$$

Therefore for $n \geqslant 1$ we have

$$\widehat{\mathcal{T}}_{2n}(s) \sim -\sum_{m=0}^{n-1}\frac{v_0^{2n-2m}}{(2n-2m)!}(\alpha)_{2n-2m}\widehat{\mathcal{T}}_{2m}(s). \tag{E7}$$

When $n = 0$, it can be obtained that

$$\widehat{\mathcal{T}}_0(s) = s^{\alpha-1}\widehat{T}_0(s) = \frac{s^{\alpha-1}}{\sqrt{\pi}[1-\widehat{\phi}(s)]} \sim \frac{1}{\sqrt{\pi}\Gamma(1-\alpha)s}. \tag{E8}$$

Algorithm 1. Simulate trajectory of LW in constant external force field.

```
Input: x_0: initial position; t: observation time; v_0: initial velocity for
    each step; F_0: constant external force; φ(τ): PDF for τ in (1);
Output: The position x_t of process at time t;
set x_t = x_0, set T = 0;
while 1 == 1 do
    generate τ from φ(τ), set T = τ + T;
    if T ⩽ t then
        generate a random variable uniformly distributed in interval
    (0,1) to represent the direction of initial velocity, r = rand(0, 1);
      if r ⩽ 0.5 then
        x_t = 1/2 * F_0 * τ^2 + x_t + v_0 * τ;
      else
        x_t = 1/2 * F_0 * τ^2 + x_t − v_0 * τ;
      end if
    else
      τ' = t − T + τ, calculate the residual time from T_N to t;
      generate r = rand(0, 1);
      if r ⩽ 0.5 then
        x_t = 1/2 * F_0 * τ'^2 + x_t + v_0 * τ';
      else
        x_t = 1/2 * F_0 * τ'^2 + x_t − v_0 * τ';
      end if
      break
    end if
end while
return x_t
```

According to (E7) we see that for $n \geqslant 1$

$$\Gamma(1-\alpha)s^{-2n}\widehat{\mathcal{T}}_{2n}(s) \sim -\sum_{m=0}^{n-1}\frac{v_0^{2n-2m}}{(2n-2m)!}(\alpha)_{2n-2m}\widehat{\mathcal{T}}_{2m}(s)\Gamma(1-\alpha)s^{-2n}. \quad \text{(E9)}$$

Finally, according to (E3), for $n \geqslant 1$ we have

$$\widehat{R}_{2n}(s) \sim \sum_{m=0}^{n-1}\frac{v_0^{2n-2m}}{(2n-2m)!}[(\alpha-1)_{2n-2m}-(\alpha)_{2n-2m}] \times \Gamma(1-\alpha)\widehat{\mathcal{T}}_{2m}(s)s^{-2n}. \quad \text{(E10)}$$

The explicit asymptotic expression for each $\widehat{\mathcal{T}}_{2n}(s)$ is hard to obtain based on (E7), so from here numerical methods of evaluation have to be chosen.

APPENDIX F: CALCULATIONS OF LAPLACE TRANSFORMS UTILIZED IN SEC. IV

When $\phi(\tau)=\alpha(1+\tau)^{-1-\alpha}$, some Laplace transforms needed in Sec. IV are calculated here. The first Laplace transform is

$$\begin{aligned}\mathscr{L}_\tau\{\cos^2(\omega\tau)\phi(\tau)\}(s) &= \frac{\alpha}{2}e^sE_{1+\alpha}(s)+\frac{\alpha}{4}e^{s-2i\omega}E_{1+\alpha}(s-2i\omega)\\ &+\frac{\alpha}{4}e^{s+2i\omega}E_{1+\alpha}(s+2i\omega),\end{aligned} \quad \text{(F1)}$$

where $E_\theta(x)=\int_1^\infty t^{-\theta}e^{-xt}dt$ is the exponential integral. Next by taking a Taylor expansion on the right-hand side of (F1) and ignoring higher order terms, the leading term is

$$\frac{\alpha}{2}e^sE_{1+\alpha}(s) \sim \frac{1}{2}+\frac{s}{2(1-\alpha)}+\frac{\alpha}{2}\Gamma(-\alpha)s^\alpha. \quad \text{(F2)}$$

Further, the following relation is needed to calculate the Taylor expansion of the next two terms of (F1),

$$\begin{aligned}&e^{-2i\omega}E_{1+\alpha}(-2i\omega)+e^{2i\omega}E_{1+\alpha}(2i\omega)\\ &=\int_1^\infty\frac{e^{-2i\omega(t+1)}}{t^{1+\alpha}}dt+\int_1^\infty\frac{e^{2i\omega(t+1)}}{t^{1+\alpha}}dt\\ &=2\int_1^\infty\frac{\cos[2\omega(t+1)]}{t^{1+\alpha}}dt\\ &=2\mathcal{I}_{\alpha+1}(\omega),\end{aligned} \quad \text{(F3)}$$

where $\mathcal{I}_\theta(\omega)$ is defined in (45). Therefore

$$\begin{aligned}&e^{s-2i\omega}E_{1+\alpha}(s-2i\omega)+e^{s+2i\omega}E_{1+\alpha}(s+2i\omega)\\ &\sim[e^{-2i\omega}E_{1+\alpha}(-2i\omega)+e^{2i\omega}E_{1+\alpha}(2i\omega)]\\ &+s[-e^{-2i\omega}E_\alpha(-2i\omega)+e^{-2i\omega}E_{1+\alpha}(-2i\omega)]\\ &+s[-e^{2i\omega}E_\alpha(2i\omega)+e^{2i\omega}E_{1+\alpha}(2i\omega)]\\ &=2\mathcal{I}_{\alpha+1}(\omega)+2[\mathcal{I}_\alpha(\omega)+\mathcal{I}_{\alpha+1}(\omega)]s.\end{aligned} \quad \text{(F4)}$$

Finally, from (F1) we have

$$\begin{aligned}&\mathscr{L}_\tau\{\cos^2(\omega\tau)\phi(\tau)\}(s)\\ &\sim\frac{1}{2}+\frac{\alpha}{2}\mathcal{I}_{\alpha+1}(\omega)+\frac{\alpha}{2}\Gamma(-\alpha)s^\alpha\\ &+\left[\frac{1}{2(1-\alpha)}+\frac{\alpha}{2}\mathcal{I}_\alpha(\omega)+\frac{\alpha}{2}\mathcal{I}_{\alpha+1}(\omega)\right]s.\end{aligned} \quad \text{(F5)}$$

Similarly,

$$\begin{aligned}&\mathscr{L}_\tau\{\sin^2(\omega\tau)\phi(\tau)\}(s)\\ &=\frac{\alpha}{2}e^sE_{1+\alpha}(s)-\frac{\alpha}{4}[e^{s-2i\omega}E_{1+\alpha}(s-2i\omega)\\ &+e^{s+2i\omega}E_{1+\alpha}(s+2i\omega)]\\ &\sim\frac{1}{2}-\frac{\alpha}{2}\mathcal{I}_{\alpha+1}(\omega)+\frac{\alpha}{2}\Gamma(-\alpha)s^\alpha\\ &+\left[\frac{1}{2(1-\alpha)}-\frac{\alpha}{2}\mathcal{I}_\alpha(\omega)-\frac{\alpha}{2}\mathcal{I}_{\alpha+1}(\omega)\right]s.\end{aligned} \quad \text{(F6)}$$

Further, the corresponding survival probability of $\phi(\tau)$ is

$$\Psi(\tau)=\int_\tau^\infty\alpha(1+\tau')^{-1-\alpha}d\tau'=(1+\tau)^{-\alpha}. \quad \text{(F7)}$$

Then

$$\begin{aligned}&\mathscr{L}_\tau\{\cos^2(\omega\tau)\Psi(\tau)\}(s)\\ &=\frac{1}{2}e^sE_\alpha(s)+\frac{1}{4}[e^{s-2i\omega}E_\alpha(s-2i\omega)\\ &+e^{s+2i\omega}E_\alpha(s+2i\omega)]\\ &\sim\frac{1}{2(\alpha-1)}+\frac{1}{2}\mathcal{I}_\alpha(\omega)+\frac{1}{2}\Gamma(1-\alpha)s^{\alpha-1}\\ &+\frac{s}{2}\left[-\frac{1}{(\alpha-2)(\alpha-1)}+\mathcal{I}_\alpha(\omega)-\mathcal{I}_{\alpha-1}(\omega)\right].\end{aligned} \quad \text{(F8)}$$

Finally the following Laplace transform is

$$\begin{aligned}&\mathscr{L}_\tau\{\sin^2(\omega\tau)\Psi(\tau)\}(s)\\ &=\frac{1}{2}e^sE_\alpha(s)-\frac{1}{4}[e^{s-2i\omega}E_\alpha(s-2i\omega)\\ &+e^{s+2i\omega}E_\alpha(s+2i\omega)]\\ &\sim\frac{1}{2(\alpha-1)}-\frac{1}{2}\mathcal{I}_\alpha(\omega)+\frac{1}{2}\Gamma(1-\alpha)s^{\alpha-1}\\ &+\frac{s}{2}\left[-\frac{1}{(\alpha-2)(\alpha-1)}-\mathcal{I}_\alpha(\omega)+\mathcal{I}_{\alpha-1}(\omega)\right].\end{aligned} \quad \text{(F9)}$$

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