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Wealth distribution, Pareto law, and stretched exponential decay of money: Computer simulations analysis of agent-based models

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HIGHLIGHTS

- We examine both the closed and open agent-based kinetic exchange trading economic models.
- Wealth and person distributions are analyzed and the conditions for the Pareto law are determined.
- For open economic systems, a stretched exponential law for time evolution of total money is found.
- Time decay of total money does not depend on the density of trap agents, but on saving propensity.
- The relaxation exponents of the total money for fixed and distributed saving schemes are obtained.

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ABSTRACT

We study by Monte Carlo simulations a kinetic exchange trading model for both fixed and distributed saving propensities of the agents and rationalize the person and wealth distributions. We show that the newly introduced wealth distribution – that may be more amenable in certain situations – features a different power-law exponent, particularly for distributed saving propensities of the agents. For open agent-based systems, we analyze the person and wealth distributions and find that the presence of trap agents alters their amplitude, leaving however the scaling exponents nearly unaffected. For an open system, we show that the total wealth – for different trap agent densities and saving propensities of the agents – decreases in time according to the classical Kohlrausch–Williams–Watts stretched exponential law. Interestingly, this decay does not depend on the trap agent density, but rather on saving propensities. The system relaxation for fixed and distributed saving schemes are found to be different.

1. Introduction

Economical inequality and long-tailed income distributions – inevitable and vital paradigms in modern societies [1–6] – offer interesting subjects of investigation for societal and physical-mathematical sciences, see also Refs. [7–9]. For a long time economists have been trying to pinpoint the exact reasons of monetary inequalities, and to quantitatively answer the question how income and wealth are distributed.

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Historically, a number of attempts were made to answer these questions. More than a century ago, Pareto performed extensive studies in Europe and found that the income distributions possess power law tails [10], in contrast to the Gaussian financial market models of Regnault and Bachelier [11,12]. Mandelbrot then picked up on Pareto's ideas and coined the term *strong Pareto law* for the income distribution [13]. Gibrat worked on similar aspects of the wealth distribution and proposed a *law of proportionate growth* [14]. Gibrat realized that the Pareto's law did not fit the whole range of the data, being valid only for the "rich end". Champernowne considered the same problem proposing a probabilistic theory reproducing the Pareto's law [15].

Today, we realize that the distributions of income indeed reveal some globally stable and robust features [16–18]. In general, the bulk of the distributions of income can be fitted reasonably well by the log-normal (Boltzmann–Gibbs) and Gamma distributions [3,19–24]. Economists prefer the log-normal form [19,20] for the positively-defined measure of money, whereas physicists use the Gamma distribution [3,21,22]. The latter generally providing a better fit of the data [23], see also Ref. [25] There is more consensus on the upper end of the distribution described by the Pareto law [24,26–28]. Recently, the income data analyses were extensively reviewed [2,5].

The lack of easily-available data for the wealth distribution, analogous to yearly income-tax returns offering a measure of income [26], compels researchers to resort to indirect methods. For the income distribution, the analysis indicates that [2,3,5,19–23,26,29,30]

$$P(m) \simeq m^{-(1+\mu)} e^{-m/m_c}, \tag{1}$$

for incomes m smaller than a characteristic income dented m_c, and the power-law form

$$P(m) \simeq m^{-(1+\nu)}, \tag{2}$$

for $m \geq m_c$. Here, μ and ν are the respective scaling exponents. The exponent ν is the Pareto exponent, typically ranging between one and three [22,30–33], so that a typical income $\langle m \rangle$ exists. For a historical overview of Pareto's data and the recent progress we refer to Refs. [5,34,35] and to the study [33] for a double power law of $P(m)$. The subject of income distribution got massive attention in the last decade [2,5,7–9,18,23,26,27,29,30,33,34,36–66]. The crossover point m_c in Eq. (1) can be practically extracted via fitting the initial Gamma distribution and the power-law tail in Eqs. (1) and (2) [22]. An alternative way to joint together the bulk Gamma distribution and the Pareto law for the tails is to choose m_c such that the jointed distribution is continuous and differentiable at this point.

The allocation of wealth and income in a society is the subject of intense research in econophysics. Despite the decades of development and data analysis, the precise mathematical mechanisms are not fully understood. To reveal them, a number of models were proposed based on statistical mechanics analysis, see also Ref. [67]. In particular, to explain the functioning principles of trading systems, a number of ideal-gas saving-based models of closed economic systems were proposed [29,48–53,55–59]. In these models, the total amount of money M plays the role of the kinetic energy of the gas particles and the elastic collisions of molecules represent the money-conserving trading events between the agents [50].

Two variants of saving-based models are of interest: the models in which all agents have the same saving factor λ and models with a *quenched*, random distribution of λ [50], as opposing to the *annealed* situation [56]. The authors of Ref. [50] considered an ideal-gas model with a fixed total money in the system of N agents. A fully-connected network of agents can be formed via $N - 1$ links representing the "channels" for money transfer [50], mimicking money transport mechanisms in economical trading structures. No production or migration of money outside of the system occurs, and no debt is allowed. The economic activity is restricted to agent-to-agent trading events, see also Refs. [68,69]. This is similar to an isolated thermodynamic system at equilibrium.

We here consider the one-dimensional closed (conservative) and open (non-equilibrium) economical model. We denote the person and wealth distributions as $F(w)$ and $W(w)$. $F(w)$ quantifies the number of agents possessing the wealth in the interval from w to $w + dw$, [43]

$$F(w)dw = (N_{w,w+dw}/N)dw. \tag{3}$$

On the other hand,

$$W(w)dw = w(N_{w,w+dw})dw. \tag{4}$$

is the wealth of persons in the same wealth interval. In the limit of substantial dw/w fractions, the distribution $W(w)$ can be considered as the *integral* over the $F(w)$ distribution in the range of wealth $\{w, w + dw\}$, in the sense of a Riemann sum. In simulations, we divided the interval of final wealth values w into ten subdivisions and recovered $W(w)$ distributions presented below directly from the data. Clearly, the results for $W(w)$ will depend on the number of subdivisions of the wealth interval (not shown).

The paper is organized as follows. In Section 2 we introduce our agent-based model and present the main results of simulations for the closed system. In Section 3 we investigate the person $F(w)$ and wealth $W(w)$ distributions in an open system, and discuss the evolution of the total wealth $w_\Sigma(t)$ over time. We present some new results for open trading systems and unveil a Kohlrausch–Williams–Watts (KWW)-type stretched exponential decay for the cumulative wealth $w_\Sigma(t)$ for the non-conservative situation. The conclusions are given in Section 4.

2. Close conservative system

Here we set the model and discuss two saving scenarios in a closed system: uniform and non-uniform, as proposed previously [23,29,45,51]. We quantify the state of the system via the social-class distribution $F(w)$ and the wealth distribution $W(w)$. For a closed system, they reach their equilibrium form after sufficiently long simulation times, so below in this Section we consider $F(w) = F^{\text{eq}}(w)$ and $W(w) = W^{\text{eq}}(w)$.

2.1. Uniform saving strategy

In the random-exchange model, the ith agent possesses the amount of wealth $w_i(t)$ at time t and the total amount of wealth

$$w_\Sigma(t) = \sum_{i=1}^{N} w_i(t) \tag{5}$$

is conserved, $w_\Sigma(t) = \text{const}$. In any trading event, a pair of nearest neighbors agents i and j randomly exchange their wealth,

$$w_i(t+1) = w_j(t), \quad w_j(t+1) = w_i(t). \tag{6}$$

Some amount of *wealth* is transferred from a loser to a winner agent [43]. Nearest-neighbor only exchanges on the lattice are considered here; this can be regarded artificial from the economic perspective and clearly present some limitations of this simplistic model for the description of real trading systems. The findings of these simple agents-based models for the person distribution are, however, consistent with the Pareto law. We expect the model incorporating money exchanges between arbitrary – and not only nearest-neighbor – pairs of agents to yield very similar results for the equilibrium distributions. The relaxation of the system to these distributions might, however, be slower in the nearest-neighbor only case, particular for non-uniform initial distribution of money among the agents. After a trading event the time t in the system is changed by one unit. Obviously, this operation conserves the wealth held by a given pair of agents,

$$w_i(t) + w_j(t) = w_i(t+1) + w_j(t+1). \tag{7}$$

We now introduce the uniform saving propensity factor, λ, representative for a *homogeneous ensemble* of agents, as considered previously [23,29,45,51]. The agent i at time step t saves a given fraction $0 \leq \lambda \leq 1$ of wealth, $\lambda w_i(t)$, and trades the remaining amount $(1-\lambda)w_i(t)$, randomly. Economists call the fraction λ the *marginal propensity to save* [52]. Mathematically, the exchange action is expressed as [29,50,51]

$$\begin{aligned} w_i(t+1) &= \lambda w_i(t) + \varepsilon_{ij} K_{ij}(t), \\ w_j(t+1) &= \lambda w_j(t) + (1-\varepsilon_{ij}) K_{ij}(t), \end{aligned} \tag{8}$$

where we introduce the notation

$$K_{ij}(t) = (1-\lambda)[w_i(t) + w_j(t)]. \tag{9}$$

The amount of wealth exchanged between the pair of agents is then

$$\Delta w_{ij}(t) = (1-\lambda)\{\varepsilon_{ij}[w_i(t) + w_j(t)] - w_i(t)\}, \tag{10}$$

where ε_{ij} is a random number in the range $\{0, 1\}$. This randomness reflects the *stochastic nature* of each trading. We follow here the mathematical formulation and the notations of Refs. [22,29,50]. The exchange matrix $\mathbf{M}(\varepsilon)$ is symmetric, so that $\varepsilon_{ij} = \varepsilon_{ji}$.

The market is called non-interacting at $\lambda = 1$: it leaves the initial distribution of wealth unchanged. The value $\lambda = 0$ corresponds to the limit of the full-risk trading when no savings are put aside [52]. For uniform saving propensities λ the steady-state distribution $F(w)$ decays exponentially for both small and large w [50,54,56]. Depending on the value of λ, the most probable wealth per person shifts from $w = 0$ for $\lambda = 0$ (corresponding to the Gibbs exponential distribution, see Fig. 1) to a finite value for $\lambda \to 1$ [2,52] with a more Gaussian-like distribution, see Ref. [58] for details. The choice of $\lambda \to 1$ corresponds to almost *no trading events* taking place between the agents, and thus the final wealth distributions contain just small deviations/fluctuations around the initial wealth of the traders, see Fig. 1 where the value of

$$w(t=0) = 1$$

was chosen. The uniform initial distribution of wealth is unstable, however, for $\lambda \to 1$ – during a given simulation time – the departure of the system from the initial condition is rather small. In contrast, for small values of the saving propensity λ very different final distributions are being formed during the same T, i.a. the exponential $F(w)$ distribution at $\lambda \to 0$.

We analyzed the features of this trading model by Monte Carlo simulations. Initially, the wealth is assumed to be *equally distributed* between N agents with the amount $w_i(0) = 1$. The factor λ assumes a fixed value between 0 and 1 and stays constant during the simulations. At each step, k pairs of agents are randomly selected and the wealth exchange is taking

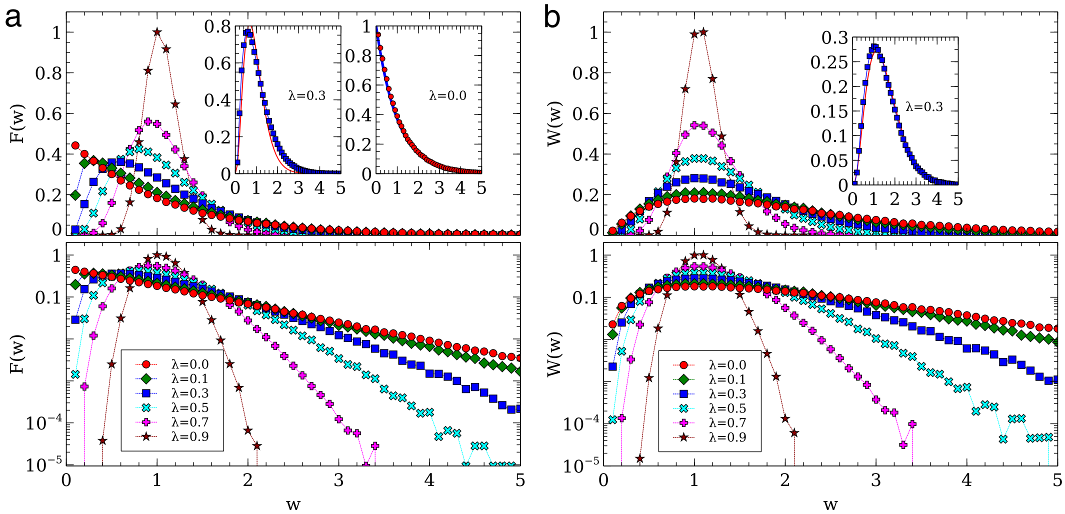

Fig. 1. Equilibrium distributions of (a) Person $F(w)$ and (b) Wealth $W(w)$ in the steady state, for different uniform saving propensities λ as indicated in the plot, computed after $T = 10^4$ time steps and shown in the linear–linear (top panels) and log-linear (bottom panels) scales. Here and below the number of traders is $N = 10^2$ and the initial money per agent is $w(t = 0) = 1$. The fit by the exponential asymptote $F(w) = (N/w_\Sigma)\exp[-w/(w_\Sigma/N)] = e^{-w}$ – expected [56] for the case $\lambda = 0$ with $N = 10^2$ agents and the total money $w_\Sigma = 100$ – is shown in the inset of panel (a). Also, the fits of the $F(w)$ and $W(w)$ simulation data by the normalized Gamma distribution $\Gamma(w) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} w^{\alpha-1} e^{-\frac{w}{\beta}}$ for $\lambda = 0.3$ are shown in the insets of top panels. The parameters for panel (a) are $\alpha = 3.31$, $\beta = 0.30$, while for panel (b) the best fit is achieved at $\alpha = 3.61$ and $\beta = 0.43$.

place between them. Each agent is involved in a single transaction per time step (multiple trading events can be made possible too).

The convergence of our simulations with respect to the initial distributions of money is relatively fast and other than uniform initial distributions (e.g. random) provide *the same* results for the steady-state person and wealth distributions. This robustness with respect to the initial conditions was also pointed out in Ref. [52]. Note that in Ref. [23] the results for $F(w)$ in a similar system were presented with additional *averaging* over different random initial distribution of money between the agents.

We run simulations until the distributions reach a *steady state*. We take *configurational averaging* over $R = 5000$ realizations of exchange amounts, in order to extract the averaged $F(w)$ and $W(w)$. In our analysis, the number of agents is $N = 10^2$ and the simulation time is $t \gtrsim 10^4$ steps. The steady state is reached for all sets of the model parameters.

We show the person $F(w)$ and wealth $W(w)$ distributions for different λ values in Fig. 1(a) and (b). We find that their general shapes are similar, but at large w the decay of $F(w)$ is faster. Apparently, the random-exchange model with a fixed saving propensity does *not* produce a Pareto-like distribution, in agreement with previous evidence [50,56]. As one can see from the log-linear plots of Fig. 1(a), (b), both $F(w)$ and $W(w)$ distributions decay exponentially as functions of w for small λ values, but for large λ they look more symmetric. Their shapes can be fitted with the Gamma distribution,

$$F(w) \simeq w^{-(1+\nu)} e^{-w/w_F}, \tag{11}$$

and

$$W(w) \simeq w^{-(1+\beta)} e^{-w/w_W}. \tag{12}$$

Here $\nu(\lambda)$ and $\beta(\lambda)$ are the respective scaling exponents, while $w_F(\lambda)$ and $w_W(\lambda)$ are the respective decay "lengths" for the tails. The distribution $F(w)$ was used previously [29,48–53,55–57] and conjectured to follow the Gamma distribution, while the wealth distribution $W(w)$ is introduced here. We find that the decay length of $F(w)$ becomes shorter with increasing λ, see Fig. 1(a). The shape of $F(w)$ after a given number of simulation steps is narrower for λ values closer to unity.

2.2. Distributed saving strategy

We now turn to the more interesting situation when the saving propensity λ is distributed [29,50,51]. Indeed, in a real economy the factor λ varies from person to person, depending on individual strategies and readiness to risk. We incorporate this feature via agent-dependent saving factors, λ_i. The trading process can thus be written for these conditions as [29,50,51]

$$\begin{aligned} w_i(t+1) &= \lambda_i w_i(t) + \varepsilon_{ij} \bar{K}_{ij}(t), \\ w_j(t+1) &= \lambda_j w_j(t) + (1-\varepsilon_{ij}) \bar{K}_{ij}(t), \end{aligned} \tag{13}$$

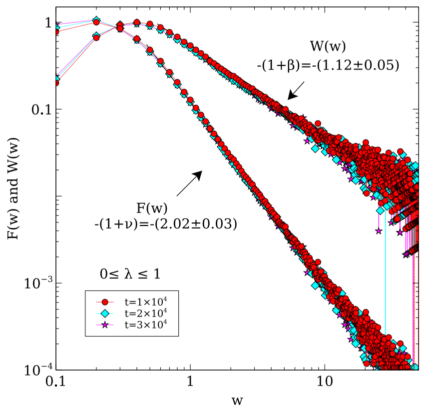

Fig. 2. Steady-state, equilibrium person $F(w)$ and wealth distributions $W(w)$ for random distributed saving propensities $0 \leq \lambda_i \leq 1$, as obtained from computer simulations of Eqs. (13), (14), (15) after $T = 10^4$, 2×10^4, and 3×10^4 time steps. The asymptotes of Eqs. (16) and (17) are shown.

with

$$\bar{K}_{ij}(t) = (1 - \lambda_i)w_i(t) + (1 - \lambda_j)w_j(t). \tag{14}$$

This operation conserves the wealth shared by agents *i* and *j*, while the exchanged amount of wealth amounts to [22,50]

$$\Delta w = \varepsilon_{ij}(1 - \lambda_j)w_j(t) - (1 - \lambda_i)(1 - \varepsilon_{ij})w_i(t). \tag{15}$$

The factors λ_i are distributed independently, randomly and uniformly in the interval $\{0, 1\}$. Thus, a particular trader *i* saves a random fraction λ_i of its wealth in a trading event. From the physical point of view, this case may be considered to have a *quenched disorder* of λ_i. Clearly, other than uniform distributions $p(\lambda)$ are also possible (results not shown).

As in Section 2.1, at each step the traders are pairwise randomly selected and the exchange among them takes place following the scheme (13). Employing Monte Carlo simulations, we compute the $F(w)$ and $W(w)$ distributions, see Fig. 2. In contrast to the uniform−λ scenario and Eq. (11), the person distribution now obeys the power-law (2),

$$F(w) \simeq w^{-(1+\nu)}, \tag{16}$$

with the Pareto exponent $\nu \approx 1.02 \pm 0.03$, similar to the conclusions of Refs. [29,50,51]. This Pareto law with $\nu \approx 1$ covers the tail of $F(w)$. The wealth distribution also decay as a power law (2),

$$W(w) \simeq w^{-(1+\beta)}, \tag{17}$$

but with $\beta \approx 0.12 \pm 0.05$. Both $F(w)$ and $W(w)$ follow their power-law decays starting from a small $w^\star$ value and the power law exponent of $W(w)$ is by about unity slower than that of $F(w)$. We remind the reader here that for a limited number of divisions of the total wealth interval, the numerically found $W(w)$ distributions can be approximated by the integral of the respective person distribution $F(w)$ in the range $\{w, w + dw\}$ as we indeed observe, so that $W(w)$ is the cumulative function of $F(w)$. We checked that the exponents of $F(w)$ and $W(w)$ are *robust* and remain unaltered also in longer simulations, see Fig. 2. Thus, the system already reached the stationary state after the chosen numbers of time steps T in simulations.

3. Open non-conservative system with traps

We now turn to open systems, where the total money (5) decreases with time and $F(w)$ and $W(w)$ cannot reach equilibrium, in contrast to the closed system in the previous Section. We show below, however, that the scaling for the tails of both distributions stays rather robust for different lengths of our simulations. In terms of physical gas-like models [43] the open trading system corresponds to dissipative, non-elastic collisions between the gas particles, representing money-exchanging economic agents. Note that the average wealth per agent can still be viewed as the effective temperature in this kinetic analogy [43,50].

We incorporate a decay of total money in the model via trap agents acting as *sink points*. They mimic e.g. non-secure transactions or possible unfair behavior of the agents. According to the rules, a regular agent trading with a trap agent loses the entire (non-saved) amount of wealth and gets nothing in exchange. Trap agents are randomly distributed on the lattice with the density $0 < \rho < 1$, and interact, for simplicity, only with nearest-neighbor agents on the lattice. Trap agents are not allowed to be nearest neighbors. This open system is inherently non-equilibrium and eventually the total wealth is expected to decay to zero, $w_\Sigma \to 0$, both for uniform and distributed saving propensities of regular agents in the system. We compute the distributions $F(w)$ and $W(w)$, as well as the time evolution of the total wealth $w_\Sigma(t)$ for varying trap densities ρ.

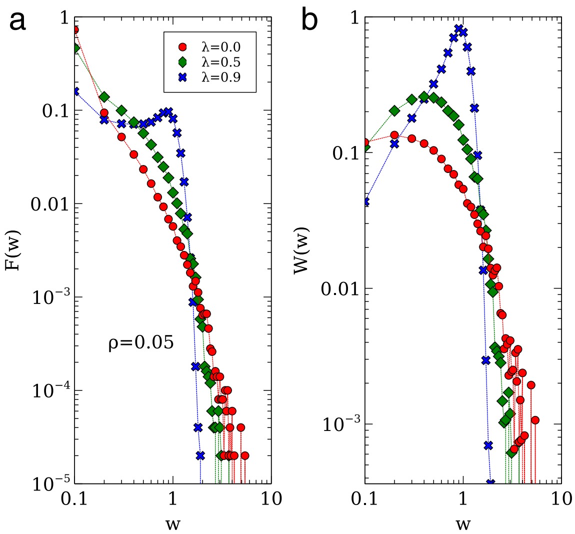

Fig. 3. Effect of trap agents on person $F(w)$ and wealth distribution $W(w)$, for the trap density of $\rho = 0.05$ and varying λ, computed after $T = 10^3$ steps and plotted in log–log scale for $N = 100$ and $w(t = 0) = 1$.

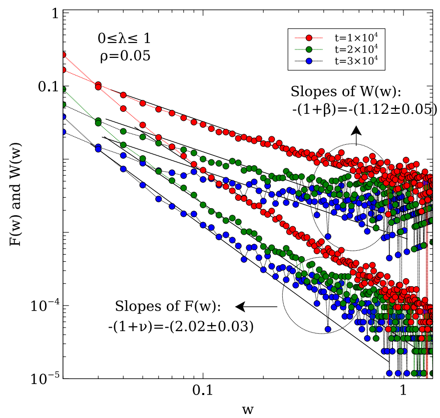

Fig. 4. Tails of wealth and money distributions in the presence of trap agents with density $\rho = 0.05$ and non-uniform saving propensities $0 \le \lambda_i \le 1$. The distributions are computed after $t = 10^4$, 2×10^4, and 3×10^4 time steps in simulations. Solid lines indicate the fitted power laws, with the exponent values provided in the plot. Other parameters are the same as in Fig. 3.

3.1. Uniform saving strategy

As before, the initial wealth of all agents is equally distributed, with trap agents having no money at $t = 0$. In the course of trading, trap agents receive random fractions of wealth from their neighboring agents. In Fig. 3 we show the distributions $F(w)$ and $W(w)$ for different λ values and the trap density $\rho = 0.05$. We find that for larger saving propensities λ the person and wealth distributions have a more pronounced peak at the most probable amount of wealth and a faster decay away from this value. For $\lambda \to 1$ and little trading activity in the system the peaks in Fig. 3 remain at the initial wealth amount $w \approx 1$, as they should. Similar to the closed system in Section 2.1, the wealth distribution does not follow the standard Pareto law. The existence of a distributed parameter λ is necessary for the Pareto law to emerge in this setup, see below. Note also that in the presence of trap agents and total wealth decreasing with time we do not normalize the distributions in the corresponding plots.

3.2. Non-uniform saving strategy

We now introduce distributed saving propensities into the model, so a distribution of λ_i values is used. Fig. 4 shows the person and wealth distributions for a fixed trap agent density, computed for varying simulation times T. We find that the power-law regimes of $F(w)$ and $W(w)$ distributions have rather constant exponents (independent of T), while the amplitudes of both distributions *decrease* due to the *leaking-out* of wealth from the system caused by the presence of traps. The scaling

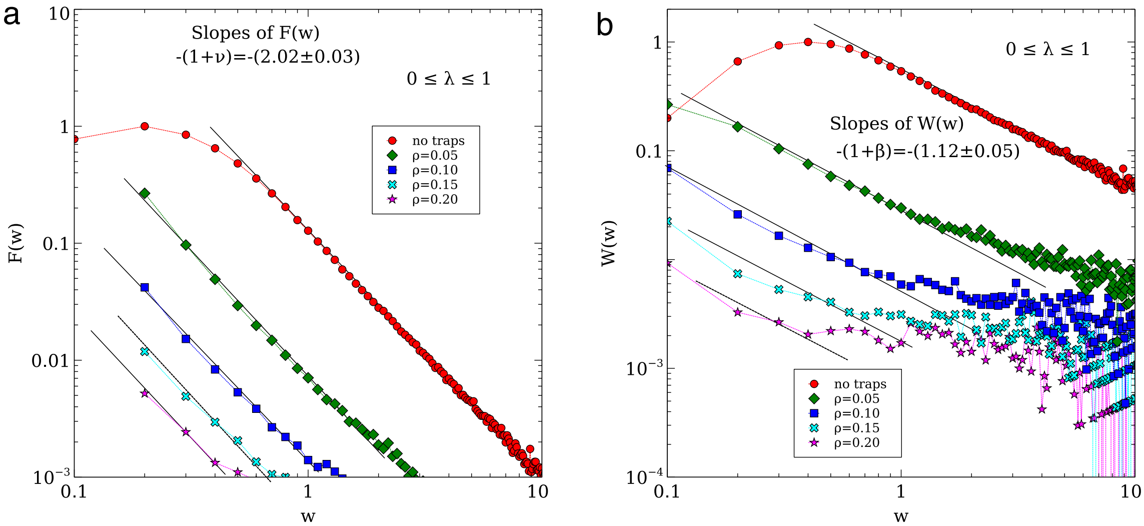

Fig. 5. Distributions of persons $F(w)$ and wealth $W(w)$ for distributed saving propensities λ_i and trap densities of $\rho = 0, 0.05, 0.1, 0.15$, and 0.20, computed after $T = 10^4$ steps. Solid lines are the power laws. Only distributions in the absence of traps are normalized.

exponents observed here agree with those in the absence of trap agents, compare Fig. 4 to Fig. 2. This indicates the *stability* of the underlying power-law Pareto scalings both for conservative and non-conservative systems, a rather important finding. Obviously, for the open system at very long times the statistics will deteriorate and eventually all wealth is drained out of the system.

To get further insight into the effect of trap agents, we plot in Fig. 5 the person and wealth distributions for different trap agent densities ρ and non-uniform λ values. We find that the slopes of $F(w)$ and $W(w)$ are almost equal to those in the closed trading system with randomly distributed λ_i values, Fig. 2. Note that in open systems the overall wealth decreases with time, and thus the relative fluctuations of $W(w)$ grow at later times, as we see for large w values in Fig. 5. Also, as expected, the amplitude of the $F(w)$ and $W(w)$ distributions decays with time, and after a given number of simulations steps T the probability magnitude is naturally smaller for the system with higher trap agent densities ρ, see Fig. 5.

3.3. Time evolution of the cumulative wealth

We now compute the total *normalized* wealth of the open system at the conditions of varying trap densities ρ, that at time t is given by

$$w_\Sigma(t) = \frac{1}{w_\Sigma(0)} \sum_{i=1}^{N} w_i(t). \tag{18}$$

Our simulations results are shown in Fig. 6 for the case of uniform saving propensities with $\lambda = 0.0, 0.5, 0.9$ and for distributed propensities, $0 \le \lambda \le 1$. The computations are done at fixed trap agent densities of $\rho = 0.05$ and 0.10. All curves are normalized to unity at $t = 0$ and the cumulative wealth decreases monotonically with time, as expected. When the trap agent density increases the decay of $w_\Sigma(t)$ gets faster, as it should. Also in accord with intuition we observe that as the saving propensity λ *increases* the wealth decays *slower* with time, compare the curves in different panels of Fig. 6. This is due to the fact that for increasing λ values in any given transaction a smaller amount of wealth is being traded.

In Fig. 7 all curves for $w_\Sigma(t)$ are plotted in terms of $\log[-\log w_\Sigma(t)]$ versus $\log(t)$. As on this representation the observed functional behavior is linear (to a very good approximation), we conclude that the time evolution of the total wealth obeys the stretched exponential relaxation law,

$$w_\Sigma(t) \simeq \exp\left[-(t/\tau)^\gamma\right]. \tag{19}$$

Here, the stretching exponent is in the interval $0 \le \gamma \le 1$ and τ is the relaxation time. The relation (19) is often referred to as the KWW law [70,71], widely used to describe the relaxation dynamics in glassy, polymeric or rubbery systems as well as protein folding dynamics, among others [72–74].

We show the time evolution of $w_\Sigma(t)$ for different, uniform saving propensities λ in Fig. 7(a), (b), (c) for systematically varying trap agent densities ρ. We observe that for different λ the slope of the $-\log[w_\Sigma(t)]$ curves is *independent* on the trap agent density, compare the γ values given in the figure caption. In the long-time limit only the magnitude of $w_\Sigma(t)$ is affected by the ρ values. This is the most important result of this Section. In Fig. 7(d), for non-uniform saving propensities

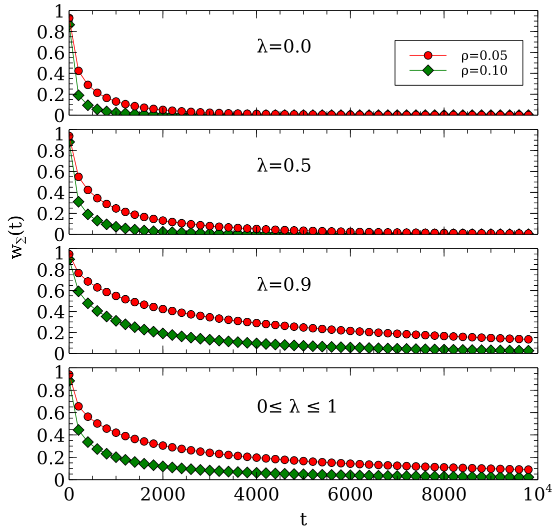

Fig. 6. Time evolution of total wealth $w_\Sigma(t)$ normalized to unity at $t = 0$ as given by Eq. (18) for trap agent densities of $\rho = 0.05$ and 0.1, computed for different uniform and distributed saving propensities λ, with the values as indicated in the plots.

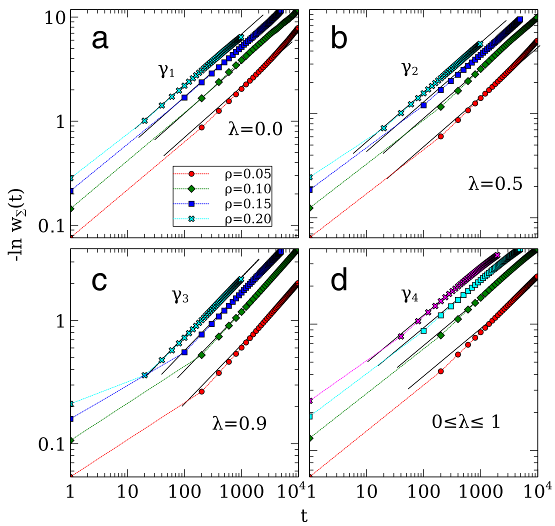

Fig. 7. Stretched exponential KWW-like decay of $-\log[w_\Sigma(t)]$ versus time shown in a log–log scale, for the trap agent densities of $\rho = 0.05$, 0.1, 0.15, 0.20, for uniform (a) $\lambda = 0.0$, (b) 0.5, (c) 0.9, and (d) For non-uniform $0 \le \lambda \le 1$. The plots are after $T = 10^4$ steps. Solid lines are long-time power law fits, with the exponents $\gamma_1 = 0.483$, $\gamma_2 = 0.492$, $\gamma_3 = 0.518$, and $\gamma_4 = 0.402$.

the exponent is $\gamma_4 = 0.402$. We note that the slopes of the function $-\log[w_\Sigma(t)]$ versus t in the log–log scale have a slightly increasing trend for increasing uniform λ, compare the values of $\gamma_{1,2,3}$ in Fig. 7. For non-uniform saving propensities, the stretching exponent γ_4 is measurably smaller than for any of the three values of λ in the uniform case, see Fig. 7.

In general, the dependence of the stretching exponent on the trap agent density ρ is rather weak, while the relaxation time τ in Eq. (19) clearly depends on ρ, as intuitively expected. For larger densities of trap agents the approach to the asymptote (19) is faster. Also, because of a non-conservative nature of this agent-based system, with increasing density of traps ρ the total wealth $w_\Sigma(t)$ at a given time gets smaller, and thus the magnitude of $-\log[w_\Sigma(t)]$ grows as a function of trap density, compare the curves for different ρ values in Fig. 7. This trend is similar to that presented in Fig. 5(b). Note, finally, that for progressively smaller ρ values the decay of the total money gets delayed to later times in simulations, so that for $\rho \to 0$ situations during the length of simulations we used the effective stretching exponent can be considered $\gamma \to 0$.

In Fig. 8 we directly compare the cumulative wealth decay for a given trap density both for uniform and distributed saving propensities λ of the agents. This plot demonstrates the significantly slower decay dynamics with a smaller stretching exponent γ for non-uniform λ. Moreover, the decay time τ clearly depends on λ, as expected. The inset of Fig. 8 illustrates the universality of the stretching exponent in the system with for distributed λ values, as computed for varying simulations

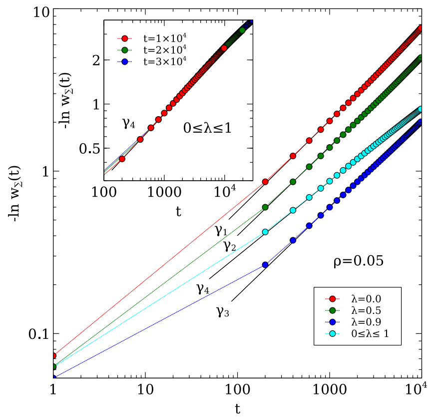

Fig. 8. Stretched exponential decay of $-\log[w_\Sigma(t)]$ at $\rho = 0.05$, plotted for different saving propensities λ. Data points are provided each 200 time steps. In the inset, the cumulative wealth for distributed λ is shown after $T = 10^4$, 2×10^4, and 3×10^4 time steps. Solid lines are the power-law fits.

times, T. The re-scaling of all the curves indicates a *constant* stretching exponent γ independent on the length of simulations: the steady state is reached.

4. Conclusions

We generalized the ideal-gas random-exchange trading model via introducing a distinction between the person $F(w)$ and wealth $W(w)$ distributions. We demonstrated by Monte Carlo simulations that for uniform saving propensity of the agents a fast decay of both distributions occurs, while for non-uniform λ a Pareto law for $F(w)$ with the exponent $\nu = 1.02 \pm 0.03$ in Eq. (16) emerges, in line with Refs. [29,50,51]. Importantly, we showed that for both uniform and non-uniform saving propensities the distributions $F(w)$ and $W(w)$ behave as power laws but differently. In particular, for the case of distributed saving propensities for $W(w)$ the scaling exponent $\beta = 0.12 \pm 0.05$ in Eq. (17) was obtained, i.e. by about one smaller than the exponent ν of the person distribution in Eq. (16). This fact is consistent with the notion that in the limit of substantial windows dw the distribution $W(w)$ can be regarded as the integral of $F(w)$ distribution in the range of wealth $\{w, w + dw\}$. We believe that depending on the questions asked for the financial system under scrutiny the distribution $W(w)$ may be more amenable to interpretation. Moreover, for poorer data the cumulative character of $W(w)$ may return less noisy shapes of the distribution.

We then introduced open systems, in which trap agents at a given density ρ act as wealth sinks. Interestingly, our analysis demonstrated that the presence of trap agents only affects the amplitudes of the distributions $F(w)$ and $W(w)$, while the above mentioned scaling exponents ν and β of the power-law regions remain unaffected by the value of ρ.

A remarkable result was obtained for the time evolution of the cumulative wealth $w_\Sigma(t)$ in non-conservative systems with trap agents present at different densities ρ. Specifically, we found that the wealth decreases with time according to the KWW stretched exponential law. The decay rate was found to depend on ρ, as expected. However, the stretching exponent γ of the KWW law (19) turns out to be *independent* of ρ. Conversely, the value of γ was found to be a slightly increasing function of λ for the case of uniform saving propensities λ. For non-uniform λ the stretching exponent $\gamma = \gamma_4$ assumes even smaller value.

We observed that the wealth distribution is independent of the saving propensity and trap agents density in the open trading system. Therefore, no special optimal *saving strategy* follows from our current analysis. However, in the current model the wealth inequality in the spirit of Pareto's law originates from the *randomness* of the saving propensity, for both closed and open trading systems. In this sense, the saving propensity can be viewed as a control parameter of inequality in a society.

Note that the general trend in all industrialized countries is an overall *increase* of wealth and money for most market participants, instead of *decrease* of wealth as in our model with trap agents. This increase can be modeled via pumping a given percentage of total money *into the system* at each trading step. This corresponds to a non-zero *inflation rate* in the system, the inherent feature and the vital control parameter of economic growth and wealth distribution in capital-based societies. The law for the wealth evolution (19) in this case will depend on the amount of money growth due to pumping from external sources versus money decrease because of annihilation of wealth by the trap agents in each exchange event. Also, a modified agent-based model in which one agent gets finally all the money seems appropriate too: this scenario would mimic such economic players as global insurance companies, pension funds, and leading banks.

We finish by noting that the observed KWW relaxation dynamics will deserve closer attention in the future. In particular, it is of interest to investigate the connections of the agent-based models with anomalous diffusion dynamics [75], following the observation that the Pareto law is indeed linked to anomalous diffusion [76].

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