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From continuous-time random walks to the fractional Jeffreys equation: Solution and properties

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ABSTRACT

Jeffreys equation provides an increasingly popular extension of the diffusive laws of Fourier and Fick for heat and particle transport. Similar to generalisations of the diffusion equation, we here investigate the connection between a time-fractional generalisation of the Jeffreys equation and a continuous-time random walk process based on a generalised waiting time density with diverging mean. We demonstrate that the mean squared displacement exhibits a variety of anomalous behaviors, such as retarding and accelerating subdiffusion, as well as a crossover from superdiffusion to subdiffusion. Moreover, we provide two alternative approaches, namely, a fractional Taylor series and distributed-order derivatives, that transform Fourier's or Fick's law into the time-fractional Jeffreys equation. Our discussion provides physics-based support for the fractional Jeffreys equation and shows its versatility for practical applications.

1. Introduction

Fourier's law for heat conduction and Fick's law for particle dispersion both lead to the (parabolic) diffusion equation for the corresponding fields in the hydrodynamic limit. When finite propagation speed effects become relevant, the description in terms of the diffusion equation becomes problematic [1]. An alternative approach is provided by the hyperbolic Cattaneo or telegrapher's equations including a second-order time derivative [1]. However, long before these modifications of the diffusion equation approach, Jeffreys presented a relation for the rheology of the Earth's core [2], a third-order partial differential equation. The linear Jeffreys equation (JE) consists of the constitutive law [2][1]

$$J^{'}\left(x^{'},t^{'}\right)+\tau_j\frac{\partial J^{'}\left(x^{'},t^{'}\right)}{\partial t^{'}}=-K\left(1+\tau_p\frac{\partial}{\partial t^{'}}\right)\frac{\partial P^{'}\left(x^{'},t^{'}\right)}{\partial x^{'}},\tag{1}$$

and the standard continuity equation

$$\frac{\partial P'(x',t')}{\partial t'}=-\frac{\partial J'(x',t')}{\partial x'},\tag{2}$$

where $J'(x',t')$ is the particle flux and $P'\left(x',t'\right)$ is the probability density function (PDF). Moreover, here x' denotes the spatial variable, t' is time, and K is the diffusivity. Conventionally, the PDF is a positive quantity and its integral along the entire domain equals unity. The temporal constants τ_j and τ_p in general are not equivalent, $\tau_j\neq\tau_p$. Eliminating the flux $J'(x',t')$ from Eqs. (1) and (2) results in the Jeffreys (or Jeffreys-type) equation

$$\frac{\partial P'(x',t')}{\partial t'}+\tau_j\frac{\partial^2 P'(x',t')}{\partial t'^2}=K\left(1+\tau_p\frac{\partial}{\partial t'}\right)\frac{\partial^2 P'(x',t')}{\partial x'^2}.\tag{3}$$

[1] The primes in these equations is used to distinguish the dimensional formulation from the dimensionless equations below.

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When the time constants in this equation are equal or can be neglected, $\tau_j = \tau_p = 0$, Eq. (1) reduces to the well-known Fick's first law, and from Eq. (3) we arrive back at the conventional diffusion equation. When only $\tau_p = 0$, Eq. (3) reduces to the telegrapher's equation. In the 1990s the Jeffreys equation found applications in heat transfer [1], and was used in the Dual-Phase-Lag model [3] which was shown to capture different heat transport processes in metals, superfluid helium, porous, and amorphous media; see [4] on the lagging behavior concept and its applications, and [5] on the microscale heat transfer models and references therein. The Jeffreys Eq. (3) has a similar structure to the energy equation governing lattice and the electron temperature distributions in the two-step model, and the energy equation governing the temperature distribution in the Guyer-Krumhansl model, see [1,3,4].

In the case of particle transport the lagging behavior assumes a non-instantaneous relation between the particle flux and the concentration gradient,

$$J'(x', t' + \tau_j) = -K \frac{\partial P'(x', t' + \tau_p)}{\partial x'}, \tag{4}$$

thus, it distinguishes two main types of particles flow: the flux-precedence (*flux-driven*) diffusion ($\tau_j < \tau_p$) and the concentration gradient-precedence (*gradient-driven*) diffusion ($\tau_j > \tau_p$). The latter is the more familiar situation, in which the flux is the causal result of the gradient. The flux-driven situation may be found in self-propelled systems such as the diffusion of energetic particles or the run-and-tumble motion, in which the particle flux occurs first, regardless of the presence of concentration differences. As a natural result of the flux-driven situation, the concentration of particles will be changed subsequently. The flux-driven model is conventionally invoked in modeling "*ultrafast thermal phenomena*" in heat transfer problems. In effect, Eq. (4) introduces a rather intuitive explanation for existing physical situations, in particular, when the time of observation t' is of the same order of the time constants τ_j and τ_p. In the opposite case, when the observation time is much greater than the time scales τ_j and τ_p, then $t' + \tau_j \approx t' + \tau_p \approx t'$, and Eq. (4) reduces to Fick's first law $J'(x', t') = -K\partial P'(x', t')/\partial x'$. When τ_j and τ_p are sufficiently small with respect to observational time, Eq. (4) can be approximated using a Taylor's series, yielding

$$\begin{aligned} &J'\left(x', t'\right) + \tau_j \frac{\partial J'\left(x', t'\right)}{\partial t'} + O\left(\tau_j^2\right) \\ &\quad = -K\left(1 + \tau_p \frac{\partial}{\partial t'} + O\left(\tau_p^2\right)\right) \frac{\partial P'\left(x', t'\right)}{\partial x'}, \end{aligned} \tag{5}$$

which in turn can be viewed as the Jeffreys constitutive law (1) if the second order terms in τ_j and τ_p are neglected. It is worth mentioning that some generalized cases of the Dual-Phase-Lag, Eq. (4), including those representing flux-driven and gradient-driven flows ($\tau_j \lessgtr \tau_p$) under certain conditions obey the criteria of stability, well-posedness, and the spatial decay estimate [6].

In [7], the mean squared displacement (MSD) resulting from the JE (3) was demonstrated to exhibit non-anomalous behavior, analogous to Brownian motion governed by the underdamped Langevin equation: ballistic motion at short times and normal diffusion at long times. An element of particular interest is the immobilization of particles described by the JE and its connection with a two-phase mobile-immobile reaction-diffusion model of mass transfer [7]. Models of diffusion including the immobile phase are employed in such diverse physical processes as, e.g., groundwater transport [8], glassy dynamics [9], and protein binding interactions in live cells [10]. The immobilization of particles also plays a key role in the aging continuous time random walk model [11] and in stochastic resetting contexts [12]. All these circumstances prompt further studies and generalizations of the JE that are able to capture anomalous diffusion processes, the subject of this work.

For the description of anomalous transport processes in a range of different systems, the theory of diffusion and Fokker-Planck equations with partial fractional derivatives in time and space, called *fractional kinetics*, has been established as a versatile tool encoding the non-linear growth in time of the MSD $\propto t'^{\mu}$ with $\mu \neq 1$. One distinguishes *subdiffusion* for $0 < \mu < 1$, *superdiffusion* for $1 < \mu < 2$, and (sometimes) *hyperdiffusion* with $\mu > 2$. The limiting cases of Brownian motion (normal diffusion) and ballistic motion respectively correspond to $\mu = 1$ and $\mu = 2$. Finally, if the MSD grows in time as a positive power of the logarithm, the transport process is called *ultraslow*. Anomalous diffusion processes have been ubiquitously unveiled in a large range of systems, see [13,14] and the references therein for numerous examples and their classification. Applications of fractional kinetics range from charge transport in amorphous semiconductors to underground water pollution and motion of subcellular units in biology [15–17]. The continuous time random walk (CTRW), a process where a wandering particle waits for a random time between random jumps [18,19] represents the probabilistic basis for fractional kinetics: Indeed, fractional diffusion equations with time or space fractional derivatives can be derived as long time-space limits of the CTRW with heavy-tailed distributions of waiting times or jump lengths, respectively [20–24]. A further important generalization invokes mixtures of fractional derivatives (the distributed order fractional kinetics) [25–33]. Fractional generalizations of the Cattaneo (telegrapher's) equation were also discussed [34–37]. A fractional generalization of the Jeffreys Eq. (1) was phenomenologically found in [38], while connecting the process to a fractional two-step model was established in [39], see also [40–42]. We also note that anomalous diffusion on a comb with the Dual-Phase-Lag constitutive relation and its fractional generalization have recently been explored in a series of papers [43], while a similar model was employed to describe the flow of an incompressible Oldroyd-B fluid [44].

However, in contrast to fractional kinetics, the microscopic probabilistic CTRW foundation of the fractional models capturing the lagging behavior has not been addressed. In the present work we consider a CTRW model with a specific heavy-tailed waiting time density which asymptotically leads to the fractional generalization of the JE suggested in [40]. We also show that the fractional Jeffreys equation (FJE) is a particular case of the *generalized non-Markovian Fokker-Planck equation* that was derived by Robert Zwanzig in his seminal 1961 paper [45]. We thus establish a connection with the projection operator formalism which found its applications in diverse problems of non-equilibrium statistical mechanics [46]. We demonstrate that the FJE exhibits a variety of anomalous diffusion regimes and moreover suggest alternative phenomenological approaches to the origin of the FJE. The FJE is thus a well-founded extension of the JE that provides a versatile framework for the description of non-standard transport processes.

The paper is organized as follows. After some preliminary remarks on the necessary positivity of a probability density function (PDF) in Section 2, Section 3 is devoted to the derivation of an exact solution of the CTRW process corresponding to the FJE, based on a specific choice of the waiting time PDF. Important particular cases are discussed. In Section 4 we provide a closed-form solution to the FJE in terms of the Fox H-function for the flux-driven case and discuss the various different anomalous diffusion regimes described by the FJE. In Section 5 we consider phenomenological approaches that demonstrate how the FJE emerges from the (classical) Fourier or Fick laws. These derivations provide additional insight to the reader about the physical relevance of the FJE considered in this work. We draw our conclusions and outline future plans in Section 6.

2. Preliminary remarks on the positivity of the solution of the fractional Jeffreys equation

One of the essential requirements for the PDF governed by a partial differential equation (or an integrodifferential equation) is that the fundamental solution is non-negative everywhere and at all times. Some successful mathematical models that work well in the one-dimensional setting, however, lose their non-negativity in higher dimensions. Generally speaking, mathematical models possessing wave features may result in solutions taking negative values in higher dimensions under specific choices of the model parameters as discussed, e.g., for the fractional wave equation [47,48], the ordinary telegrapher's equation [49], or the ordinary JE (3), see the discussion in [50]. Accordingly, the domain of validity of the fractional generalizations should be carefully determined to guarantee the non-negativity of the associated PDF. It has been shown [40] that fractionalizing (1) to

$$\left(1+\tau_j^{\alpha}\,{}_0^{RL}\mathcal{D}_{t'}^{\alpha}\right)J'\left(x',t'\right)=-K_{\gamma}\,{}_0^{RL}\mathcal{D}_{t'}^{1-\gamma}\left(1+\tau_p^{\beta}\,{}_0^{RL}\mathcal{D}_{t'}^{\beta}\right)\frac{\partial P'\left(x',t'\right)}{\partial x'},\tag{6}$$

ensures the non-negative solution in the one-dimensional setting under the sufficient condition

$$0<\alpha,\beta,\gamma\le 1,\qquad \tau_j>0,\quad \tau_p>0,\qquad \beta\le\gamma,\tag{7a}$$

while for the two- and three-dimensional settings, in which the first derivative $\partial/\partial x'$ with respect to the spatial variable is replaced by the gradient operator $\nabla=(\partial/\partial x_1',\cdots,\partial/\partial x_d')$ where $d=2,3$, the non-negativity of solutions requires the sufficient conditions

$$0<\alpha,\beta,\gamma<1,\quad \tau_j>0,\quad \tau_p>0,\quad 0<\alpha+\gamma\le 1,\quad \beta\le\gamma.\tag{7b}$$

A particular case of Eq. (6) is the fractional telegrapher equation ($\tau_p=0$) subject to the sufficient conditions $0<\alpha,\gamma<1$, $\tau_j>0$, $0<\alpha+\gamma\le 1$. In Eq. (6) ${}_0^{RL}\mathcal{D}_t^{\alpha}$ is the Riemann-Liouville fractional derivative defined by [51]

$${}_0^{RL}\mathcal{D}_t^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(\zeta)}{(t-\zeta)^{\alpha}}d\zeta, & 0<\alpha<1;\\ \frac{df(t)}{dt}, & \alpha=1,\end{cases}\tag{8}$$

and as usual we used the generalized diffusion coefficient K_γ with physical units $Length^2/Time^{\gamma}$ to keep the dimension in order ($\gamma=1$ corresponds to the conventional diffusivity $K_1=K$).

Before we proceed, for convenience let us replace Eqs. (6) and (1) with the dimensionless analogues

$$\left(1+{}_0^{RL}\mathcal{D}_t^{\alpha}\right)J(x,t)=-{}_0^{RL}\mathcal{D}_t^{1-\gamma}\left(1+\chi^{\beta}\,{}_0^{RL}\mathcal{D}_t^{\beta}\right)\frac{\partial P(x,t)}{\partial x},\tag{9}$$

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial J(x,t)}{\partial x},\tag{10}$$

where the dimensionless quantities

$$t=\frac{t'}{\tau_j},\quad x=\frac{x'}{\sqrt{K\tau_j}},\quad P=\frac{P'}{P_0},\quad J=J'\sqrt{\frac{\tau_j}{KP_0^2}}\tag{11}$$

have been used. Here P_0 is a characteristic constant with dimension $[P_0]=[P]$, and we have chosen $K_\gamma=K\tau_j^{1-\gamma}$, without loss of generality. In Eq. (9), $\chi=\tau_p/\tau_j$, thus the cases $\chi\lesseqgtr 1$ correspond to $\tau_p\lesseqgtr\tau_j$.

3. CTRW process

3.1. Model

We consider a random walker (test particle) that is initially placed at the origin $x_0=0$ at $t_0=0$ and has equal probabilities to go right or left along the real line $\mathbb{R}$. The particle is trapped at the origin for a time t_1, then it hops (with infinite speed) to the next position x_1 where it is immobilised for a time t_2. Next the particle leaves x_1 and makes a jump of length x_2, and so forth. After a time $T_n=t_1+t_2+\cdots+t_n$, the particle arrives at $X_n=x_1+x_2+\cdots+x_n$, $n\in\mathbb{N}$. It is assumed that $t_1,\ t_2,\ t_3,\dots$ are independent and identically distributed (iid) positive random waiting times subject to the PDF $\psi(t)$, and $x_1,\ x_2,\ x_3,\dots$ are iid random jump lengths with the PDF $\lambda(x)$. Thus we are dealing with a renewal process [19,52], and the PDF to find the particle at position x at time t can be found from the master equation [20,53]:

$$P_{CTRW}(x,t)=\Psi(t)\delta(x)+\int_0^t\psi(t-\tau)\left[\int_{-\infty}^{\infty}\lambda(x-\xi)P_{CTRW}(\xi,\tau)d\xi\right]d\tau,\tag{12}$$

where $\delta(\cdot)$ is the Dirac delta function. Here, the continuum limit has been implicitly applied, i.e.,

$$\sum_i\lambda(x-\xi_i)P_{CTRW}(\xi_i,\tau)\to\int_{-\infty}^{\infty}\lambda(x-\xi)P_{CTRW}(\xi,\tau)d\xi,\tag{13}$$

and $P_{CTRW}(x.t)dx$ is defined as the probability to find the particle within the interval $(x,x+dx]$ at the time instant t. The function $\Psi(t)$ is the probability that the random walker has not jumped (i.e., here is still trapped at the origin $x_0=0$) till time t, i.e.,

$$\Psi(t)=1-\int_0^t\psi(\tau)d\tau=\int_t^{\infty}\psi(\tau)d\tau.\tag{14}$$

Introducing the Fourier-Laplace transform[2] in Eqs. (12) and (14), we obtain the well-known Montroll-Weiss equation

$$\hat{\tilde{P}}_{CTRW}(q,s)=\frac{1-\tilde{\psi}(s)}{s}\frac{1}{1-\tilde{\psi}(s)\hat{\lambda}(q)},\tag{15}$$

in which the first factor is the Laplace transform of $\Psi(t)$.

In what follows we consider the generalized waiting time PDF whose Laplace transform reads

$$\tilde{\psi}(s)=\exp\left(-\frac{s^{\gamma}(s^{\alpha}+1)}{1+\chi^{\beta}s^{\beta}}\right),\tag{16}$$

where $0<\alpha,\beta,\gamma\le 1$ and $\chi>0$. The jump length PDF is supposed to be Gaussian, i.e., $\lambda(x)=(4\pi)^{-1/2}\exp(-x^2/4)$, and thus

$$\hat{\lambda}(q)=\exp(-q^2).\tag{17}$$

Evidently, the waiting time PDF (16) generalizes the one-sided (Lévy) stable PDF $\ell_\gamma(t), t\ge 0$, see [74], which is usually defined through its Laplace transform $\tilde{\ell}_\gamma(s)=\exp(-s^{\gamma})$ with the stable (Lévy) index $0<\gamma<1$ and $\ell_\gamma(t)=0$ for $t<0$. $\ell_\gamma(t)$ is commonly used to derive the time-fractional diffusion equation from the CTRW model. The function $\psi(t)$ is clearly normalized, $\tilde{\psi}(s=0)=1$. To address the positivity of $\psi(t)$ defined by Eq. (16) the following proposition holds.

Proposition 1. *The sufficient condition for the waiting time distribution, whose Laplace transform is given by Eq. (16), to be a PDF are* $0<\alpha+\gamma\le 1$ *and* $\beta\le\gamma$, *or* $\alpha=\beta=\gamma=1$ *and* $\chi>1$.

[2] Here the tilde denotes a Laplace transform,

$$\tilde{f}(x,s)=\mathcal{L}\{f(x,t);t\}(x,s)=\int_0^{\infty}f(x,t)\exp(-st)dt,$$

and the hat represents a Fourier transform,

$$\hat{f}(q,t)=\mathcal{F}\{f(x,t);x\}(q,t)=\int_{-\infty}^{\infty}f(x,t)\exp(\imath qx)dx.$$

Here $s\in\mathbb{C}$ is the Laplace variable, and $q\in\mathbb{R}$ is the Fourier variable ("wave number").

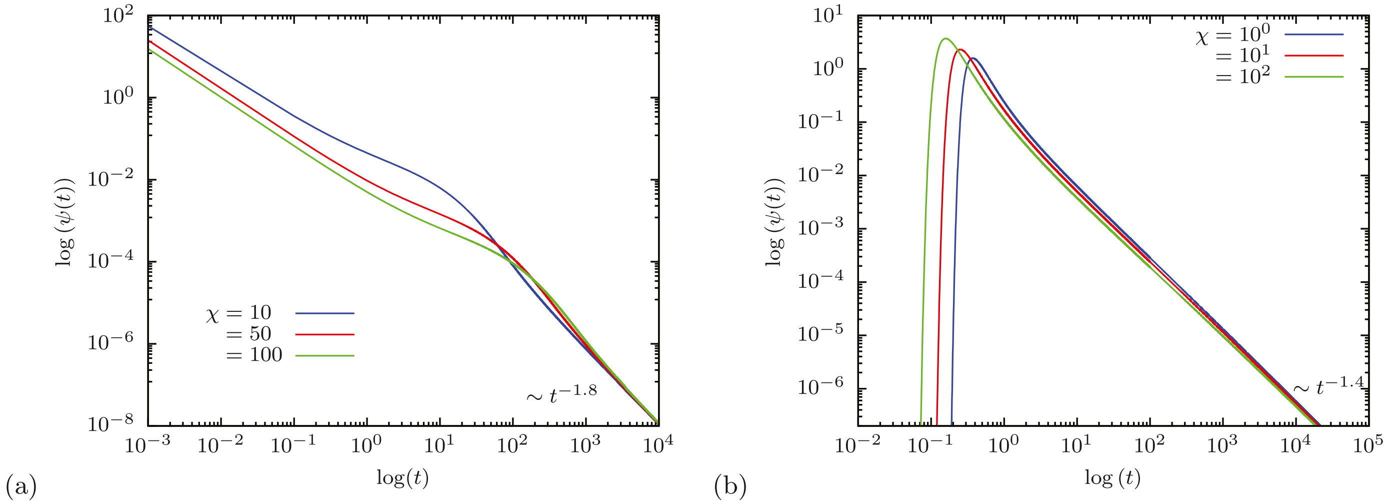

Fig. 1. Log-log plot of the waiting time PDF (16) for different values of the coefficient χ and: (a) $\alpha = 0.2$, $\beta = 0.8$, $\gamma = 0.8$; (b) $\alpha = 0.6$, $\beta = 0.2$, $\gamma = 0.4$.

Proof. See Appendix E. □

In Appendix C, we determine the inverse Laplace transform of the waiting time PDF (16), see Eq. (C.3), along with its asymptotics. In the long-time limit,

$$\psi(t) \sim \ell_\gamma(t) \sim \frac{\gamma t^{-\gamma-1}}{\Gamma(1-\gamma)}, \quad t \to \infty, \tag{18}$$

which can be derived using relations (16) or (C.3). Eq. (18) indicates that $\psi(t)$ is a fat-tailed PDF under the sufficient conditions of Proposition 1 with a divergent mean $\langle t_\psi \rangle = \int_0^\infty t\psi(t)dt \to \infty$.[3] The short-time behavior of $\psi(t)$ takes the form

$$\psi(t) \sim \chi^{\beta/(\alpha+\gamma-\beta)} \ell_{\alpha+\gamma-\beta}\left(\chi^{\beta/(\alpha+\gamma-\beta)} t\right), \quad t \to 0. \tag{19}$$

In other words, the generalized waiting time PDF (16) behaves like a one-sided Lévy stable PDF in both the long and short time limits, however, with different Lévy indices. Further properties of the waiting time PDF (16) are discussed in Appendix F.

In Fig. 1 we show the waiting time PDF (16) for different values of the model parameters. The long time limit clearly shows the expected power-law behavior in accordance with the analytical result (18). At the short and intermediate times, the behavior varies quantitatively with the value of χ with fixed fractional parameters. Indeed, while the long-time behavior (18) depends only on the fractional parameter γ, the short-time behavior (19) explicitly depends on all parameters. We note that in Fig. 1 (a) we do not see the expected decay to zero when time t tends to zero, in contrast to panel (b). This is due to the fact that the time scale for this decay to zero is affected by the coefficient χ as well as the specific choise of the fractional parameters. In our two examples, the choice $\alpha = 0.2$, $\beta = 0.8$, and $\gamma = 0.8$ leads to the coefficient χ^4 in the argument of the stable density, whereas $\alpha = 0.6$, $\beta = 0.2$, and $\gamma = 0.4$ leads to $\chi^{1/4}$. Thus, for the chosen χ values the decay to zero is shifted to much shorter times in panel (a).

3.2. Exact solution and MSD

Using the fact that $0 < \widetilde{\psi}(s)\widehat{\lambda}(q) < 1$ for $\Re\{s\} > 0$ and $q \in \mathbb{R}$, see Eqs. (16) and (17), the Montroll-Weiss Eq. (15) can be rewritten as the series

$$\widehat{\widetilde{P}}_{CTRW}(q,s) = \frac{1-\widetilde{\psi}(s)}{s} \sum_{u=0}^{\infty} \left[\widetilde{\psi}(s)\widehat{\lambda}(q)\right]^u. \tag{20}$$

Substituting Eqs. (16) and (17) into (20) and inverting the Laplace-Fourier transform, we obtain the exact solution of the CTRW process in the subordination form

$$P_{CTRW}(x,t) = \sum_{u=0}^{\infty} N(u,t) G(x,u), \tag{21}$$

where we used the auxiliary functions

$$\widetilde{N}(u,s) = \widetilde{g}(u,s) - \widetilde{g}(u+1,s), \quad \widetilde{g}(u,s) = \frac{1}{s}\exp\left(-u\frac{s^\gamma(s^\alpha+1)}{1+\chi^\beta s^\beta}\right) \tag{22}$$

and

$$\widehat{G}(q,u) = \exp(-uq^2). \tag{23}$$

In Eq. (21), $N(u,t)$ is the probability that exactly u jumps are performed within the interval $(0,t)$, while $G(x,u)$ in Eq. (23) is the probability that the random walker arrives at x after exactly u steps. The exact solution of our CTRW model can be brought in a more explicit form by reverting expressions (22) and (23) to the physical domain,

$$N(u,t) = g(u,t) - g(u+1,t), \tag{24}$$

and

$$G(x,u) = \frac{1}{\sqrt{4\pi u}}\exp\left(-\frac{x^2}{4u}\right), \tag{25}$$

where

$$g(u,t) = \mathcal{L}^{-1}\{\widetilde{g}(u,s)\}, \tag{26}$$

with the properties

$$g(0,t) = 1, \quad g(1,t) = \int_0^t \psi(\tau)d\tau, \quad \Psi(t) = g(0,t) - g(1,t). \tag{27}$$

Thus, the solution of the CTRW process can be written as

$$P_{CTRW}(x,t) = \Psi(t)\delta(x) + \sum_{u=1}^{\infty}[g(u,t) - g(u+1,t)]G(x,u). \tag{28}$$

[3] In the case $\alpha = \beta = \gamma = 1$ and $\chi > 1$, the waiting time PDF $\widetilde{\psi}_1(s) = \exp(-s(s+1)/[1+\chi s])$ generalizes the Dirac delta waiting time PDF $\delta(t-1)$ for Brownian motion, which can be obtained from $\widetilde{\psi}_1(s)$ by setting $\chi = 1$. The mean of $\psi_1(t)$ does not diverge, namely, $\langle t_{\psi_1} \rangle = 1$.

The MSD of the exact solution (28) defined by $\langle X^2_{CTRW}(t)\rangle = \int_{-\infty}^{\infty} x^2 P_{CTRW}(x,t)dx$, can be readily calculated, yielding

$$\langle X^2_{CTRW}(t)\rangle = 2\sum_{u=1}^{\infty} u[g(u,t) - g(u+1,t)]. \tag{29}$$

The long time behavior of the MSD (29), which will be relevant for our final result, can be evaluated by inverting the MSD to the Laplace domain,

$$\langle \widetilde{X}^2_{CTRW}(s)\rangle = \frac{2}{s}\mathfrak{S}\left(\frac{s^\gamma(s^\alpha+1)}{1+\chi^\beta s^\beta}\right)\left[1-\exp\left(-\frac{s^\gamma(s^\alpha+1)}{1+\chi^\beta s^\beta}\right)\right], \tag{30}$$

where $\mathfrak{S}(\zeta) = \sum_{u=1}^{\infty} u\exp(-u\zeta)$. Using the relation $\mathfrak{S}(\zeta) \sim 1/\zeta^2$ for $\zeta \to 0$, we get

$$\langle \widetilde{X}^2_{CTRW}(s)\rangle \sim 2\frac{1+\chi^\beta s^\beta}{s^{\gamma+1}(s^\alpha+1)} \tag{31}$$

for $\Re\{s\} \to 0$. After an inverse Laplace transform, we obtain the asymptotic behavior

$$\langle X^2_{CTRW}(t)\rangle \sim 2t^{\alpha+\gamma}E_{\alpha,\alpha+\gamma+1}(-t^\alpha) + 2\chi^\beta t^{\alpha+\gamma-\beta}E_{\alpha,\alpha+\gamma-\beta+1}(-t^\alpha)$$
$$\sim \frac{2t^\gamma}{\Gamma(1+\gamma)}, \tag{32}$$

which can be shown to work well for relatively large values of time, and it leads to the stated power-law behavior for $t \to \infty$. Here we used the generalized Mittag-Leffler function as defined in Appendix A.

3.3. Special and limiting cases

Here we consider relevant special cases for particular choices of the parameters in expression (16) for the waiting time PDF:

(i) *Normal diffusion equation.* Let $\alpha = \beta$, $\chi = 1$ and $\gamma = 1$. Then, $\widetilde{\psi}(s) = \exp(-s)$, and $\psi(t) = \delta(t-1)$. This is the simplest case of the CTRW with a constant time pace. In the long space-time limit $\widetilde{\psi}(s) \sim 1-s$, $\widehat{\lambda}(q) \sim 1-q^2$, we use the Montroll-Weiss Eq. (15) to see that the process is described by the normal diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2 P(x,t)}{\partial x^2}. \tag{33}$$

(ii) *Fractional diffusion equation.* When $\alpha = \beta$, $\chi = 1$, and $0 < \gamma < 1$, Eq. (16) reduces to the one-sided Lévy stable PDF, $\widetilde{\ell}_\gamma(s) = \exp(-s^\gamma)$. The CTRW solution corresponding to the combination of $\widetilde{\ell}_\gamma(s)$ and $\widehat{\lambda}(q)$ is given by [54]

$$P_{CTRW}(x,t) = \left[1-L_\gamma(t)\right]\delta(x) + \sum_{u=1}^{\infty}\left[L_\gamma\left(\frac{t}{u^{1/\gamma}}\right) - L_\gamma\left(\frac{t}{(u+1)^{1/\gamma}}\right)\right]G(x,u), \tag{34}$$

where $G(x,u)$ is given by Eq. (25) and $L_\gamma(t) = \int_0^t \ell_\gamma(\tau)d\tau$ is the one-sided cumulative α-stable distribution. In the long time-space limit we have

$$\widetilde{\ell}_\gamma(s) \sim 1-s^\gamma, \quad \widehat{\lambda}(q) \sim 1-q^2, \tag{35}$$

which together with Eq. (15) leads to the time-fractional diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = {}^{RL}_{\ 0}\mathcal{D}_t^{1-\gamma}\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{36}$$

(iii) *Bifractional diffusion equation for retarding subdiffusion.* When $\chi = 0$, the waiting time PDF (16) reduces to

$$\widetilde{\psi}_{\gamma,\alpha}(s) = \exp(-s^\gamma[s^\alpha+1]), \tag{37}$$

which represents a PDF if $0 < \alpha+\gamma \le 1$. Using the useful asymptotic equivalence $e^{-x} \sim 1/(1+x)$ for small x we find that

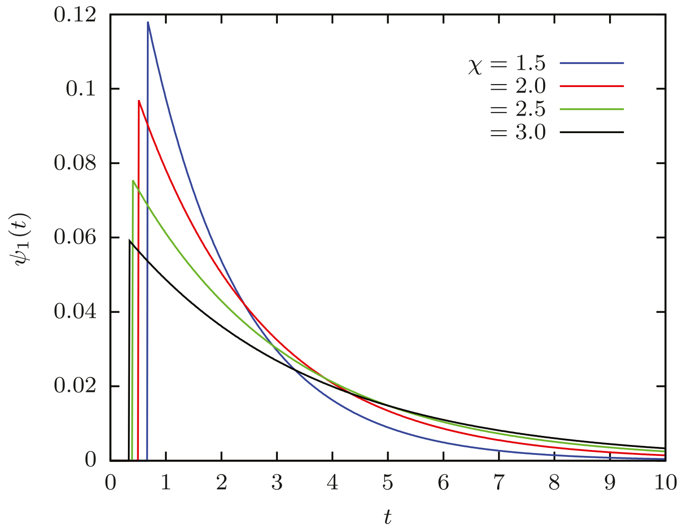

Fig. 2. Variation of the waiting time PDF for $\alpha = \beta = \gamma = 1$, with the coefficient χ. The case $\chi = 1$ retrieves the Dirac delta function $\delta(t-1)$.

for sufficiently small Laplace variable s (corresponding to long times t)

$$\widetilde{\psi}_{\gamma,\alpha}(s) \sim \frac{1}{1+s^\gamma+s^{\alpha+\gamma}}, \tag{38}$$

which has been obtained using a different approach in [36], see also [29] for an analogue asymptotic form of the waiting time PDF for retarding subdiffusion. Inverse Laplace transform of Eq. (38) produces

$$\psi_{\gamma,\alpha}(t) = \sum_{n=0}^{\infty}(-1)^n t^{(\alpha+\gamma)(n+1)-1}E^{n+1}_{\alpha,(\alpha+\gamma)(n+1)}(-t^\alpha), \tag{39}$$

which for large t reads (see Eq. (A.14))

$$\psi_{\gamma,\alpha}(t) \sim \sum_{n=0}^{\infty}(-1)^n\frac{t^{(\alpha+\gamma)(n+1)-1-\alpha(n+1)}}{\Gamma([\alpha+\gamma][n+1]-\alpha[n+1])}$$
$$= t^{\gamma-1}E_{\gamma,\gamma}(-t^\gamma) \sim \frac{\gamma t^{-\gamma-1}}{\Gamma(1-\gamma)}, \tag{40}$$

as it should be.

In the case $\chi = 0$ and $\alpha = \gamma = 1$ the waiting time PDF (16) reads $\widetilde{\psi}(s) = \exp[-s(s+1)]$, which is not guaranteed to be a completely monotonic function (see Appendix D), and thus the function $\psi(t)$ does not represent a PDF.

(iv) *Ordinary Jeffreys equation.* When $\alpha = \beta = \gamma = 1$ and $\chi > 1$ the waiting time PDF (16) represents a PDF according to Proposition 1 and further generalizes the classical random walker model (i), see Fig. 2. At long time and for large distances we have the following behavior in the Laplace and Fourier domain, respectively,

$$\widetilde{\psi} = \widetilde{\psi}_1(s) \sim 1 - \frac{s(s+1)}{1+\chi s}, \quad \widehat{\lambda}(q) \sim 1-q^2. \tag{41}$$

With the Montroll-Weiss Eq. (15) and using the asymptotic behavior (49) this leads to

$$\widehat{\widetilde{P}}(q,s) = \frac{s+1}{s(s+1)+(1+\chi s)q^2}. \tag{42}$$

Inverting Eq. (42) we arrive at the ordinary Jeffreys equation, or the parabolic flux-driven DPL model,

$$\frac{\partial^2 P(x,t)}{\partial t^2} + \frac{\partial P(x,t)}{\partial t} = \left(1+\chi\frac{\partial}{\partial t}\right)\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{43}$$

subject to the initial condition

$$P(x,0^+)=\delta(x),\quad \left.\frac{\partial P(x,t)}{\partial t}\right|_{t\to 0^+}=\chi u_2(x), \tag{44}$$

and free boundary conditions, where $u_n(x)$ is the nth differentiator function, see Appendix B for a discussion on such a type of initial conditions. It is a simple mathematical task to check the equivalence between Eq. (43) subject to (44) and the coupled system

$$\left(1+\frac{\partial}{\partial t}\right)J(x,t)=-\left(1+\chi\frac{\partial}{\partial t}\right)\frac{\partial P(x,t)}{\partial x}, \tag{45}$$

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial J(x,t)}{\partial x}, \tag{46}$$

subject to the initial conditions

$$P(x,0^+)=\delta(x),\quad J(x,0^+)=-\chi u_1(x) \tag{47}$$

and free boundary conditions, where $u_1(x)$ is the unit doublet function (B.1). When $\chi=1$, the second condition of (47) reduces to $J(x,0^+)=-u_1(x)$ which has been suggested to occur in normal diffusion problems [37]. Note that Eq. (46) is the limiting case of the fractional version (9) when $\alpha\to 1$, $\beta\to 1$ and $\gamma\to 1$.

(v) *Fractional Jeffreys equation.* In the general case $0<\alpha,\beta,\gamma<1$ with $\alpha+\gamma<1$, $\beta\le\gamma$, and $\chi>0$, and for large values of time t and distance x, the waiting time density (16) and the jump length density (17) have the asymptotic behaviors

$$\widetilde{\psi}(s)\sim 1-\frac{s^{\gamma}(s^{\alpha}+1)}{1+\chi^{\beta}s^{\beta}},\quad \widehat{\lambda}(q)\sim 1-q^2. \tag{48}$$

Further, we assume that the asymptotic behavior of the CTRW process is $\widehat{\widetilde{P}}(q,s)$, namely,

$$\widehat{\widetilde{P}}_{CTRW}(q,s)\sim\widehat{\widetilde{P}}(q,s). \tag{49}$$

Therefore Eqs. (15), (48) and (49) provide the asymptotic behavior of the CTRW process (28) in Laplace-Fourier domain,

$$\widehat{\widetilde{P}}(q,s)=\frac{s^{\gamma-1}(s^{\alpha}+1)}{s^{\gamma}(s^{\alpha}+1)+(1+\chi^{\beta}s^{\beta})q^2}. \tag{50}$$

Rearranging terms, this is equivalent to

$$(s^{\alpha}+1)\left[s\widetilde{P}(x,s)-\delta(x)\right]=s^{1-\gamma}(1+\chi^{\beta}s^{\beta})\frac{\partial^2\widetilde{P}(x,s)}{\partial x^2}. \tag{51}$$

Returning to the time domain, this result is equivalent to the integro-differential equation

$$\left(1+{}_0^{RL}\mathcal{D}_t^{\alpha}\right)\frac{\partial P(x,t)}{\partial t}={}_0^{RL}\mathcal{D}_t^{1-\gamma}\left(1+\chi^{\beta}\,{}_0^{RL}\mathcal{D}_t^{\beta}\right)\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{52}$$

subject to the initial condition

$$P(x,0^+)=\delta(x), \tag{53}$$

and free boundary conditions. Eq. (52) also appears when we eliminate the flux $J(x,t)$ in Eq. (9). Conversely, Eq. (51) can be rearranged and inverted to the form

$${}_0^{C}\mathcal{D}_t^{\alpha+\gamma}P(x,t)+{}_0^{C}\mathcal{D}_t^{\gamma}P(x,t)=\left(1+\chi^{\beta}\,{}_0^{RL}\mathcal{D}_t^{\beta}\right)\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{54}$$

where ${}_0^{C}\mathcal{D}_t^{\alpha}$ stands for the Caputo fractional derivative of order α defined by [51]

$${}_0^{C}\mathcal{D}_t^{\alpha}f(t)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t\frac{\partial_\zeta f(\zeta)}{(t-\zeta)^{\alpha}}d\zeta, & 0<\alpha<1\\[2mm] \dfrac{df(t)}{dt}, & \alpha=1\end{cases} \tag{55}$$

The form (54) of the fractional Jeffreys equation involves both Caputo and Riemann-Liouville fractional derivatives in the same equation. It can be viewed as a possible generalization of the fractional telegrapher's equation that uses the Caputo fractional derivative, see Eq. (62) below and [36] for a short review on the fractional (Cattaneo) telegrapher's equation.

In the above cases, we have used the properties $\mathcal{L}\{{}_{0^+}^{RL}\mathcal{D}_t^{\alpha}f(t);s\}=s^{\alpha}\widetilde{f}(s)$ and $\mathcal{L}\{{}_{0^+}^{C}\mathcal{D}_t^{\alpha}f(t);s\}=s^{\alpha}\widetilde{f}(s)-s^{\alpha-1}f(0^+)$, where $\alpha\in(0,1)$.

Remark 1. The FJE (52) can be interpreted as a special case of the generalized diffusion equation with generalized memory kernel [55],

$$\int_0^t M(t-\tau)\frac{\partial P(x,\tau)}{\partial\tau}d\tau=\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{56}$$

by considering the specific choice of memory kernel

$$M(t)=\frac{1}{\chi^{\beta}}\left[t^{\beta-\alpha-\gamma}E_{\beta,1+\beta-\alpha-\gamma}\left(-(t/\chi)^{\beta}\right)+t^{\beta-\gamma}E_{\beta,1+\beta-\gamma}\left(-(t/\chi)^{\beta}\right)\right]. \tag{57}$$

Indeed, the memory kernel (57) can be easily derived by rewriting Eq. (51) in the form

$$\frac{1}{\chi^{\beta}}\left(\frac{s^{\alpha+\gamma-1}}{s^{\beta}+\frac{1}{\chi^{\beta}}}+\frac{s^{\gamma-1}}{s^{\beta}+\frac{1}{\chi^{\beta}}}\right)\left[s\widetilde{P}(x,s)-\delta(x)\right]=\frac{\partial^2\widetilde{P}(x,s)}{\partial x^2}, \tag{58}$$

and inverting to the real domain by using Eq. (A.16) for $\gamma=1$. Furthermore, the memory kernel (57) of the FJE comprises two existing special cases:

(i) When $\alpha=\beta$ and $\chi=1$, we have a power-law memory kernel since

$$M(t)=t^{-\gamma}E_{\alpha,1-\gamma}(-t^{\alpha})+t^{\alpha-\gamma}E_{\alpha,1+\alpha-\gamma}(-t^{\alpha})=\frac{t^{-\gamma}}{\Gamma(1-\gamma)}, \tag{59}$$

where we used the property $E_{\alpha,\beta}(z)=zE_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)}$. This memory kernel yields a fractional diffusion equation of the "natural" type,

$${}_0^{C}\mathcal{D}_t^{\gamma}P(x,t)=\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{60}$$

which is equivalent to Eq. (36).

(ii) When $\chi=0$, the memory kernel (57) reduces to

$$\begin{aligned}M(t)&=\lim_{\chi\to 0}\frac{1}{\chi^{\beta}}\left[t^{\beta-\alpha-\gamma}E_{\beta,1+\beta-\alpha-\gamma}\left(-(t/\chi)^{\beta}\right)+t^{\beta-\gamma}E_{\beta,1+\beta-\gamma}\left(-(t/\chi)^{\beta}\right)\right]\\&=\frac{1}{\chi^{\beta}}\left[\frac{t^{\beta-\alpha-\gamma}(t/\chi)^{-\beta}}{\Gamma(1-\alpha-\gamma)}+\frac{t^{\beta-\gamma}(t/\chi)^{-\beta}}{\Gamma(1-\gamma)}\right]\\&=\frac{t^{-\alpha-\gamma}}{\Gamma(1-\alpha-\gamma)}+\frac{t^{-\gamma}}{\Gamma(1-\gamma)},\end{aligned} \tag{61}$$

where we utilized the asymptotic behavior of the Mittag-Leffler function for large negative argument, see Eqs. (A.14) and (A.15). This memory kernel yields the distributed order diffusion equation (since $\alpha+\gamma\le 1$), or the special case of the fractional telegrapher's equation

$${}_0^{C}\mathcal{D}_t^{\alpha+\gamma}P(x,t)+{}_0^{C}\mathcal{D}_t^{\gamma}P(x,t)=\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{62}$$

which describes retarding subdiffusion process [25], as it should be.

Remark 2. We note that the conditions of Proposition 1 coincide with the sufficient conditions providing the non-negativity for the Green's function of the FJE in two- and three-dimensional space, see Theorem 3.7 in [40]. Consequently, our results in the one-dimensional CTRW can be generalized to higher dimensions, and we arrive at the multidimensional FJE

$$\left(1+{}_0^{RL}\mathcal{D}_t^{\alpha}\right)\frac{\partial P(\mathbf{r},t)}{\partial t}={}_0^{RL}\mathcal{D}_t^{1-\gamma}\left(1+\chi^{\beta}\,{}_0^{RL}\mathcal{D}_t^{\beta}\right)\Delta P(\mathbf{r},t), \tag{63}$$

where $\Delta=\nabla^2$ is the d-dimensional Laplacian operator. This equation can be implemented in the CTRW approach in terms of the jump-length PDF

$$\lambda(r)=(4\pi)^{-d/2}\exp(-r^2/4), \tag{64}$$

where $r=|\mathbf{r}|$ and $\mathbf{r}\in\mathbb{R}^d$.

4. Fractional Jeffreys equation: Analytical solution and numerical computations

We now study the detailed properties of the FJE in terms of the solution for the PDF $P(x,t)$ and the MSD.

4.1. Solution for a flux-driven case

We derive the solution for the CTRW process (52) with initial condition (53),or alternatively Eq. (9) with $P(x,0^+)=\delta(x)$. To this end we rewrite Eq. (50) in the form

$$\hat{\tilde{P}}(q,s)=\frac{s^{\gamma-1}(s^{\alpha}+1)}{s^{\alpha+\gamma}+\chi^{\beta}s^{\beta}q^2}[1+\varrho(q,s;\chi)]^{-1},$$
$$\varrho(q,s;\chi)=\frac{s^{\gamma}+q^2}{s^{\alpha+\gamma}+\chi^{\beta}s^{\beta}q^2},\ s>0. \tag{65}$$

The auxiliary function $\varrho(q,s;\chi)$ satisfies $0<\varrho(q,s;\chi)<1$ for $|q|>1$, $s>1$, and $\chi>1$, but also for $0<s<1$ (i.e., at long times) if the parameter χ is large enough. Then, after expanding Eq. (65) in powers of ϱ we get for $\chi\gg1$

$$\hat{\tilde{P}}(q,s)=\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\binom{n}{k}q^{2k}\left\{\frac{s^{(\alpha-\beta+\gamma)-[1+(\beta-\gamma)n+\gamma k]}}{(s^{\alpha-\beta+\gamma}+\chi^{\beta}q^2)^{n+1}}+\frac{s^{(\alpha-\beta+\gamma)-[\alpha+1+(\beta-\gamma)n+\gamma k]}}{(s^{\alpha-\beta+\gamma}+\chi^{\beta}q^2)^{n+1}}\right\}. \tag{66}$$

Taking the inverse Laplace transform, using relation (A.17), yields

$$\hat{P}(q,t)=\sum_{n=0}^{\infty}\frac{(-t^{\alpha})^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(t^{\gamma}q^2\right)^k\left\{H_{1,2}^{1,1}\left[\chi^{\beta}t^{\alpha-\beta+\gamma}q^2\middle|_{(0,1);\,(-\alpha n-\gamma k,\,\alpha-\beta+\gamma)}^{(-n,1)}\right]\right.$$
$$\left.+t^{\alpha}H_{1,2}^{1,1}\left[\chi^{\beta}t^{\alpha-\beta+\gamma}q^2\middle|_{(0,1);\,(-\alpha[n+1]-\gamma k,\,\alpha-\beta+\gamma)}^{(-n,1)}\right]\right\} \tag{67}$$

Finally, by help of relation (A.6) we arrive at

$$P(x,t)=\frac{1}{\sqrt{4\pi\chi^{\beta}t^{\alpha-\beta+\gamma}}}\sum_{n=0}^{\infty}\frac{(-t^{\alpha})^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{t^{\beta-\alpha}}{\chi^{\beta}}\right)^k[P_1(x,t)+t^{\alpha}P_2(x,t)], \tag{68}$$

where

$$P_1(x,t)=H_{2,3}^{2,1}\left[\frac{x^2}{4\chi^{\beta}t^{\alpha-\beta+\gamma}}\middle|_{(0,1),\,\left(\frac{1}{2}+n-k,1\right);\,\left(\frac{1}{2},1\right)}^{\left(\frac{1}{2}-k,1\right);\,\left(1+\alpha n+(\beta-\alpha)k-\frac{\alpha-\beta+\gamma}{2},\,\alpha-\beta+\gamma\right)}\right], \tag{69a}$$

$$P_2(x,t)=H_{2,3}^{2,1}\left[\frac{x^2}{4\chi^{\beta}t^{\alpha-\beta+\gamma}}\middle|_{(0,1),\,\left(\frac{1}{2}+n-k,1\right);\,\left(\frac{1}{2},1\right)}^{\left(\frac{1}{2}-k,1\right);\,\left(1+\alpha n+(\beta-\alpha)k+\frac{\alpha+\beta-\gamma}{2},\,\alpha-\beta+\gamma\right)}\right]. \tag{69b}$$

Eqs. (68), (69a) and (69b) require $\chi\gg1$ and $\beta<\alpha+\gamma$. It is noteworthy to remember that the sufficient condition for the one-dimensional Green's function $P(x,t)$ to be a PDF is $\beta\le\gamma$ [40], see also [56] for a different treatment.

For the case $\alpha=\beta=\gamma=1$ and $\chi\gg1$, Eq. (9) subject to (47) reduces to the so-called parabolic dual-phase-lag with flux-precedence flow [4]. The Green's function for (68) and (69) reduces then to (see Eq. (A.11))

$$P(x,t)=\frac{1}{\sqrt{4\pi\chi t}}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\chi^{-k}[P_1(x,t)+tP_2(x,t)], \tag{70}$$

where

$$P_1(x,t)=\frac{\pi}{\sin\left(\pi\left[\frac{1}{2}+n-k\right]\right)}\left\{\frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(\frac{1}{2}+n\right)}\,{}_2\bar{F}_2\left(\frac{1}{2}+k,\frac{1}{2}-n;\frac{1}{2}-n+k,\frac{1}{2};-\frac{x^2}{4\chi t}\right)-\frac{\Gamma(1+n)}{\Gamma(k)}\left(\frac{x^2}{4\chi t}\right)^{1/2+n-k}\right.$$
$$\left.\times\,{}_2\bar{F}_2\left(1+n,1-k;\frac{3}{2}+n-k,1+n-k;-\frac{x^2}{4\chi t}\right)\right\} \tag{71a}$$

$$P_2(x,t) = \frac{\pi}{\sin\left(\pi\left[\frac{1}{2}+n-k\right]\right)}\left\{\frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(\frac{3}{2}+n\right)}\,{}_2\overline{F}_2\left(\frac{1}{2}+k,-\frac{1}{2}-n;\frac{1}{2}-n+k,\frac{1}{2};-\frac{x^2}{4\chi t}\right)-\frac{\Gamma(1+n)}{\Gamma(1+k)}\left(\frac{x^2}{4\chi t}\right)^{1/2+n-k}\right.$$
$$\left.\times\,{}_2\overline{F}_2\left(1+n,-k;\frac{3}{2}+n-k,1+n-k;-\frac{x^2}{4\chi t}\right)\right\} \tag{71b}$$

where ${}_2\overline{F}_2(\cdot)$ is the regularized hypergeometric function, see Appendix A.

To complete our analysis we provide the exact formula for the diffusive flux $J(x,t)$ in the Laplace-Fourier domain (see Eqs. (9) and (50))

$$\widehat{\widetilde{J}}(q,s) = -\imath q\frac{1+\chi^\beta s^\beta}{s^\gamma(s^\alpha+1)+(1+\chi^\beta s^\beta)q^2}, \tag{72}$$

which may be written in the physical domain as

$$J(x,t) = -\frac{\partial\varphi(x,t)}{\partial x}, \tag{73}$$

where

$$\widehat{\widetilde{\varphi}}(q,s) = \frac{1+\chi^\beta s^\beta}{s^\gamma(s^\alpha+1)+(1+\chi^\beta s^\beta)q^2}. \tag{74}$$

Under the condition $\chi \gg 1$ and $\beta < \alpha+\gamma$ and with the help of Eqs. (A.17) and (A.6), one can invert (74) to obtain

$$\varphi(x,t) = \frac{t^{(\alpha-\beta+\gamma)/2-1}}{\sqrt{4\pi\chi^\beta}}\sum_{n=0}^{\infty}\frac{(-t^\alpha)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{t^{\beta-\alpha}}{\chi^\beta}\right)^k\left[t^\beta\varphi_1(x,t)+\chi^\beta\varphi_2(x,t)\right], \tag{75}$$

where

$$\varphi_1(x,t) = H_{2,3}^{2,1}\left[\frac{x^2}{4\chi^\beta t^{\alpha-\beta+\gamma}}\left|\begin{matrix}\left(\frac{1}{2}-k,1\right);\left(\alpha n+[\beta-\alpha]k+\frac{\alpha+\beta+\gamma}{2},\alpha-\beta+\gamma\right)\\(0,1),\left(\frac{1}{2}+n-k,1\right);\left(\frac{1}{2},1\right)\end{matrix}\right.\right], \tag{76a}$$

$$\varphi_2(x,t) \quad = \quad H_{2,3}^{2,1}\left[\frac{x^2}{4\chi^\beta t^{\alpha-\beta+\gamma}}\left|\begin{matrix}\left(\frac{1}{2}-k,1\right);\left(\alpha n+[\beta-\alpha]k+\frac{\alpha-\beta+\gamma}{2},\alpha-\beta+\gamma\right)\\(0,1),\left(\frac{1}{2}+n-k,1\right);\left(\frac{1}{2},1\right)\end{matrix}\right.\right]. \tag{76b}$$

Hence the diffusive flux $J(x,t)$ can be determined by substituting Eqs. (75), (76a), and (76b) into Eq. (73) and using the useful relation (A.5). After some manipulations we find

$$J(x,t) = \frac{t^{(\alpha-\beta+\gamma)/2-1}}{x\sqrt{\pi\chi^\beta}}\sum_{n=0}^{\infty}\frac{(-t^\alpha)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{t^{\beta-\alpha}}{\chi^\beta}\right)^k\left[t^\beta J_1(x,t)+\chi^\beta J_2(x,t)\right], \tag{77}$$

where

$$J_1(x,t) = H_{2,3}^{2,1}\left[\frac{x^2}{4\chi^\beta t^{\alpha-\beta+\gamma}}\left|\begin{matrix}\left(\frac{1}{2}-k,1\right);\left(\alpha n+[\beta-\alpha]k+\frac{\alpha+\beta+\gamma}{2},\alpha-\beta+\gamma\right)\\(1,1),\left(\frac{1}{2}+n-k,1\right);\left(\frac{1}{2},1\right)\end{matrix}\right.\right], \tag{78a}$$

$$J_2(x,t) = H_{2,3}^{2,1}\left[\frac{x^2}{4\chi^\beta t^{\alpha-\beta+\gamma}}\left|\begin{matrix}\left(\frac{1}{2}-k,1\right);\left(\alpha n+[\beta-\alpha]k+\frac{\alpha-\beta+\gamma}{2},\alpha-\beta+\gamma\right)\\(1,1),\left(\frac{1}{2}+n-k,1\right);\left(\frac{1}{2},1\right)\end{matrix}\right.\right]. \tag{78b}$$

Finally, the diffusive flux of the ordinary JE (parabolic Dual-Phase-Lag) can be analytically expressed for $\chi \gg 1$ as

$$J(x,t) = \frac{1}{\sqrt{\pi\chi x^2 t}}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\chi^{-k}\left[tJ_1(x,t)+\chi J_2(x,t)\right], \tag{79}$$

where

$$J_1(x,t) = \frac{\pi}{\sin\left(\pi\left(-\frac{1}{2}+n-k\right)\right)}\left\{\frac{\Gamma\left(\frac{3}{2}+k\right)}{\Gamma\left(\frac{1}{2}+n\right)}\left(\frac{x^2}{4\chi t}\right){}_2\overline{F}_2\left(\frac{3}{2}+k,\frac{1}{2}-n;\frac{3}{2}-n+k,\frac{3}{2};-\frac{x^2}{4\chi t}\right)-\frac{\Gamma(1+n)}{\Gamma(1+k)}\left(\frac{x^2}{4\chi t}\right)^{1/2+n-k}{}_2\right.$$
$$\left.\times\overline{F}_2\left(1+n,-k;\frac{1}{2}+n-k,1+n-k;-\frac{x^2}{4\chi t}\right)\right\}, \tag{80a}$$

$$J_2(x,t) = \frac{\pi}{\sin\left(\pi\left(-\frac{1}{2}+n-k\right)\right)}\left\{\frac{\Gamma\left(\frac{3}{2}+k\right)}{\Gamma\left(n-\frac{1}{2}\right)}\left(\frac{x^2}{4\chi t}\right){}_2\overline{F}_2\left(\frac{3}{2}+k,\frac{3}{2}-n;\frac{3}{2}-n+k,\frac{3}{2};-\frac{x^2}{4\chi t}\right)-\frac{\Gamma(1+n)}{\Gamma(k)}\left(\frac{x^2}{4\chi t}\right)^{1/2+n-k}{}_2\right.$$
$$\left.\times\overline{F}_2\left(1+n,1-k;\frac{1}{2}+n-k,1+n-k;-\frac{x^2}{4\chi t}\right)\right\}. \tag{80b}$$

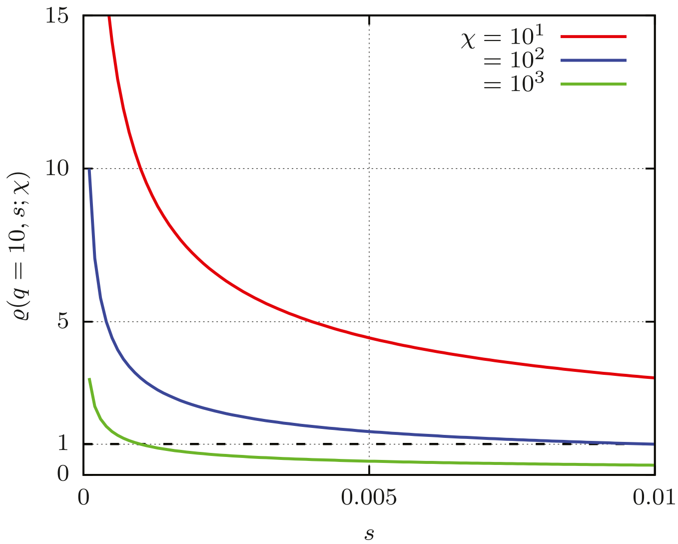

Fig. 3. Auxiliary function $\varrho(q, s; \chi)$ for $q = 10$, $\alpha = \beta = \gamma = 0.5$ and different values of the ratio χ.

4.2. Numerical schemes

We here briefly discuss the numerical schemes used throughout the paper. For the closed-form solution (28) of the CTRW process the following approximation is used,

$$P_{CTRW}(x,t) = \Psi(t)\delta(x) + \sum_{u=1}^{u_{\max}} G(x,u) \sum_{n=1}^{50} \frac{(-1)^n}{n!} \left(\frac{t^{\beta-\gamma}}{\chi^\beta}\right)^n \left[u^n - (u+1)^n\right] \sum_{k=0}^{n} \binom{n}{k} t^{-\alpha k} \sum_{\nu=0}^{50} \frac{(-1)^\nu}{\nu!} \left(\frac{t}{\chi}\right)^{\beta\nu} \times \frac{\Gamma(n+\nu)}{\Gamma(n)\Gamma[\beta\nu + (\beta-\gamma)n - \alpha k + 1]} \tag{81}$$

where $G(x,t)$ is given by (25). In (81) we truncated the last two series at the 51st term, while the first (main) series is truncated at $u_{\max}$. We defined u as the number of steps (jumps) performed within the interval $(0,t)$. Therefore, the number $u_{\max}$ should be chosen greater than the time t, $u_{\max} \gg t$; for example, when $t = 10$ we may choose $u_{\max} = 10, 20, \ldots,$ until the numerical results stabilize at constant values. This task can be easily implemented using any symbolic mathematics program. To avoid the singularities of the gamma function in the denominator of (81), we replace, e.g., $\alpha = 0.5$ by $\alpha = 0.5000001$. A specific choice for the ratio χ, e.g., $\chi = 100$, confines the computational range for expression (81). In other words, for $t \geq 100$ the above series may diverge. In this case the direct numerical inversion of the Laplace transform is used, as we will show below.

The H-function representation of the solution (68) and (69) to FJE can be computed by using the series expansion (A.4), however, the condition $0 < \varrho(q,s;\chi) < 1$ should be considered, where $\varrho(q,s;\chi)$ is given by (65). We note that the auxiliary function $\varrho(q,s;\chi)$ does not lie within the interval (0,1) for all values of the Laplace parameter s. In Fig. 3, we draw $\varrho(q,s;\chi)$ for $q = 10$ and different values of the ratio χ. Notably, for small values of the Laplace parameter s, i.e., for large values of the non-dimensional time variable t, the auxiliary function does not satisfy the condition $0 < \varrho(q,s;\chi) < 1$, which indicates that the solution (68) and (69) may diverge at large values of t. Indeed, we find that the solution (68) and (69) with $\chi = 100$ works well within the interval $t \in (0, 20]$.

When the series (81) for the CTRW process and the series (68) for the FJE fail to converge, a direct numerical inversion of Laplace transform is the method of choice. The numerical scheme for inverting the Laplace transform depends on computing the "Riemann sum approximation" [4]

$$f(x,t) = \frac{\exp(at)}{t} \left\{ \frac{1}{2}\tilde{f}(x, s=a) + \Re\left[\sum_{k=1}^{N_{sum}} (-1)^k \tilde{f}\left(x, s = a + \frac{\imath\pi k}{t}\right)\right]\right\}, \tag{82}$$

where N_{sum} is the number of summation terms (taken here from 10^5 to 10^7) and the choice $a = 4.7/t$ was found to be optimal. The series (82) can be easily computed using, e.g., PTC Mathcad or Wolfram Mathematica. In the Laplace domain the solution of the CTRW process is given as

$$\tilde{P}_{CTRW}(x,s) = \tilde{\Psi}(s)\delta(x) + \sum_{u=1}^{u_{max}} \tilde{N}(u,s) G(x,u), \tag{83}$$

where $\tilde{N}(u,s)$ and $G(x,u)$ are respectively defined by (22) and (25), and the one-dimensional solution of FJE is given as

$$\tilde{P}(x,s) = \frac{1}{2s}\sqrt{\frac{s^\gamma(s^\alpha+1)}{1+\chi^\beta s^\beta}} \exp\left(-|x|\sqrt{\frac{s^\gamma(s^\alpha+1)}{1+\chi^\beta s^\beta}}\right). \tag{84}$$

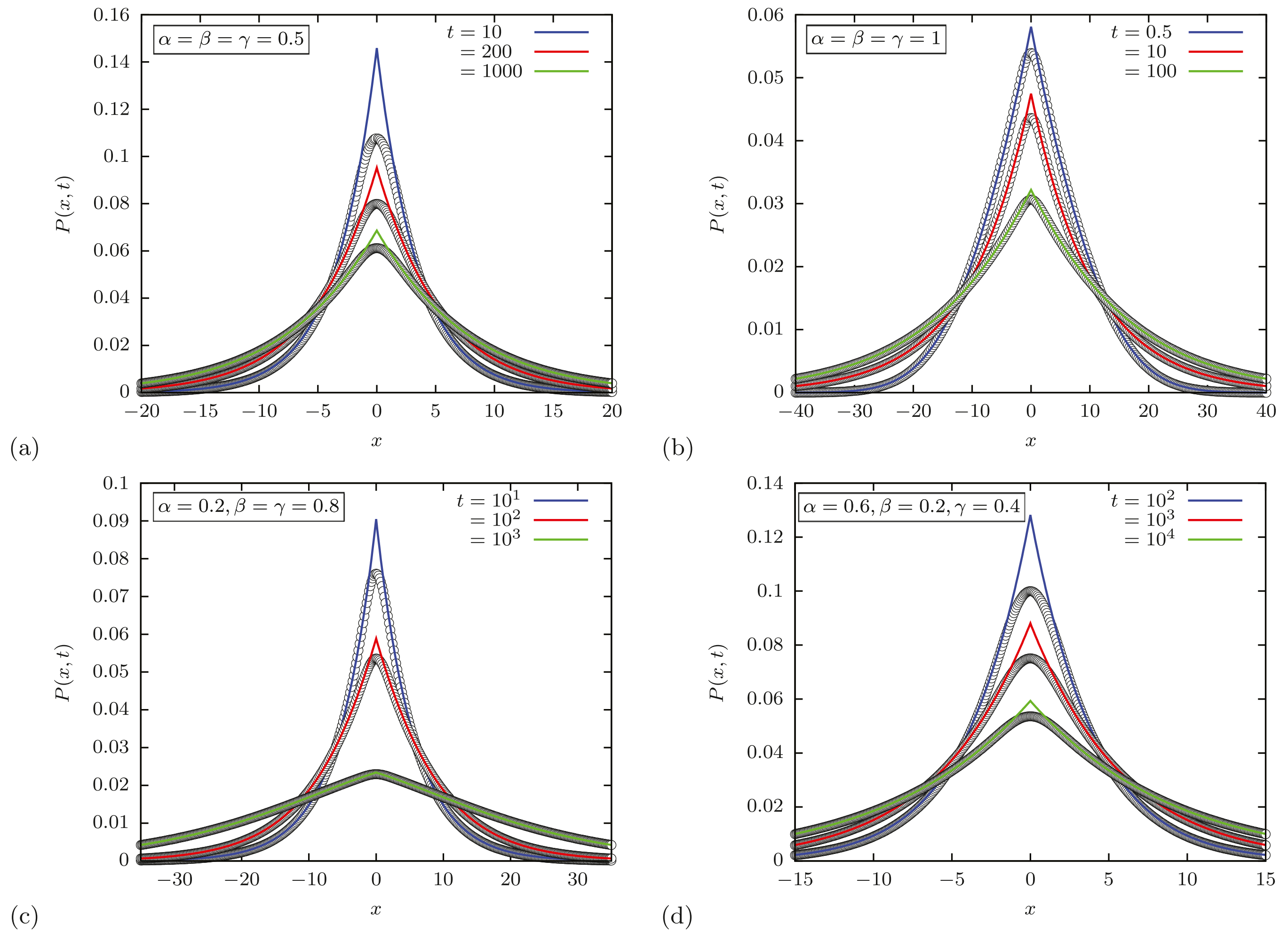

Fig. 4. PDF $P(x,t)$ of the FJE and JE (solid colored lines) at different times t and the corresponding PDF $P_{CTRW}(x,t)$ of the CTRW process (black circles). The parameters are: (a) $\alpha=\beta=\gamma=0.5$; (b) $\alpha=\beta=\gamma=1$; (c) $\alpha=0.2$, $\beta=\gamma=0.8$; (d) $\alpha=0.6$, $\beta=0.2$, $\gamma=0.4$. In all cases we set $\chi=100$.

In Fig. 4 we compare the solutions of the FJE and JE, Eqs. (68), (69a), and (69b); and Eqs. (70), (71a) and (71b), respectively with the solutions of the CTRW process (26) to (28). Typically the PDFs $P(x,t)$ and $P_{CTRW}(x,t)$ show good agreement at longer distances x. The differences between the two solutions near the origin $x=0$ have been extensively discussed, see [22] and references therein. We excluded the first term of the series (21), or the term $\Psi(t)\delta(x)$ in Eq. (81), to avoid Dirac delta effect at the origin $x=0$. In the ordinary JE, Fig. 4 (b) shows that the solution fits well the solution of the CTRW process even in the short time limit.

4.3. MSD

The MSD $\langle x_P^2(t)\rangle$ can be expressed in terms of the PDF's Fourier transform as

$$\langle \tilde{x}_P^2(s)\rangle = -\left.\frac{\partial^2 \widehat{\widetilde{P}}(q,s)}{\partial q^2}\right|_{q=0}. \tag{85}$$

From Eq. (50) we find after Laplace inversion

$$\langle x_P^2(t)\rangle = 2\left[t^{\alpha+\gamma}E_{\alpha,\alpha+\gamma+1}(-t^{\alpha}) + \chi^{\beta}t^{\alpha+\gamma-\beta}E_{\alpha,\alpha+\gamma-\beta+1}(-t^{\alpha})\right]. \tag{86}$$

The particular case with $\alpha=\beta$ and $\gamma=1$ corresponds to the result obtained in [41]. From comparison with Eqs. (32) and (86) we see that

$$\langle X_{CTRW}^2(t)\rangle \sim \langle x_P^2(t)\rangle \tag{87}$$

in the long time limit.

The asymptotic behavior of result (86) can be evaluated by help of Eq. (A.15). At short times we get

$$\langle x_P^2(t)\rangle \sim 2\left(\frac{t^{\alpha+\gamma}}{\Gamma(1+\alpha+\gamma)} + \chi^{\beta}\frac{t^{\alpha+\gamma-\beta}}{\Gamma(1+\alpha+\gamma-\beta)}\right) \sim \frac{2\chi^{\beta}t^{\alpha+\gamma-\beta}}{\Gamma(1+\alpha+\gamma-\beta)}, \quad t\to 0^{+}, \tag{88}$$

while at long times we have

$$\langle x_P^2(t)\rangle \sim 2\left(\frac{t^{\gamma}}{\Gamma(1+\gamma)} + \chi^{\beta}\frac{t^{\gamma-\beta}}{\Gamma(1+\gamma-\beta)}\right) \sim \frac{2t^{\gamma}}{\Gamma(1+\gamma)}, \quad t\to\infty. \tag{89}$$

The asymptotic behavior in Eqs. (88) and (89) contains interesting special cases, see also Fig. 5:

(i) The case $\alpha=\beta=\gamma\neq 1$ encodes the asymptotic behavior

$$\langle x_P^2(t)\rangle \sim \begin{cases} \dfrac{2\chi^{\alpha}t^{\gamma}}{\Gamma(1+\gamma)}, & t\to 0^{+} \\ \dfrac{2t^{\gamma}}{\Gamma(1+\gamma)}, & t\to\infty \end{cases} \tag{90}$$

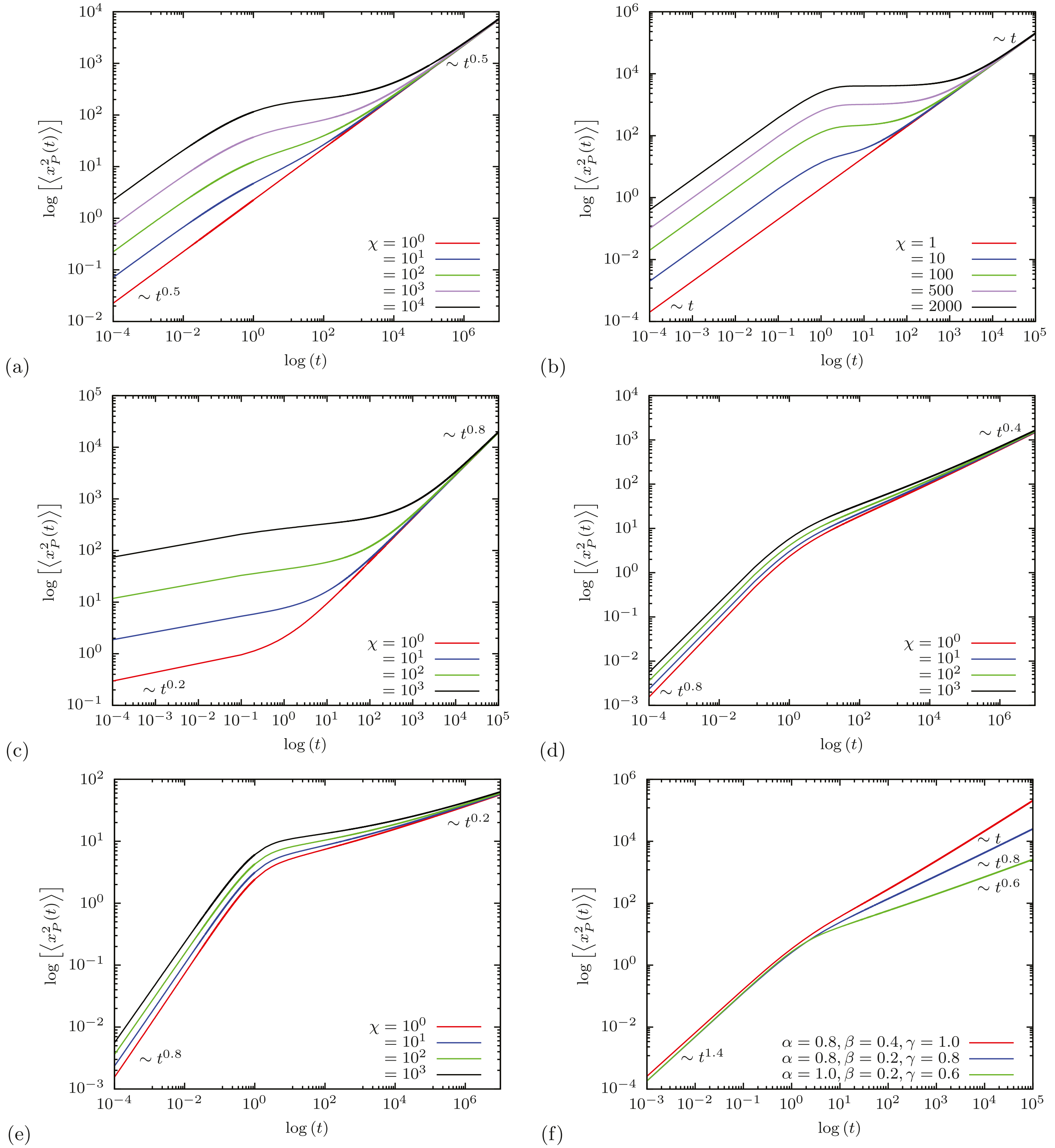

Fig. 5. MSD $\langle x_P^2(t)\rangle$ of the Green's function $P(x,t)$ of the FJE and JE for different values of the coefficient χ and the fractional parameters. In (a) $\alpha=\beta=\gamma=0.5$; (b) $\alpha=\beta=\gamma=1$; (c) $\alpha=0.2$, $\beta=0.8$, $\gamma=0.8$; (d) $\alpha=0.6$, $\beta=0.2$, $\gamma=0.4$; (e) $\alpha=0.8$, $\beta=0.2$, $\gamma=0.2$; (f) $\chi=10$.

This behavior is unique and could not be captured by other fractional kinetic tools such as the double-order fractional diffusion equation of modified or natural types [25–29,55,57], see also pp. 107-127 in [16]. We show this case for $\alpha=\beta=\gamma=0.5$ and different values of the coefficient χ in Fig. 5 (a). The case $\chi=1$ (red line) corresponds to the fractional diffusion Eq. (36) with $\gamma=0.5$, i.e., the MSD reads $\langle x_P^2(t)\rangle \sim t^{1/2}$. As the parameter χ increases a cage-like behavior arises in the intermediate time regime between two anomalous diffusion domains characterized by the power-law $t^{1/2}$. The PDF corresponding to this case is shown in Fig. 4 (a) including the comparison with the corresponding CTRW process for $\chi=100$.

The case $\alpha=\beta=\gamma=1$ and $\chi>1$ corresponds to the JE, that was shown to appear as the long time limit of a CTRW dynamics characterized by the waiting time PDF $\widetilde{\psi}_1(s) =$

$\exp(-s[s+1]/[1+\chi s])$. We then have

$$\langle x_P^2(t)\rangle \sim \begin{cases} 2\chi t, & t\to 0^+, \\ 2t, & t\to\infty. \end{cases} \quad (91)$$

Fig. 5(b) shows this behavior for different values of χ. This type of diffusion is reminiscent of the dynamics in jammed particle packs [58] where there exist two Fickian diffusion regimes separated by an intermediate, cage-like regime, see the second figure in [58]. We present the PDF for this case in Fig. 4 (b). One can see that this PDF is close to the CTRW solution even in the regime of relatively short times.

(ii) When $\alpha<\beta$ the FJE describes an accelerating subdiffusion process. Such a dynamics is described in terms of double-order (or multi-term) fractional diffusion equations of modified type [27–29]. Fig. 5 (c) illustrates this accelerating subdiffusion, while the corresponding PDF along with its CTRW analog are presented in Fig. 4 (c) for different times and for $\chi=100$ (green curve).

(iii) When $\alpha>\beta$, $\alpha+\gamma-\beta<1$, and $\chi>0$, the solution of the FJE represents a retarded subdiffusion process [25,29], see Figs. 5 (d) and 5 (e) for illustrating examples. We further plot the PDF corresponding to the green curve in Fig. 5 (d) and the PDF of the CTRW model in Fig. 4 (d) for different times.

(iv) When $\alpha>\beta$, $\alpha+\gamma-\beta>1$, and $\chi>0$, the solution of the FJE represents a crossover from superdiffusion to subdiffusion, whilst the fractional telegrapher's behavior is recovered whenever $\chi=0$ [37], see Fig. 5 (f).

(v) When $\gamma=1$ and $\alpha<\beta$, we observe a crossover from subdiffusion to normal diffusion.

(vi) Finally when $\gamma=1$ and $\alpha>\beta$, we have a crossover from superdiffusion to normal diffusion.

We note that the cases (iv)-(vi) do not obey the conditions of Proposition 1, hence, they do not follow the CTRW process discussed in Section 3. However, they agree with the conditions of theorem 3.7 in [40] on the non-negativity of the one-dimensional solution of the FJE, see Eq. (7a).

5. From Fick's first law to the fractional Jeffreys law

We now complement the above discussion by some useful phenomenological approaches demonstrating the connection between the fractional Jeffreys constitutive law and Fick's first law.

5.1. Fractional diffusion-wave equation

Let us replace Fick's first law $J(x,t)=-\partial P(x,t)/\partial x$ with the bi-fractional version

$${}_0^{RL}\mathcal{D}_t^{1-\alpha}J(x,t)=-{}_0^{RL}\mathcal{D}_t^{1-\beta}\frac{\partial P(x,t)}{\partial x}, \quad (92)$$

where $0<\alpha,\beta\le 1$. This generalization will be useful to introduce its distributed-order version. By transforming the distribution $P(x,t)$, in view of relation (92) and the continuity equation $-\partial J(x,t)/\partial x=\partial P(x,t)/\partial t$ to the Laplace-Fourier domain, we get

$$\widehat{\widetilde{P}}(q,s)=\frac{s^{\beta-\alpha}}{s^{1+\beta-\alpha}+q^2}. \quad (93)$$

Therefore, the solution of (92) with the continuity equation in the real domain is given by

$$P(x,t)=\frac{1}{\sqrt{4\pi t^{1+\beta-\alpha}}}H_{1,2}^{2,0}\left[\frac{x^2}{4t^{1+\beta-\alpha}}\left|\begin{matrix}\left(\frac{1}{2}+\frac{\alpha-\beta}{2},1+\beta-\alpha\right)\\(0,1),\left(\frac{1}{2},1\right)\end{matrix}\right.\right]. \quad (94)$$

Here, we discuss three different cases of Eq. (93) and its solution (94):

(i) the case $\alpha=\beta$ from Eq. (93) yields

$$\widehat{\widetilde{P}}(q,s)=\frac{1}{s+q^2}\ \Rightarrow\ \widehat{\widetilde{P}}(q,s)-1=-q^2\widehat{\widetilde{P}}(q,s), \quad (95)$$

which is the standard diffusion Eq. (33) in Fourier-Laplace domain with initial condition $P(x,t=0)=\delta(x)$. Its solution is the Gaussian PDF, which directly follows from the solution (94) for $\alpha=\beta$,

$$P(x,t)=\frac{1}{\sqrt{4\pi t}}H_{0,1}^{1,0}\left[\frac{x^2}{4t}\left|\begin{matrix}--\\(0,1)\end{matrix}\right.\right]=\frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right). \quad (96)$$

(ii) We now consider $\alpha>\beta$ with $0<1+\beta-\alpha<1$. By setting $\gamma=1+\beta-\alpha$, i.e., $0<\gamma<1$, from Eq. (93) we obtain

$$\widehat{\widetilde{P}}(q,s)=\frac{s^{\gamma-1}}{s^\gamma+q^2}\ \Rightarrow\ s\widehat{\widetilde{P}}(q,s)-1=-s^{1-\gamma}q^2\widehat{\widetilde{P}}(q,s). \quad (97)$$

This is the fractional diffusion Eq. (36) in Fourier-Laplace domain. Its solution follows from (94) and is given by [15,47]

$$P(x,t)=\frac{1}{\sqrt{4\pi t^\gamma}}H_{1,2}^{2,0}\left[\frac{x^2}{4t^\gamma}\left|\begin{matrix}\left(1-\frac{\gamma}{2},\gamma\right)\\(0,1),\left(\frac{1}{2},1\right)\end{matrix}\right.\right]. \quad (98)$$

Its tail assumes a stretched Gaussian shape [15].

(iii) Finally we take $\alpha<\beta$ for which $1<1+\beta-\alpha<2$. In our setting $\mu=1+\beta-\alpha$, which means $1<\mu<2$, and from Eq. (93) we have

$$\widehat{\widetilde{P}}(q,s)=\frac{s^{\mu-1}}{s^\mu+q^2}\ \Rightarrow\ s^\mu\widehat{\widetilde{P}}(q,s)-s^{\mu-1}=-q^2\widehat{\widetilde{P}}(q,s). \quad (99)$$

By inverse Fourier-Laplace transform one obtains the fractional wave equation

$${}_0^C\mathcal{D}_t^\mu P(x,t)=\frac{\partial^2P(x,t)}{\partial x^2},\quad 1<\mu<2, \quad (100)$$

where we impose initial conditions of form $P(x,t=0)=\delta(x)$ and $\frac{\partial P(x,t=0)}{\partial t}=0$. Its solution directly follows from Eq. (94), see [15,47,59],

$$P(x,t)=\frac{1}{\sqrt{4\pi t^\mu}}H_{1,2}^{2,0}\left[\frac{x^2}{4t^\mu}\left|\begin{matrix}\left(1-\frac{\mu}{2},\mu\right)\\(0,1),\left(\frac{1}{2},1\right)\end{matrix}\right.\right]. \quad (101)$$

We note that the non-negativity of the PDF (97) can be shown by applying the subordination approach only for $\alpha>\beta$, i.e., for $0<\gamma=1+\beta-\alpha<1$. The subordination approach does not work for $\alpha<\beta$ (i.e., for $1<1+\beta-\alpha<2$). However, one can show that the solution is nonnegative, see, for instance [30,48].

The MSD for this process can be obtained from solution (94),

$$\langle x^2(t)\rangle=\frac{2t^{1+\beta-\alpha}}{\Gamma(2+\beta-\alpha)}. \quad (102)$$

Therefore, the corresponding process for $\alpha>\beta$ is subdiffusive, $\langle x^2(t)\rangle=2t^\gamma/\Gamma(1+\gamma)$ $(0<\gamma<1)$ while for $\alpha<\beta$ it is superdiffusive, $\langle x^2(t)\rangle=2t^\mu/\Gamma(1+\mu)$ $(1<\mu<2)$. The case with $\alpha=\beta$ describes normal diffusion, $\langle x^2(t)\rangle=2t$.

5.2. Lagging behavior in fractional diffusion

When we choose $\alpha=1$ (then $\beta=\gamma$) in (92), we obtain the fractional diffusion equation of modified "dimensional" form [37]

$$J'(x',t')=-K_\gamma\,{}_0^{RL}\mathcal{D}_{t'}^{1-\gamma}\frac{\partial P'(x',t')}{\partial x'}, \quad (103)$$

where t' is the observation time. Considering the non-simultaneous relation between the flux and the distribution gradient [4], Eq. (4), the modified form (103) can be generally expressed as

$$J'(x',t'+\overline{\tau}_j)=-K_\gamma {}_0^{RL}\mathcal{D}_{t'}^{1-\gamma}\frac{\partial P'(x',t'+\overline{\tau}_p)}{\partial x'}, \tag{104}$$

where $\overline{\tau}_j$ and $\overline{\tau}_p$ are constants with dimension of time. Applying the fractional Taylor's series [60] to both sides of (104), one obtains

$$\left(1+\frac{\overline{\tau}_j^{\alpha}}{\Gamma(1+\alpha)}{}_0{}^{RL}\mathcal{D}_{t'}^{\alpha}+O\left(\overline{\tau}_j^{\alpha}\right)\right)J'\left(x',t'\right)$$
$$=-K_\gamma {}_0^{RL}\mathcal{D}_{t'}^{1-\gamma}\left(1+\frac{\overline{\tau}_p^{\beta}}{\Gamma(1+\beta)}{}_0{}^{RL}\mathcal{D}_{t'}^{\beta}+O\left(\overline{\tau}_p^{\beta}\right)\right)\frac{\partial P'\left(x',t'\right)}{\partial x'}, \tag{105}$$

where $0<\alpha,\beta\le 1$.

5.3. Distributed-order equation

We can generalize the constitutive law (92) of the fractional diffusion-wave equation to the generalized "distributed-order" version [61]

$$\int_0^1 \kappa_1(\nu){}_0^{RL}\mathcal{D}_t^{1-\nu}J(x,t)d\nu=-\int_0^1 \kappa_2(\sigma){}_0^{RL}\mathcal{D}_t^{1-\sigma}\frac{\partial P(x,t)}{\partial x}d\sigma. \tag{106}$$

When $\kappa_1(\nu)=\delta(\nu-\alpha)$ and $\kappa_2(\sigma)=\delta(\sigma-\beta)$Eq. (106) reduces to (92). We focus on three important special cases of (106), namely:

(i) With the specific choice

$$\kappa_1(\nu)=\delta(\nu-1),\quad \kappa_2(\sigma)=B_1\delta(\sigma-\beta_1)+B_2\delta(\sigma-\beta_2), \tag{107}$$

where $B_1+B_2=1$ and $0<\beta_1<\beta_2\le 1$ we get

$$J(x,t)=-\left(B_1{}_0^{RL}\mathcal{D}_t^{1-\beta_1}+B_2{}_0^{RL}\mathcal{D}_t^{1-\beta_2}\right)\frac{\partial P(x,t)}{\partial x}, \tag{108}$$

which constitutes the double-order fractional diffusion equation of modified type when combined with the continuity equation. This description represents accelerating subdiffusion [27,29].

(ii) The concrete form

$$\kappa_1(\nu)=A_1\delta(\nu-\alpha_1)+A_2\delta(\nu-\alpha_2),\quad \kappa_2(\sigma)=\delta(\sigma-\beta), \tag{109}$$

where $A_1+A_2=1$ and $0<\alpha_1<\alpha_2\le 1$ leads to

$$\left(A_1{}_0^{RL}\mathcal{D}_t^{1-\alpha_1}+A_2{}_0^{RL}\mathcal{D}_t^{1-\alpha_2}\right)J(x,t)=-{}_0^{RL}\mathcal{D}_t^{1-\beta}\frac{\partial P(x,t)}{\partial x}, \tag{110}$$

which represents a double-order fractional diffusion equation of natural type if we choose $0<\beta<\alpha_1<\alpha_2\le 1$, and a double-order fractional wave equation if $0<\alpha_1<\alpha_2<\beta\le 1$.

(iii) Let us lastly consider the more general form

$$\kappa_1(\nu)=A_1\delta(\nu-\alpha_1)+A_2\delta(\nu-\alpha_2),\kappa_2(\sigma)=B_1\delta(\sigma-\beta_1)+B_2\delta(\sigma-\beta_2), \tag{111}$$

where $A_1+A_2=1$, $B_1+B_2=1$, $0<\alpha_1<\alpha_2\le 1$, and $0<\beta_1<\beta_2\le 1$. We get

$$\left(A_1{}_0^{RL}\mathcal{D}_t^{1-\alpha_1}+A_2{}_0^{RL}\mathcal{D}_t^{1-\alpha_2}\right)J(x,t)=-\left(B_1{}_0^{RL}\mathcal{D}_t^{1-\beta_1}+B_2{}_0^{RL}\mathcal{D}_t^{1-\beta_2}\right)\frac{\partial P(x,t)}{\partial x}, \tag{112}$$

which provides a special form of the fractional Jeffreys constitutive law (9).

6. Conclusions

The classical third-order JE has been established as an efficient tool for the description of a wide range of phenomena in heat and mass transfer in physical and engineering sciences. Recently a fractional generalization of the JE was suggested, based on partial time derivatives replacing the integer order derivatives. This generalization extends the reach of applicability of the ordinary JE to anomalous transport phenomena. However, the microscopic probabilistic foundation for this fractional generalization has remained obscure. Here we investigated a continuous time random walk approach to the FJE in the diffusion limit. The derivation relies on a specific form of the waiting time PDF, for which we provide sufficient conditions for its nonnegativity. As long as the mean waiting time remains finite, the random walk model leads back to the ordinary JE. The random walk model with infinite mean waiting time, however, gives rise to the FJE. Different forms of such waiting time PDF with divergent mean have a long standing use in the continuous time random walk model and were connected with time-fractional diffusion equations used in the description of subdiffusive anomalous transport.

In addition to establishing the probabilistic foundation for the FJE we here derived the exact solution of the corresponding CTRW. The special cases of our approach include normal and time-fractional diffusion equations, bifractional equations for retarding subdiffusion, the ordinary JE, the fractional telegrapher's and the FJE itself. Our aim in this work is to establish a general framework which may serve as a common basis for the description of a broad range of anomalous diffusive processes in complex systems, in particular, for those cases in which clear crossovers between different diffusion regimes are observed. Prominent examples of complex transport phenomena exhibiting different anomalous behaviors during different stages of evolution are found in groundwater transport [62], glassy dynamics [63], lipid molecule dynamics in bilayer membranes [64], drug molecular transport in between silica slabs [65], and exciton diffusion in nanoplatelets [66], to name a few. Varying the fractional exponents in the FJE allows one to capture such phenomena as accelerating or retarded subdiffusion, crossovers from super- to normal diffusion as well as between super- and subdiffusion. Moreover, we also showed that under certain conditions the FJE describes a caging behavior between diffusive regimes. These remarkable properties require further studies and open broad perspectives for applications.

We also demonstrated that the FJE belongs to the class of generalized Fokker-Planck equations with a specific memory kernel expressed via the combination of two Mittag-Leffler functions. We provided a solution for the ordinary JE and the FJE for the flux-driven case in terms of infinite series of the hypergeometric and Fox H-functions, respectively. Hence, we were able to compare between the exact solutions of the CTRW process and the FJE, which, by construction, agree with each other in the diffusion limit of sufficiently long time and large spatial distances. We also disclosed a variety of anomalous diffusion regimes governed by the FJE. Which of them is realized depends on the particular values of the fractional derivatives entering the FJE. The latter are defined in terms of the parameters in the waiting time PDF of the underlying CTRW model.

It will be interesting to generalize the concept of the FJE to situations in external potentials, thus extending the framework of the fractional Fokker-Planck equation [15]. Moreover, it will be of much practical use to establish the first-passage dynamics encoded in the FJE.

Declaration of Competing Interest

All authors declare no conflict of interest.

CRediT authorship contribution statement

Emad Awad: Conceptualization, Writing – original draft, Writing – review & editing. **Trifce Sandev:** Conceptualization, Writing – original draft, Writing – review & editing. **Ralf Metzler:** Conceptualization, Writing – original draft, Writing – review & editing. **Aleksei Chechkin:** Conceptualization, Writing – original draft, Writing – review & editing.

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Appendix A. Special functions

We here summarize the special functions used throughout the paper. The Fox H-function is defined in terms of the Mellin-Barnes integral [67]

$$H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi\imath}\int_\Omega\Theta(s)x^s ds, \tag{A.1}$$

where m, n, p, and q are integers satisfying $0\le n\le p$, $1\le m\le q$, $a_i,b_j\in\mathbb{C}$, $A_i,B_j\in\mathbb{R}_+$, $i=1,\ldots,p$, $j=1,\ldots,q$, and the function $\Theta(s)$ is given by

$$\Theta(s)=\frac{\prod_{j=1}^m\Gamma(b_j-\ B_js)\prod_{j=1}^n\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^q\Gamma(1-b_j+B_js)\prod_{j=n+1}^p\Gamma(a_j-A_js)}, \tag{A.2}$$

where $\Gamma(\cdot)$ is the Gamma function. The contour Ω on the right side of Eq. (A.1) separates the poles of $\Gamma(b_j+B_js)$, $j=1,\ldots,m$ from the poles of $\Gamma(1-a_i-A_is)$, $i=1,\ldots,n$. If the poles of $\prod_{j=1}^m\Gamma(b_j-B_js)$ are simple, the following series expansion holds true

$$H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]=\sum_{h=1}^m\sum_{\nu=0}^\infty\frac{(-1)^\nu x^{\frac{b_h+\nu}{B_h}}}{\nu!B_h}\frac{\prod_{j=1,j\neq h}^m\Gamma\left(b_j-B_j\frac{b_h+\nu}{B_h}\right)\prod_{j=1}^n\Gamma\left(1-a_j+A_j\frac{b_h+\nu}{B_h}\right)}{\prod_{j=m+1}^q\Gamma\left(1-b_j+B_j\frac{b_h+\nu}{B_h}\right)\prod_{j=n+1}^p\Gamma\left(a_j-A_j\frac{b_h+\nu}{B_h}\right)}. \tag{A.3}$$

Eq. (A.3) leads to the concrete expansion

$$H^{2,1}_{2,3}\left[x\left|\begin{matrix}\left(\frac12-k,1\right);(a,b)\\(0,1),\left(\frac12+n-k,1\right);\left(\frac12,1\right)\end{matrix}\right.\right]=\sum_{\nu=0}^\infty\frac{(-x)^\nu}{\nu!}\left\{\frac{\Gamma\left(\frac12+n-k-\nu\right)\Gamma\left(\frac12+k+\nu\right)}{\Gamma\left(\frac12+\nu\right)\Gamma(a-b\nu)}+\frac{\Gamma\left(-\frac12-n+k-\nu\right)\Gamma(1+n+\nu)}{\Gamma(1+n-k+\nu)\Gamma\left(a-\frac{b}{2}-b[n-k+\nu]\right)}x^{\frac12+n-k}\right\} \tag{A.4}$$

The following relation has been used in deriving the diffusive flux,

$$\frac{d}{dx}H^{2,1}_{2,3}\left[ax^2\left|\begin{matrix}\left(\frac12-k,1\right);(\alpha,\beta)\\(0,1),\left(\frac12+n-k,1\right);\left(\frac12,1\right)\end{matrix}\right.\right]=-\frac{2}{x}H^{2,1}_{2,3}\left[ax^2\left|\begin{matrix}\left(\frac12-k,1\right);(\alpha,\beta)\\(1,1),\left(\frac12+n-k,1\right);\left(\frac12,1\right)\end{matrix}\right.\right]. \tag{A.5}$$

The inverse Fourier transform of the H-function is given by [37]

$$\int_{-\infty}^{\infty}|q|^\lambda H^{1,1}_{1,2}\left[a|q|^\delta\left|\begin{matrix}(-n,1)\\(0,1);(\beta,\gamma)\end{matrix}\right.\right]\exp(-\imath qx)dq=\frac{\sqrt{\pi}}{a^{(\lambda+1)/\delta}}H^{2,1}_{2,3}\left[\frac{|x|^\delta}{2^\delta a}\left|\begin{matrix}\left(1-\frac{\lambda+1}{\delta},1\right);\left(1-\beta-\frac{\lambda+1}{\delta}\gamma,\gamma\right)\\\left(0,\frac{\delta}{2}\right),\left(1+n-\frac{\lambda+1}{\delta},1\right);\left(\frac12,\frac{\delta}{2}\right)\end{matrix}\right.\right], \tag{A.6}$$

where $a,\gamma,\delta\in\mathbb{R}_+$, $\beta,n\in\mathbb{C}$, $\lambda\in\mathbb{R}_+\cup\{0\}$.

When $A_i=B_j=1$, $i=1,\ldots,p$, $j=1,\ldots,q$, the H-function (A.1) reduces to the Meijer G-function [68]

$$H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_1,1),\ldots,(a_p,1)\\(b_1,1),\ldots,(b_q,1)\end{matrix}\right.\right]=G^{m,n}_{p,q}\left[x\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right]. \tag{A.7}$$

defined by the Mellin-Barnes integral

$$G^{m,n}_{p,q}\left[x\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right]=\frac{1}{2\pi\imath}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)}x^s ds, \tag{A.8}$$

where L is a contour encircling all poles of $\Gamma(b_j-s)$, $j=1,\ldots,m$ in the negative direction, but not encircling any pole of $\Gamma(1-a_i+s)$, $i=1,\ldots,n$. Then the integral converges for $q>p\ge0$. If the poles of $\prod_{j=1}^m\Gamma(b_j-s)$ are simple and the integral (A.8) is convergent ($q>p$), then the Meijer G-function can be expressed as a sum of residues in terms of generalized hypergeometric functions ${}_pF_{q-1}$ (see p. 145 in Ref. [69]),

$$G^{m,n}_{p,q}\left[x\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right]=\sum_{h=1}^m\frac{\prod_{j=1,j\neq h}^m\Gamma\left(b_j-b_h\right)\prod_{j=1}^n\Gamma\left(1-a_j+b_h\right)x^{b_h}}{\prod_{j=m+1}^q\Gamma\left(1-b_j+b_h\right)\prod_{j=n+1}^p\Gamma\left(a_j-b_h\right)}\,{}_pF_{q-1}\left(\left.\begin{matrix}1-a_1+b_h,\ldots,1-a_p+b_h\\1-b_1+b_h,\ldots,1-b_{h-1}+b_h,1-b_{h+1}+b_h,\ldots,1-b_q+b_h\end{matrix}\right|-x\right), \tag{A.9}$$

where $q>p$.

Utilizing the regularized hypergeometric function [70], related to the generalized hypergeometric function through

$$ {}_pF_q\left(\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix}|x\right) = {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x) = \Gamma(\beta_1)\cdots\Gamma(\beta_q)\,{}_p\bar{F}_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;x), \tag{A.10}$$

with relations (A.9), (A.7) and the Euler reflection formula $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, we deduce that

$$ H^{2,1}_{2,3}\left[x\left|\begin{matrix}(a_1,1);(a_2,1)\\(b_1,1),(b_2,1);(b_3,1)\end{matrix}\right.\right] = \frac{\pi}{\sin(\pi[b_2-b_1])}\times\left\{\frac{\Gamma(1-a_1+b_1)}{\Gamma(a_2-b_1)}x^{b_1}\,{}_2\bar{F}_2(1-a_1+b_1,1-a_2+b_1;1-b_2+b_1,1-b_3+b_1;-x) - \frac{\Gamma(1-a_1+b_2)}{\Gamma(a_2-b_2)}x^{b_2}\,{}_2\bar{F}_2(1-a_1+b_2,1-a_2+b_2;1-b_1+b_2,1-b_3+b_2;-x)\right\}. \tag{A.11}$$

The Prabhakar generalization of Mittag-Leffler function is defined through the series representation [71]

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad \alpha,\beta,\gamma,z\in\mathbb{C},\quad \Re\{\alpha\}>0, \tag{A.12}$$

where $(\gamma)_n$ is the ascending Pochhammer symbol defined by $(\gamma)_0=1$, $(\gamma)_n=\gamma(\gamma+1)\cdots(\gamma+n-1)=\Gamma(\gamma+n)/\Gamma(\gamma)$. The function $E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})$ is a completely monotone function (CMF) provided that $\lambda>0$, $t>0$, and [72]

$$t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})\in \mathrm{CMF}\quad \text{iff}\quad \begin{cases}0<\alpha\gamma\le\beta\le 1,\\ 0<\alpha\le 1.\end{cases} \tag{A.13}$$

The asymptotic behavior for large negative argument can be found from the formula

$$E^{\gamma}_{\alpha,\beta}(-z)\sim\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\beta-\alpha(\gamma+n))}\frac{(-z)^{-n}}{n!},\quad z>1, \tag{A.14}$$

and together with the definition of the Mittag-Leffler function (A.12) give the following asymptotics [72]

$$E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})\sim\begin{cases}\dfrac{1}{\Gamma(\beta)}-\dfrac{\lambda\gamma t^{\alpha}}{\Gamma(\alpha+\beta)}, & t\to 0^+,\\[2mm] \dfrac{(\lambda t^{\alpha})^{-\gamma}}{\Gamma(\beta-\alpha\gamma)}, & t\to\infty.\end{cases} \tag{A.15}$$

The Laplace transform of the $E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})$ is given by [71]

$$\mathcal{L}\left\{t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha});s\right\}=\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+\lambda)^{\gamma}},\quad \Re\{\beta\}>0. \tag{A.16}$$

Using the familiar relation between the Prabhakar generalization of the Mittag-Leffler function and the Fox H-function and Eq. (A.16) we obtain [67]

$$\int_0^{\infty}t^{\beta-1}H^{1,1}_{1,2}\left[\lambda t^{\alpha}\left|\begin{matrix}(1-\gamma,1)\\(0,1);(1-\beta,\alpha)\end{matrix}\right.\right]\exp(-st)dt=\Gamma(\gamma)\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+\lambda)^{\gamma}}. \tag{A.17}$$

When $\gamma=1$ we have the special case

$$E^{1}_{\alpha,\beta}(z)=E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^m}{\Gamma(\alpha n+\beta)}, \tag{A.18}$$

which known as the generalized Mittag-Leffler function. For $\beta=1$ it further reduces to the standard Mittag-Leffler function. All the above relations of $E^{\gamma}_{\alpha,\beta}(z)$ are valid for $E_{\alpha,\beta}(z)$ upon setting $\gamma=1$. The following special cases will be used [71],

$$tE_{1,2}(-t)=1-\exp(-t),\quad t^2E_{1,3}(-t)=\exp(-t)-1+t. \tag{A.19}$$

Appendix B. *n* th differentiator function as initial condition

The unit doublet function (differentiator) [73] is defined as the first derivative of the Dirac delta function:

$$u_1(x)=\frac{\partial}{\partial x}\delta(x), \tag{B.1}$$

with the properties

$$\mathcal{F}\{u_1(x);q\}=\imath q,\quad \int_{-\infty}^{\infty}f(x-\xi)u_1(\xi)d\xi=\frac{\partial}{\partial x}f(x). \tag{B.2}$$

This distribution can be generalized to the so-called "nth differentiator function"

$$u_n(x)=\frac{\partial^n}{\partial x^n}\delta(x), \tag{B.3}$$

where $n\in\mathbb{N}\cup\{0\}$. When n is an negative integer, Eq. (B.3) defines the nth integrator function, whereas $n=0$ recovers the conventional Dirac delta function (unit impulse function). The Dirac delta and the unit doublet functions can be viewed at the limiting forms

$$\delta(x)=\lim_{a\to0^+}\frac{1}{|a|\sqrt{\pi}}\exp\left(-\frac{x^2}{a^2}\right),\quad u_1(x)=-\lim_{a\to0^+}\frac{2x}{a^2|a|\sqrt{\pi}}\exp\left(-\frac{x^2}{a^2}\right). \tag{B.4}$$

Let us consider the following initial conditions for the partial differential Eq. (43)

$$P(x,0^+)=\delta(x),\quad \left.\frac{\partial P(x,t)}{\partial t}\right|_{t\to0^+}=\nu_0(x), \tag{B.5}$$

where $\nu_0(x)$ is the initial rate of the distribution $P(x,t)$. In Laplace domain, Eq. (43) subject to (B.5) reads

$$(s+1)\left[s\widetilde{P}(x,s)-\delta(x)\right]=(1+\chi s)\frac{\partial^2\widetilde{P}(x,s)}{\partial x^2}+\nu_0(x)-\chi u_2(x). \tag{B.6}$$

After additional Fourier transformation,

$$(s+1)\left[s\widehat{\widetilde{P}}(q,s)-1\right]=-(1+\chi s)q^2\widehat{\widetilde{P}}(q,s)+\widehat{\nu}_0(q)-\chi\widehat{u}_2(q), \tag{B.7}$$

which can be rearranged to

$$\widehat{\widetilde{P}}(q,s)=\frac{s+1+\widehat{\nu}_0(q)-\chi\widehat{u}_2(q)}{s(s+1)+(1+\chi s)q^2}. \tag{B.8}$$

Upon comparison of the CTRW long-time limit (42) with the above expression (B.8) we readily deduce that the partial differential Eq. (43) is the limiting case of the CTRW process provided that

$$\widehat{\nu}_0(q)=\chi\widehat{u}_2(q),\qquad \chi>1, \tag{B.9}$$

or, more precisely, it should be subject to the initial condition (44). Using similar analysis, one can validate the initial condition (47) for the flux-distribution formulation.

Appendix C. Waiting time PDF $\psi(t)$ and asymptotics

We present a closed-form solution for the waiting time PDF $\psi(t)$. Using the expansions

$$\exp(-x) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}, \quad (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k, \tag{C.1}$$

Eq. (16) can be written as

$$\tilde{\psi}(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\chi^{-\beta n}\sum_{k=0}^{n}\binom{n}{k}\frac{s^{\gamma n+\alpha k}}{\left(s^\beta+\frac{1}{\chi^\beta}\right)^n}. \tag{C.2}$$

Applying formally an inverse Laplace transformation to Eq. (C.2) and using Eq. (A.16) we get the following form for the waiting time PDF in the time domain,

$$\psi(t) = \delta(t) + \frac{1}{t}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{t^{\beta-\gamma}}{\chi^\beta}\right)^n\sum_{k=0}^{n}\binom{n}{k}t^{-\alpha k}E^n_{\beta,[\beta-\gamma]n-\alpha k}\left(-\left(\frac{t}{\chi}\right)^\beta\right). \tag{C.3}$$

Note that the second lower index of the three-parameter Mittag-Leffler function (C.3) is negative, and thus such a solution does not obey the condition (A.16). However, the series (C.3) gives the same long and short time behavior of the waiting time PDF as deduced from its Laplace transform (16). Moreover, the numerical result of the inverse Laplace transformation of Eq. (16) fits well the numerical result of (C.3) in the whole time domain.

Next, we derive the short and long time behaviors for the waiting time PDF $\psi(t)$. The long-time behavior of (C.3) can be derived from the long time behavior of the Prabhakar generalization of the Mittag-Leffler function (A.15), such that we obtain

$$\psi(t) \sim \frac{1}{t}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}t^{-\gamma n}\sum_{k=0}^{n}\binom{n}{k}\frac{t^{-\alpha k}}{\Gamma(-\gamma n-\alpha k)}, \tag{C.4}$$

where $t\to\infty$ and $t\gg\chi$, which can be approximated for long times by neglecting all terms of the finite sum except the first term, namely,

$$\psi(t) \sim \frac{1}{t}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{t^{-\gamma n}}{\Gamma(-\gamma n)}. \tag{C.5}$$

Using the expansion of the one-sided Lévy stable density,

$$\ell_\alpha(t) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{t^{-\alpha n-1}}{\Gamma(-\alpha n)}, \tag{C.6}$$

which can be easily derived by using the relation between $\ell_\alpha(t)$ and the Fox H-function [74] and expansion (A.3), we derive the long-time behavior (18), where $\ell_\alpha(t)\sim\alpha t^{-\alpha-1}/\Gamma(1-\alpha)$ as $t\to\infty$.

The short time behavior of (C.3) can be obtained from using the short time behavior of the Prabhakar generalized Mittag-Leffler function (A.15), leading to

$$\psi(t) \sim \sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,\chi^{\beta n}}\sum_{k=0}^{n}\binom{n}{k}\frac{t^{-(\gamma-\beta)n-\alpha k-1}}{\Gamma(-[\gamma-\beta]n-\alpha k)}. \tag{C.7}$$

For fixed n and short times we can neglect all terms of the finite series in (C.7) except from the last term, obtaining

$$\psi(t) \sim \sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,\chi^{\beta n}}\frac{t^{-(\alpha+\gamma-\beta)n-1}}{\Gamma(-[\alpha+\gamma-\beta]n)}. \tag{C.8}$$

Comparing this result with the Lévy density (C.6) we can derive the short time behavior (19).

It is worthwhile noting that the above asymptotic behaviors can be also deduced by employing the Tauberian theroem [52] to the waiting time density in the Laplace domain (16). For example the long time behavior corresponds to small values of the Laplace parameter, namely,

$$\tilde{\psi}(s) \sim \exp(-s^\gamma) = \tilde{\ell}_\gamma(s), \quad \Re(s)\to 0. \tag{C.9}$$

Conversely the short time behavior can be derived by letting $\Re(s)$ tend to ∞, i.e.,

$$\tilde{\psi}(s) \sim \exp\left(-\left[\frac{s}{\chi^{\beta/(\alpha+\gamma-\beta)}}\right]^{\alpha+\gamma-\beta}\right), \quad \Re(s)\to\infty, \tag{C.10}$$

which can be inverted to the time domain result (19).

The short time behavior (19) can be further expressed in a simpler form using the asymptotic behavior of the α-stable PDF for small values [54,74]:

$$\ell_\alpha(t) \sim Bt^{-\sigma}\exp(-\kappa t^{-\tau}), \quad t\to 0, \tag{C.11}$$

where

$$B = \sqrt{\frac{\alpha^{1/(1-\alpha)}}{2\pi(1-\alpha)}}, \quad \sigma = \frac{2-\alpha}{2(1-\alpha)}, \quad \kappa = (1-\alpha)\alpha^{\alpha/(1-\alpha)}, \quad \tau = \frac{\alpha}{1-\alpha}. \tag{C.12}$$

Appendix D. Bernstein functions

The Bernstein functions were used for the validation of the non-negativity of the fundamental solution of the distributed-order fractional diffusion-wave equation [48] and extended to the multi-term case [75]. They were invoked in different places of the current work. For the convenience of the reader we here collect the fundamental concepts and properties of some functions of this class, see [52,76].

(i) The function $\tilde{f}(x,\lambda):\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+^0$ is a *completely monotone function* with respect to λ if it is continuously differentiable with respect to λ and satisfies the condition $(-1)^k\partial^k\tilde{f}(x,\lambda)/\partial\lambda^k\ge 0$ for all $k\in\mathbb{N}$. Then we write $\tilde{f}(x,\lambda)\in$ CMF, where CMF is the set of all completely monotone functions. The importance of the property CMFs in our investigation is derived from the property that the generic function $f(x,t)$ is nonnegative, for all values of time $t>0$ and space $x\in\mathbb{R}$ if its Laplace transform restricted to the positive real line, $\tilde{f}(x,\lambda)=\int_0^\infty f(x,t)\exp(-\lambda t)dt$, $\lambda>0$ lies in CMF. The product and the linear combination of two completely monotone functions are also completely monotone functions.

(ii) The function $\tilde{f}(x,\lambda):\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+^0$ is a *Stieltjes function* with respect to λ if there exists a completely monotone function $f(x,t)$ with respect to $t>0$ such that $\tilde{f}(x,\lambda)=\int_0^\infty f(x,t)\exp(-\lambda t)dt$, $\lambda>0$, namely, if it is the Laplace transform of a completely monotone function. We then write $\tilde{f}(x,\lambda)\in$ SF, where SF is the set of all Stieltjes functions. It is clear that any Stieltjes function is a completely monotone function (SF $\subset$ CMF), but the contrary is not generally true. The product of two Stieltjes functions is not necessarily a Stieltjes function, however, if $\varphi,\psi\in$ SF then $[\varphi(\lambda)]^\alpha[\psi(\lambda)]^\beta\in$ SF provided that $\alpha,\beta\in(0,1)$ and $\alpha+\beta\le 1$. Choosing $\alpha=\beta=0.5$ we thus have $\sqrt{\varphi\psi},\sqrt{\varphi},\sqrt{\psi}\in$ SF.

(iii) The function $\tilde{f}(x,\lambda):\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+^0$ is a *Bernstein function* with respect to λ if it is continuously differentiable with respect to λ and satisfies the condition $(-1)^{k-1}\partial^k\tilde{f}(x,\lambda)/\partial\lambda^k\ge 0$ for all $k\in\mathbb{N}$. The set of all Bernstein functions is denoted by BF. If $f,g\in$ BF and $h\in$ CMF, then $f[g(\lambda)]\in$ BF, $h[f(\lambda)]\in$ CMF and $f(\lambda)/\lambda\in$ CMF.

(iv) The Bernstein function $\tilde{f}(x,\lambda):\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+^0$ is said to be a *complete Bernstein function*, with respect to λ exactly if $\tilde{f}(x,\lambda)/\lambda\in$ SF. Denoting by CBF the set of all complete Bernstein functions we see that CBF $\subset$ BF. The linear combination of complete Bernstein functions is also a complete Bernstein function. The set of complete Bernstein functions is not in general closed under multiplication, however, if $\varphi, \psi \in$ CBF, then $[\varphi(\lambda)]^{\alpha}[\psi(\lambda)]^{\beta} \in$ CBF provided that $\alpha, \beta \in (0,1)$ and $\alpha+\beta \leq 1$. If $\varphi(\lambda) \in$ CBF then $\exp(-a\varphi(\lambda)) \in$ CMF for $a, \lambda > 0$.

Appendix E. Proof of Proposition 1

In order to prove that $\psi(t)$ is a PDF it is sufficient to show that $\psi(t)$ is normalized and non-negative. The normalization of $\psi(t)$ is straightforward from (16),

$$\int_0^{\infty} \psi(t)dt = \int_0^{\infty} \psi(t)\exp(-st)dt\Big|_{s=0} = \tilde{\psi}(0) = 1. \tag{E.1}$$

Let us consider the function

$$\tilde{\omega}(\xi) = \frac{\xi^{\gamma}(\xi^{\alpha}+1)}{1+\chi^{\beta}\xi^{\beta}}, \quad \xi = \Re\{s\} > 0, \tag{E.2}$$

where $0<\alpha,\beta,\gamma<1$ and $\chi>0$. It is known that $\tilde{\omega}(\xi)$ is a complete Bernstein function (CBF), i.e., $\tilde{\omega}(\xi) \in \text{CBF} \subset \text{BF}$, if its inverse is a Stieltjes function (SF), i.e., $\tilde{\eta}(\xi) = [\tilde{\omega}(\xi)]^{-1} \in$ SF, see App. Appendix D for Bernstein functions (BFs). Rewriting $\tilde{\eta}(\xi)$ as

$$\tilde{\eta}(\xi) = \frac{\xi^{-\gamma}}{\xi^{\alpha}+1} + \chi^{\beta}\frac{\xi^{\beta-\gamma}}{\xi^{\alpha}+1}, \tag{E.3}$$

and inverting the Laplace transform $\tilde{\eta}(\xi) \to \eta(t)$, see Eq. (A.16), we get

$$\eta(t) = t^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-t^{\alpha}) + \chi^{\beta}t^{\alpha+\gamma-\beta-1}E_{\alpha,\alpha+\gamma-\beta}(-t^{\alpha}), \tag{E.4}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function (A.18). Using the complete monotonicity conditions on the Mittag-Leffler functions, see (A.13), one can deduce that $\eta(t) \in$ CMF exactly if conditions (7b) are satisfied. Conversely from the fact that the function $\tilde{\eta}(\xi) = \int_0^{\infty}\eta(t)\exp(-\xi t)dt$ is a Stieltjes function exactly if $\eta(t)$ is a completely monotone function, we have that $\tilde{\eta}(\xi) \in$ SF provided that the sufficient conditions (7b) are met. Therefore,

$$\tilde{\omega}(\xi) = [\tilde{\eta}(\xi)]^{-1} \in \text{CBF} \subset \text{BF}, \tag{E.5}$$

if conditions (7b) hold true. Since the composition of a completely monotone function and a Bernstein function is a completely monotone function,

$$\tilde{\psi}(\xi) = \exp(-\tilde{\omega}(\xi)) \in \text{CMF} \tag{E.6}$$

if the conditions (7b) are satisfied, which proves the non-negativity of $\psi(t)$ in the case $0<\alpha,\beta,\gamma<1$ and $\chi>0$, and thus completes the proof of the first part of the Proposition.

Secondly, we focus our attention now on the special case $\alpha=\beta=\gamma=1$, namely,

$$\tilde{\omega}_1(\xi) = \frac{\xi(\xi+1)}{1+\chi\xi}, \quad \xi = \Re\{s\} > 0. \tag{E.7}$$

By rearranging $\tilde{\omega}_1(\xi)$, we can write $\tilde{\eta}_1(\xi) = [\tilde{\omega}_1(\xi)]^{-1}$ as

$$\tilde{\eta}_1(\xi) = \frac{1+(\chi-1)\xi/(\xi+1)}{\xi}. \tag{E.8}$$

Since $\xi/(\xi+1) \in \text{CBF} \subset \text{BF}$ the numerator of $\tilde{\eta}_1(\xi)$ is also CBF whenever $\chi>1$. Thereby $\tilde{\eta}_1(\xi) \in$ SF and $\tilde{\omega}_1(\xi) = [\tilde{\eta}(\xi)]^{-1} \in \text{CBF} \subset \text{BF}$ whenever $\chi>1$. Hence,

$$\tilde{\psi}_1(\xi) = \exp(-\tilde{\omega}_1(\xi)) \in \text{CMF}, \tag{E.9}$$

provided that $\chi>1$, which proves the non-negativity of $\psi_1(t)$ in the case $\alpha=\beta=\gamma=1$ and completes the proof of the proposition.

Appendix F. Temporal scale of the corresponding CTRW for the fractional Jeffreys and Fick's equations

In the classical CTRW picture the one-sided α-stable law $\ell_{\gamma}(t)$ with the Laplace transform $\tilde{\ell}_{\gamma}(s) = \exp(-s^{\gamma})$, $0<\gamma<1$, is often chosen as the waiting time PDF for the derivation of the time-fractional diffusion equation and the approximation

$$\tilde{\ell}_{\gamma}(s) \sim 1 - s^{\gamma}, \quad s \to 0, \tag{F.1}$$

is used in the long time limit, see e.g., [15]. Such approximations of the Laplace transform corresponds to the asymptotic behavior (18) of the one-sided α-stable PDF. In our paper, in order to get the FJE from the CTRW, we used the asymptotic behavior (48). It is interesting to check what is the difference between these two asymptotic behaviors graphically. This helps to understand what is the corresponding CTRW temporal domain which can be represented exactly by means of the two fractional equations.

In Fig. 6 we compare the exact waiting time PDF (C.3) with the asymptotic forms (48) and (18). From these graphical representations one can see the same long time behavior ($t \to \infty$) for all three waiting time PDFs while the short time limit yields different behaviors. Interestingly the FJE is valid in a wider time domain

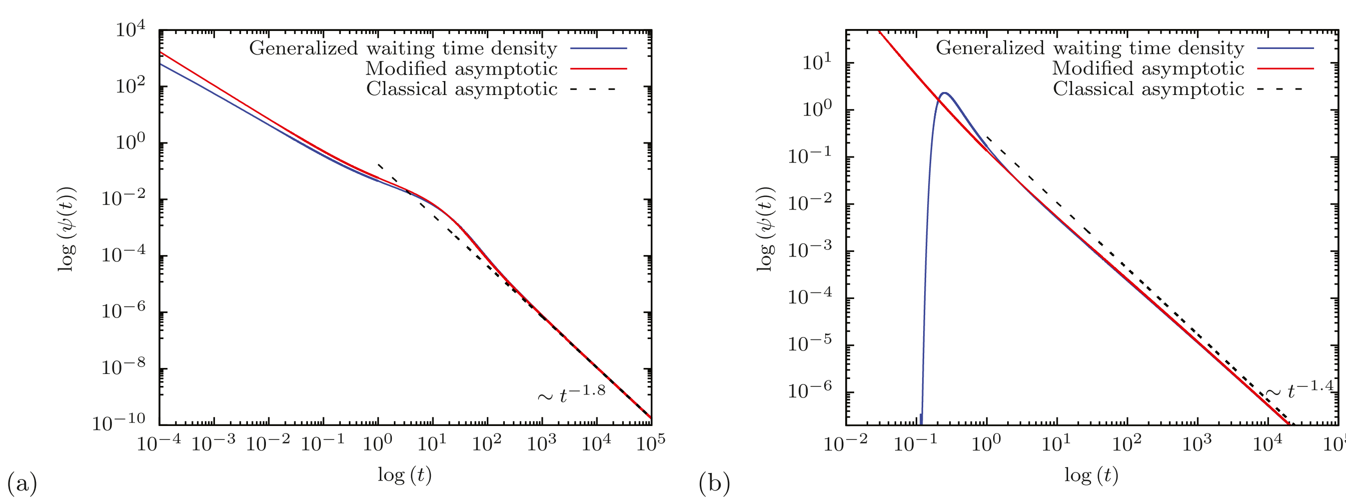

Fig. 6. Generalized waiting time PDF $\psi(t)$, Eq. (C.3) (blue solid line) compared with both the asymptotic behavior (48) which leads to the FJE (red solid line) and the classical asymptotic behavior (18) which leads to the fractional diffusion equation (dashed black line). The parameters are (a) $\alpha=0.2$, $\beta=0.8$, $\gamma=0.8$ and $\chi=10$ (acceleratingd case); (b) $\alpha=0.6$, $\beta=0.2$, $\gamma=0.4$ and $\chi=10$ (retarded case). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

of the CTRW process than the domain of validity of the fractional diffusion equation. Thus when we refer to the asymptotic behavior (48) it is worth to emphasize that it works not only in the long time limit $t \to \infty$ but also at intermediate times.

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