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Ergodicity breaking and particle spreading in noisy heterogeneous diffusion processes

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We study noisy heterogeneous diffusion processes with a position dependent diffusivity of the form $D(x) \sim D_0|x|^{\alpha_0}$ in the presence of annealed and quenched disorder of the environment, corresponding to an effective variation of the exponent α in time and space. In the case of annealed disorder, for which effectively $\alpha_0 = \alpha_0(t)$, we show how the long time scaling of the ensemble mean squared displacement (MSD) and the amplitude variation of individual realizations of the time averaged MSD are affected by the disorder strength. For the case of quenched disorder, the long time behavior becomes effectively Brownian after a number of jumps between the domains of a stratified medium. In the latter situation, the averages are taken over both an ensemble of particles and different realizations of the disorder. As physical observables, we analyze in detail the ensemble and time averaged MSDs, the ergodicity breaking parameter, and higher order moments of the time averages. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4917077>]

I. INTRODUCTION

The motion of individual molecules and submicron tracer particles of different sizes in the cytoplasm of living biological cells,¹ in artificially crowded environments *in vitro*,² in glass-like systems,³ or in large scale *in silico* studies of membrane structures⁴ was shown to follow the anomalous diffusion law

$$\langle x^2(t) \rangle \simeq t^\beta, \quad (1)$$

with the subdiffusive diffusion exponent mostly in the range $\beta = 0.4 \dots 0.9$.^{5,6} A number of mathematical models of different kinds were proposed to unveil the properties of anomalous diffusion phenomena embodied in the mean squared displacement (MSD) in Eq. (1).⁷ In most of these models, the properties of the stochastic process are homogeneous in space. Especially for smaller tracers—which may cover longer distances within the measurement time—or for techniques allowing for full maps of local diffusivities, it turns out that the diffusion coefficient becomes a function of the local tracer position. For both eukaryotic⁸ and prokaryotic⁹ cells, such local diffusivity maps indeed show significant variations. The motion of tracer particles through space may also be impeded by caging effects when the size of the particle is comparable to the local mesh size in structured environments.^{10,11} In such cases, the tracer diffusion becomes characterized by a non-uniform, position-dependent diffusivity $D(x)$. Similarly, spatially varying transport characteristics are ubiquitous in contaminant dispersion in subsurface water aquifers.¹²

In the field of stochastic dynamics, anomalous diffusion in spatially random media, disordered energy landscapes, weakly chaotic systems, and dynamic maps received considerable attention.^{13–20} More specifically, anomalous diffusion due to micro-domains was investigated,²¹ and the influence of

environmental Gaussian noise on diffusive particle trajectories in disordered systems was studied.²² Moreover, deviations from normal diffusion due to quenched and annealed disorder of the medium's diffusivity received renewed interest.^{23,24} In such studies, one is mainly interested in the quantitative behavior of particle MSD (1) as well as the ergodic properties of the system: is the information from time averages of physical observables typically garnered as time series by modern particle tracking assays equivalent to those of the corresponding ensemble averages known from the theoretical models? It turns out that a large variety of anomalous diffusion processes involve weak ergodicity breaking,^{7,25–29} the disparity between (long) time averages, and ensemble averages of physical observables such as the MSD, and that in those cases, the Khintchine theorem needs to be substituted by generalized versions.^{30,31}

Here, we study the dynamics and the ergodic properties of heterogeneous diffusion processes (HDPs) with position dependent diffusivity $D(x)$, in the presence of piece-wise deterministic quenched and annealed disorder. More specifically, we generalize the standard HDPs with power-law diffusivity

$$D(x) = D_0|x|^{\alpha_0}, \quad (2)$$

for which the anomalous diffusion exponent of the MSD assumes the form^{32–36}

$$\beta = \frac{2}{2 - \alpha_0}. \quad (3)$$

The physical dimension of the coefficient D_0 in Eq. (2) is $[D_0] = \text{cm}^{2-\alpha_0}\text{s}^{-1}$. The exponent (3) designates subdiffusion for $\alpha_0 < 0$ and superdiffusion for $0 < \alpha_0$.^{32–36} The profiles of the diffusivity for these cases are shown in Figs. 1(a) and 1(b). HDPs are weakly non-ergodic and ageing, that is, their dynamics depends explicitly on the time gap between original initiation of the system and start of the measurement.^{32–36} We

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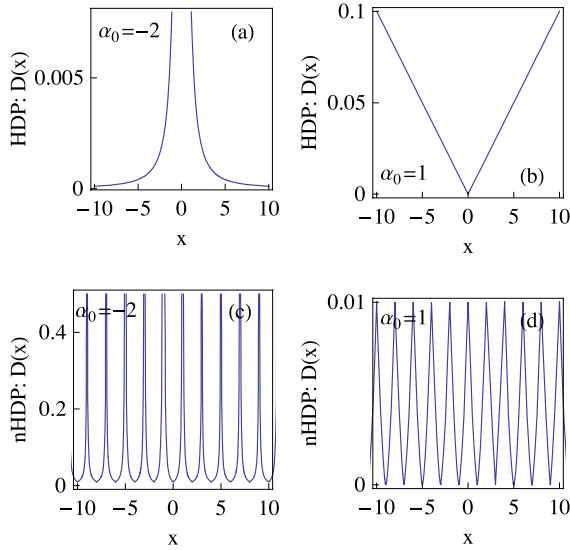


FIG. 1. Particle diffusivity for heterogeneous diffusion processes: (a) and (b) for, respectively, $\alpha_0 = -2$ and $\alpha_0 = 1$. Panels (c) and (d) show the diffusivity of HDPs with quenched disorder for the same values of α_0 and for the parameters $\sigma^2 = 0.25$ and $2\delta x = 2$ (see text for details). Slight variations of the diffusivity in panels (c) and (d) indicate the external noise superimposed in the process.

note that the ageing properties of HDPs³⁵ embodied in the ensemble and time averaged MSDs are in fact similar to those of subdiffusive continuous time random walks³⁷ and scaled Brownian motion.³⁸

In the following, we unravel how the additional disorder in the system modifies the diffusive and ergodic properties of HDPs. In particular, we examine in detail how the anomalous laws of HDPs change when (i) the HDP scaling exponent α_0 itself becomes a random variable and (ii) the domains of HDPs are periodically distributed in space throughout the system. The first scenario then corresponds to an annealed disorder, while in the second situation, the disorder is quenched and thus hard-wired into the structure of the diffusion medium. Specifically, we compute the scaling laws for the ensemble and time averaged MSDs as well as the amplitude spread of individual realizations of the process. The analysis shows that the annealed and quenched disorders introduce significant differences in the behavior of HDPs with power-law diffusivity (2). The article is structured as follows. In Sec. II, we define the observables that we will analyze. Section III specifies the model and its implementation in the simulations. In Sec. IV, we then study HDPs with annealed disorder, followed by the scenario with quenched disorder in Sec. V. Section VI concludes this work.

II. OBSERVABLES

The central quantity in the study of stochastic processes is the ensemble averaged MSD,

$$\langle x^2(t) \rangle = \int_{-\infty}^{\infty} x^2 P(x,t) dx, \quad (4)$$

calculated as the spatial average of x^2 over the probability density function $P(x,t)$ to find the particle at position x at

given time t .³⁹ However, when individual time series $x(t)$ of the particle position are measured in experiments or simulations, the typical quantity studied is the time averaged MSD,

$$\overline{\delta^2(\Delta)} = \frac{1}{T-\Delta} \int_0^{T-\Delta} [x(t+\Delta) - x(t)]^2 dt. \quad (5)$$

Here, Δ is the lag time and T is the measurement time (length) of the trajectory $x(t)$.^{7,27,28} Often, also the additional average

$$\langle \overline{\delta^2(\Delta)} \rangle = \frac{1}{N} \sum_{i=1}^N \overline{\delta_i^2(\Delta)} \quad (6)$$

of the time averaged MSD over N individual trajectories is taken.^{7,27,28} A process is called ergodic when we observe the equality

$$\langle x^2(\Delta) \rangle = \lim_{\Delta/T \rightarrow 0} \overline{\delta^2(\Delta)}. \quad (7)$$

Examples for ergodic processes are Brownian motion^{7,27-29} as well as anomalous diffusion processes with MSD (1) given by random walks on fractals⁴⁰ and processes driven by fractional Gaussian noise.⁴¹⁻⁴³ Once a process is non-stationary, equality (7) is violated, the phenomenon of weak ergodicity breaking.^{7,25-29} A whole range of anomalous diffusion processes with power-law MSD (1) belongs to this class and specifically exhibits the linear lag time dependence

$$\langle \overline{\delta^2(\Delta)} \rangle \simeq \frac{\Delta}{T^{1-\beta}} \quad (8)$$

of the time averaged MSD.⁷ As examples, we mention continuous time random walk processes with scale free distributions of waiting times,^{7,25-29,31} correlated continuous time random walks,⁴⁴ as well as diffusion processes with space³²⁻³⁶ and time^{32,38,45,46} dependent diffusion coefficients and their combinations.⁴⁷ We also mention ultraslow diffusion processes with a logarithmic form for $\langle x^2(t) \rangle$ and linear lag time dependence (8) of the time averaged MSD⁴⁸ as well as the ultraweak ergodicity breaking of superdiffusive Lévy walks.⁴⁹

For finite measurement time, even ergodic processes exhibit a statistical scatter of the amplitude of time averaged observables. This irreproducibility for the case of the time averaged MSD $\overline{\delta^2(\Delta)}$ can be quantified in terms of the distribution $\phi(\xi)$ as function of the dimensionless variable^{7,26-28}

$$\xi = \frac{\overline{\delta^2}}{\langle \overline{\delta^2} \rangle}. \quad (9)$$

The variance of $\phi(\xi)$ is measured in terms of the ergodicity breaking parameter^{7,26-28}

$$\text{EB}(\Delta) = \langle \xi^2(\Delta) \rangle - \langle \xi(\Delta) \rangle^2 \simeq \langle \xi^2 \rangle - 1. \quad (10)$$

For Brownian motion, the ergodicity breaking parameter at $\Delta/T \rightarrow 0$ has the form

$$\text{EB}_{\text{BM}}(\Delta) = \frac{4\Delta}{3T}. \quad (11)$$

Continuous time random walk processes with scale free waiting time distribution have a finite value for ergodicity breaking EB parameter even in the limit $\Delta/T = 0$,²⁶ similar to HDPs,³³⁻³⁶ while for scaled Brownian motion, the ergodicity breaking parameter approaches zero in this limit.⁴⁵⁻⁴⁷

For reference in what follows, we also mention that the probability density function of HDPs has the exponential form³³

$$P(x, t) = \frac{|x|^{-2/\alpha_0}}{\sqrt{4\pi D_0 t}} \exp\left(-\frac{|x|^{2-\alpha_0}}{(2-\alpha_0)^2 D_0 t}\right), \quad (12)$$

which is a stretched (compressed) Gaussian for superdiffusive (subdiffusive) HDPs with $0 < \alpha_0 < 2$ ($\alpha_0 < 0$). Note that, respectively, the shape (12) has a distinct cusp at the origin or is bimodal with $P(0, t) = 0$.³³

III. MODEL

We employ the same tested stochastic algorithm for the Markovian HDPs as developed in Refs. 33–36, based on the one-dimensional Langevin equation for the particle displacement $x(t)$ with the position dependent diffusivity $D(x)$,

$$\frac{dx(t)}{dt} = \sqrt{2D(x)} \times \zeta(t). \quad (13)$$

The process is driven by the white Gaussian noise $\zeta(t)$ with covariance $\langle \zeta(t)\zeta(t') \rangle = \delta(t-t')$ and zero mean $\langle \zeta(t) \rangle = 0$. We interpret Eq. (13) in the Stratonovich sense leading to the following implicit mid-point iterative scheme: at step $i+1$, the particle position is

$$x_{i+1} - x_i = \sqrt{2D\left(\frac{x_{i+1} + x_i}{2}\right)} \times (y_{i+1} - y_i), \quad (14)$$

where the increments $(y_{i+1} - y_i)$ of the Wiener process represent a δ -correlated Gaussian noise with unit variance and zero mean. Unit time intervals separate consecutive iteration steps. Below, we simulate three values for the exponent α_0 , corresponding to $\beta = 1/2$ (subdiffusive MSD), $\beta = 0$ (Brownian motion), and $\beta = 2$ (superdiffusive MSD). For standard HDPs, these cases were analyzed by us in Refs. 33–36. To avoid divergencies of the particle motion, we regularize the diffusivity at $x = 0$ by addition of a small constant, namely, $D(x) = D_0(|x|^\alpha + D_{\text{off}})$ where $D_{\text{off}} = 10^{-3}$ and $D_0 = 10^{-2}$ for all results shown below. This choice does not affect the studied scaling laws.³³

We consider two versions of this additional disorder corresponding to the annealed and quenched limits for the variation of α . To examine the effect of additional noise due to the environment, we implement a Gaussian distribution of the scaling exponent of the diffusivity with the mean α_0 ,

$$p(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\alpha - \alpha_0)^2}{2\sigma^2}\right). \quad (15)$$

Generally, the distribution $p(\alpha)$ may be asymmetric, but we restrict our discussion to symmetric forms. In the *annealed case* of noisy HDPs, the properties of the environment change rapidly in time compared to time scales of the particle motion. Physically, such noise may be due to the imprecision of the experimental setup or because of additional thermal agitation in the system. The diffusing particle thus visits regions in space with different local exponents α . In such a scheme, the particle diffusivity at position x fluctuates in time, and the value of the diffusivity will be different each time the particle revisits

the same position x . In this annealed case, large diffusivity variations occur in the entire space.

In superdiffusive HDPs, distant particle excursions take place due to the growth of $D(x)$ away from the origin and the associated acceleration of the motion, while for subdiffusive HDPs the walker is increasingly trapped in the low-diffusivity regions at larger values of the position $|x|$.^{33–36} With increasing strength σ^2 of the annealed noise given by distribution (15), the excursions of the particles in both superdiffusive and subdiffusive cases become more erratic as time evolves. The time interval δt during which the walker has a given HDP exponent α_i obviously affects the properties of noisy HDPs. These time spans δt are here taken to be uniformly distributed. To simulate annealed noisy HDPs, we use Eq. (15) with varying σ^2 . The particle performs jumps with a given scaling exponent for the time interval δt , after which a new exponent is chosen from distribution (15), and so on. The particle displacement x_i during the time span $(t_i, t_i + \delta t)$ with HDP exponent α_i is the starting condition for the next time interval. Shorter δt intervals imply more erratic motion and stronger external noise, as shown below.

For noisy HDPs in the presence of *quenched disorder*, the profile of the particle diffusivity is hard-wired into the environment. We choose a static periodic arrangement of domains as shown in Figs. 1(c) and 1(d). In each domain, the exponent α is drawn from $p(\alpha)$, and the particle performs a regular HDP. The midpoint of each domain is chosen as the origin in the local HDP coordinate system, that is, locally the functional shape of $D(x)$ is centered and decays or increases with the local scaling exponent α , as exemplified in Figs. 1(c) and 1(d). The period δx for the stratification of the environment plays the role of a switching mechanism affecting the system dynamics. At the boundary of the domains, the diffusivity and its derivative in general acquire jumps. Physically, the latter occurs in the presence of some walls, cages, etc.

We simulate quenched noisy HDPs as follows. The entire space is stratified into domains of width $2\delta x$, and the local HDP exponent is chosen from distribution (15). The length δx is a vital parameter of quenched noisy HDPs. The particle performs a HDP random walk in each space domain with $D(x, \alpha)$ and it hops to a neighboring domain once the domain boundary is reached. The centers of the domains are computed from the particle position x_i as

$$x_{c,i} = 2(\delta x) \text{int}\left[\frac{x_i}{2\delta x}\right] + \text{sign}[x_i](\delta x), \quad (16)$$

see Figs. 1(c) and 1(d). Here, $\text{int}[x]$ denotes the integer part of the argument, and an additional δx shift is used for convenience. The starting position of the particle is near the center of the first domain, at $x(0) = 0.1 + x_{c,1}$. The subsequent position x_{i+1} is evaluated from x_i with the local exponent α_i according to Eq. (14), that is,

$$x_{i+1} - x_i = \sqrt{2D_0\left(\left|\frac{x_{i+1} + x_i}{2} - x_{c,i}\right|^{\alpha_i} + D_{\text{off}}\right)}(y_{i+1} - y_i). \quad (17)$$

We vary the width of $p(\alpha)$ and the mean value of the scaling exponent α_0 . Shorter periodicities δx are equivalent to stronger external noise, as shown below. We note here that for

subdiffusive HDPs, the centers of the domains $x_{c,i}$ correspond to the regions of maximal diffusivity, while for superdiffusive HDPs, these are the spots of the lowest diffusivity.^{33,34}

IV. NOISY HDPS WITH ANNEALED DISORDER

A. Noisy Brownian motion, $\alpha_0 = 0$

For $\alpha_0 = 0$ and a small value σ^2 of the additional noise, as expected, we observe small discrepancies from the canonical Brownian motion, as evidenced in Fig. 2(b). The behavior is ergodic, and the ergodicity breaking parameter follows known behavior (11) for Brownian motion, see Fig. 3. Most importantly, the ensemble averaged MSD equals the time averaged MSD, apart from very short lag times at which the relaxation from the initial value $x(0) = x_0$ occurs (compare Ref. 35 for more details). At longer lag times, the deteriorating statistics of the δ^2 traces give rise to the typical cone-like scatter.

As the noise strength σ^2 increases, the variance of the α distribution $p(\alpha)$ is increased, see, for instance, in the panel

for the noise strength $\sigma = 0.5$ in Fig. 2(b); a more pronounced scatter of the δ^2 traces emerges and, importantly, persists in the limit $\Delta/T \rightarrow 0$. The occurrence of progressively more distant particle excursions caused by superdiffusive traces with $\alpha > 0$ gives rise to a larger spread of the amplitude scatter quantified by the distribution $\phi(\xi)$. The value of $\langle \delta^2 \rangle$ grows somewhat faster than the ensemble MSD (1) due to these outliers, giving rise to larger values of the ergodicity breaking parameter EB (not shown). The time averaged MSD $\overline{\delta^2}(\Delta)$ scales linearly with the lag time Δ , and, as it should, in the limit $\Delta \rightarrow T$, the time averaged MSD settles back to the ensemble averaged MSD, due to the pole in definition (5) of the time average.

For even larger noise strength σ^2 , the behavior of the time averaged MSD and the ergodic properties are dominated by extreme events: that is, by single or few trajectories in the data set with the largest exponent(s) yielding extremely distant particle excursions. With an increasing width of the α distribution $p(\alpha)$, the spread of the time averaged MSD grows, as well, as evidenced in Fig. 2(b). Similarly, for such large values of the noise strength σ^2 , the value of the ergodicity

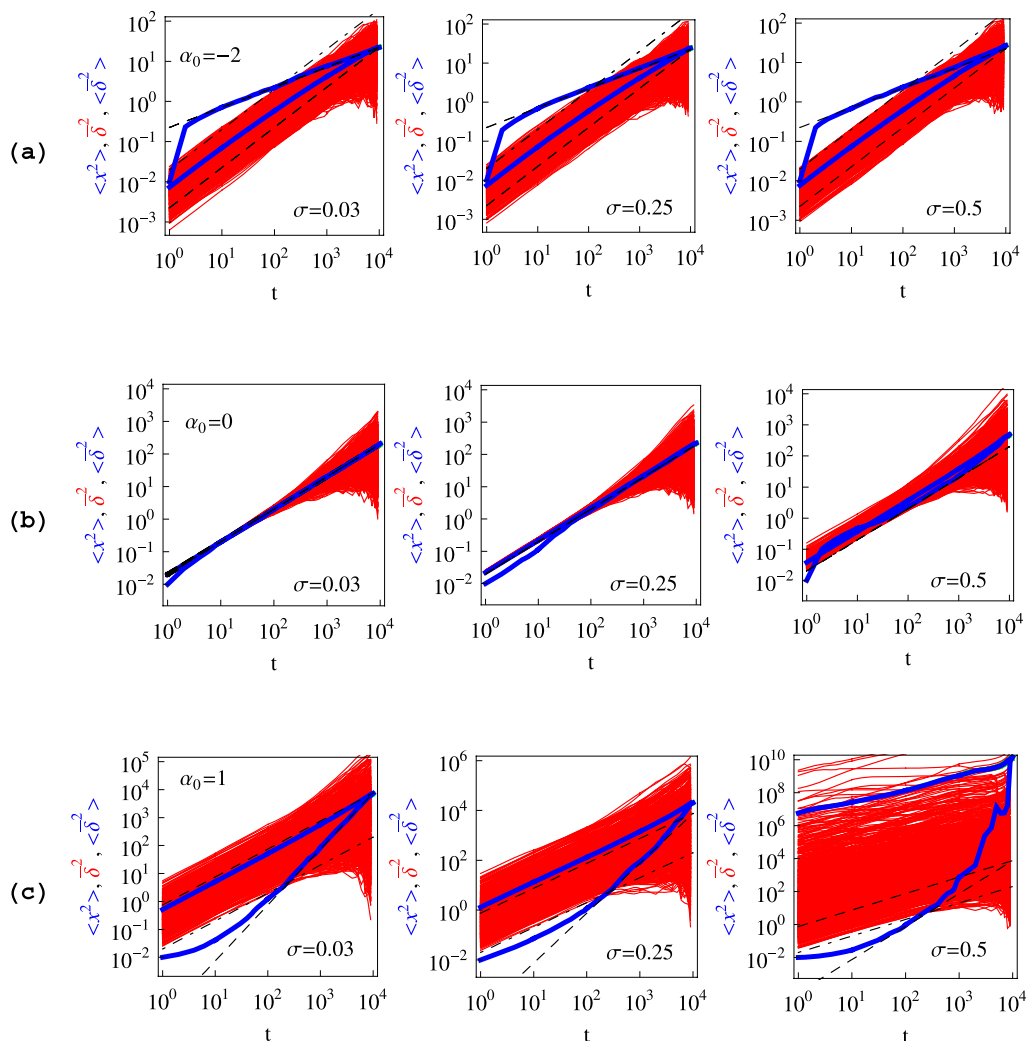


FIG. 2. Ensemble and time averaged MSDs $\langle x^2(t) \rangle$ and $\overline{\delta^2}(\Delta)$ (thick blue curves) as well as individual time averaged MSDs δ^2 (red curves) for annealed noisy HDPs. Parameters: the values of α_0 and its variance are indicated in the plots, the trace length is $T = 10^4$, and the number of sampled traces is $N = 10^3$. The initial position is $x_0 = x(t=0) = 0.1$. The top panels correspond to the noisy subdiffusive case, the middle panel represents noisy Brownian motion, and the bottom panels are the case of superdiffusive noisy HDPs. The asymptotes (1) and (8) for the ensemble and time averaged MSDs of standard HDPs are shown as the dashed curves. The Brownian asymptote $\langle x^2(t) \rangle = 2D_0t$ is the dashed-dotted line.

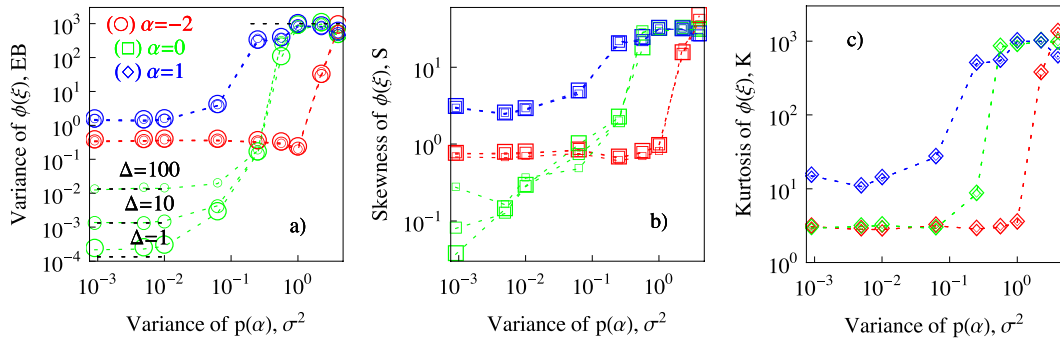


FIG. 3. Second, third, and fourth order moments of the amplitude scatter distribution $\phi(\xi)$ for annealed noisy HDPs, computed for the parameters of Fig. 2. Large, medium, and small symbols correspond to lag times $\Delta = 1, 10,$ and 100 , respectively. The dotted line in panel (a) at large σ^2 is $EB \approx N = 10^3$, indicative of the single trace dominance in this case, see text. The horizontal black dashed lines in panel (a) for small noise strength σ^2 stand for $EB_{BM}(\Delta)$ given by Eq. (11).

breaking parameter becomes proportional to the number N of recorded traces, witnessing the dominance of single traces, each having the potential to be more extreme than the others, compare Fig. 3(a). We refer the reader to Ref. 35, in which the critical properties of HDPs and the effects of the number of traces are analyzed in the limit $\alpha_0 \rightarrow 2$.

For narrow distributions $p(\alpha)$, the spread $\phi(\xi)$ of individual δ^2 traces is symmetric at short lag times Δ , developing a tail at longer lag times Δ . This behavior can be rationalized in terms of a generalized Gamma distribution (see Ref. 33). The general features of $\phi(\xi)$ are shown in Fig. 3 in terms of the higher moments of this distribution. These are the skewness

$$S(\xi) = \frac{N^{-1} \sum_{i=1}^N (\xi - 1)^3}{(N^{-1} \sum_{i=1}^N (\xi - 1)^2)^{3/2}} \quad (18)$$

and the kurtosis

$$K(\xi) = \frac{N^{-1} \sum_{i=1}^N (\xi - 1)^4}{(N^{-1} \sum_{i=1}^N (\xi - 1)^2)^2}, \quad (19)$$

which complement the variance of $\phi(\xi)$ described by ergodicity breaking parameter (10). In Fig. 3(a), we also observe that for small noise strengths σ^2 , the value $EB(\Delta = 1)$ for noisy Brownian motion approaches $EB_{BM}(\Delta = 1)$ given by Eq. (11), as expected. The values of the ergodicity breaking parameter grow with Δ , indicative of a bigger spread of δ^2 of individual traces (green points in Fig. 3(a)).

B. Subdiffusive noisy HDP, $\alpha_0 = -2$

For the subdiffusive case, the time evolution of the ensemble and time averaged MSDs is illustrated in Fig. 2(a) for different noise strengths σ^2 of the α distribution. We observe that for the subdiffusive value $\alpha_0 = -2$, the same magnitude of the α variation causes a much weaker effect as compared to the Brownian ($\alpha_0 = 0$) or superdiffusive ($\alpha_0 = 1$) situations. The scatter of δ^2 remains nearly insensitive to the lag time Δ , similar to canonical HDPs.^{33,35} The scaling of the ensemble averaged MSD also agrees with that for HDPs.³³ It is reached after less than a dozen of steps during which the relaxation of the initial condition occurs, compare Refs. 33 and 35. The scaling of the time averaged MSD $\langle \delta^2 \rangle$ remains linear and nearly unaffected by changes of σ^2 . The long time scaling of

the MSD is also weakly sensitive to σ^2 in the range considered here. Physically, for the subdiffusive case, the spread of α_i should be $\gtrsim \alpha_0$ to give rise to fast particle excursions (outliers). Thus, much larger σ^2 values are required to disturb the spread of δ^2 for strongly subdiffusive noisy HDPs as compared to superdiffusive noisy HDPs shown in Fig. 2(c). This is our first important conclusion.

We rationalize the effects of the α spread further in terms of the width and higher moments of the amplitude scatter distribution $\phi(\xi)$. The results for $\alpha_0 = 0$, sub-, and superdiffusive annealed noisy HDPs are shown in Fig. 3. We observe that all moments are typically smaller for the subdiffusive case, reflecting a less pronounced and asymmetric spread of δ^2 . The skewness of Brownian motion ($\sigma^2 \rightarrow 0$) tends to small values, as it should, while for sub- and superdiffusive noisy HDPs, it attains finite values at $\sigma^2 \rightarrow 0$ (Fig. 3(b)). This is due to the inherent asymmetry of the $\phi(\xi)$ scatter even at $\sigma^2 \rightarrow 0$: it features a tail at large ξ values, a maximum at intermediate ξ , and vanishes at $\xi \rightarrow 0$.³³ Both skewness $S(\xi)$ and kurtosis $K(\xi)$ grow dramatically with σ^2 for all values of α_0 , as demonstrated in Figs. 3(b) and 3(c). We checked that for $\sigma^2 \rightarrow 0$, the value of the ergodicity breaking parameter in the limit $\Delta/T \ll 1$ approaches that for standard HDPs,³³ as expected, while for a broad distribution of α values, the ergodicity breaking parameter increases and eventually approaches the number of traces N in the data set (single-trace domination), Fig. 3(a). The value of the ergodicity breaking parameter EB for $\alpha_0 < 0$ is nearly unaffected by variations of σ^2 over a wide range, see the red symbols in Fig. 3(a). This reflects the minor change in the spread of single traces δ^2 when σ^2 is varied for strongly subdiffusive noisy HDPs, see Fig. 2(a).

C. Superdiffusive noisy HDPs, $\alpha_0 = 1$

The ensemble and time averaged MSDs of superdiffusive noisy HDPs with $\alpha_0 = 1$ are shown in Fig. 2(c). For small noise strengths σ^2 , their scaling agrees with the results for standard HDPs, Eqs. (1) and (8). With increasing noise strength σ^2 , the time averaged MSD traces δ^2 grow dramatically, and for moderate and large lag times Δ , the time averaged MSD deviates progressively from the HDP scaling that is ballistic for $\alpha_0 = 1$ (Fig. 2(c)). The scatter of the individual time averaged MSDs δ^2 becomes progressively larger and asymmetric as the

width of $p(\alpha)$ increases. The amplitude of the time averaged MSD traces $\overline{\delta^2}$ for large values of σ^2 grows significantly above the asymptote for undisturbed HDPs due to single trajectory domination. Therefore, the moments of the scatter distribution $\phi(\xi)$ increase, see the blue symbols in Fig. 3. For large σ^2 , in the later parts of the trajectories the ensemble averaged MSD increases very fast (see the right panel in Fig. 2(c)) to meet the value of $\overline{\delta^2}$ in the limit $\Delta = T$. For superdiffusive HDPs, the moments of $\phi(\xi)$ are larger than those for subdiffusive noisy HDPs with the same σ^2 , compare the blue and red symbols in Fig. 3.

V. NOISY HDPS WITH QUENCHED DISORDER

We now turn to the situation of quenched disorder in a stratified environment, in which evenly sized domains of width δx have a diffusivity of form (2), centered within the domain, whose α value is noisy and with distribution (15). In this quenched scenario, the particle experiences the *same* value of α each time it revisits a given position in space. The situation is illustrated in Figs. 1(c) and 1(d).

A. Noisy Brownian motion, $\alpha_0 = 0$

For quenched, noisy Brownian motion, we observe that for small noise strength σ^2 the behavior, as expected, is very close to standard Brownian motion (not shown). For larger values of σ^2 , the spread of the amplitude of individual time averaged MSDs $\overline{\delta^2(\Delta)}$ is non-negligible even at short lag times Δ , as shown in Fig. 4(c). This spread is more pronounced for larger periodicities δx of the stratified medium. For small σ^2 , the ensemble averaged MSD $\langle (x - x_{c,1})^2 \rangle$ computed with respect to the center of the starting domain and the time averaged MSD (thick blue lines) almost coincides for all lag times Δ (not shown). Concurrently, the ergodicity breaking parameter follows the Brownian asymptote (11), as shown by the green symbols in Fig. 5(a). For larger values of the noise strength σ^2 , the ergodicity breaking parameter deviates pronouncedly from Eq. (11) at short lag times Δ , indicating the occurrence of weak ergodicity breaking, along with the disparity $\langle \overline{\delta^2} \rangle \neq \langle x^2 \rangle$, as witnessed by large EB values in Fig. 5(b). This inequality is particularly pronounced for larger values of the noise strength σ^2 and large periodicity δx , see the changes for varying δx in Fig. 4(c). For wider α distributions $p(\alpha)$, the ensemble averaged MSD starts close to that of the asymptote for standard Brownian motion, while at later times, there occurs a crossover to the curve for the time averaged MSD (left panel, Fig. 4(c)). This behavior is also typical for sub- and superdiffusive quenched noisy HDPs, see below. For $\sigma^2 = 1$, this transition occurs after $\sim 10^3$ time steps and becomes less pronounced for smaller periodicities δx of the medium (Fig. 4(c)).

B. Superdiffusive noisy HDPs, $\alpha_0 = 1$

In standard superdiffusive HDPs, there exists a finite probability of particle trapping in regions of low diffusivity near the origin, as witnessed by the cusp around $x = 0$ of probability density function (12).³³ For noisy HDPs, we find that for large

values of the domain size δx and small noise strengths σ^2 , the particle preferentially stays in the domain, in which it was seeded, and the resulting ensemble averaged MSD is close to that of the standard HDPs.^{33,35} Here, we again computed the MSD with respect to the center $x_{c,1}$ of the seed domain in the form $x(t = 0) - x_{c,1} = 0.1$. The time averaged MSD is equally close to asymptote (8) of the normal HDP. Ensemble and time averaged MSDs converge at long lag times $\Delta \rightarrow T$, note that the ensemble averaged MSD here is below the time averaged MSD, as evidenced by Figs. 4(a) and 4(b).

We start with a narrow spread of α in the spatial domains corresponding to $\sigma = 0.03$. In this case, we find that with decreasing domain size δx , the amplitude scatter of individual time averaged MSDs shrinks and the amplitude of the trajectory mean $\langle \overline{\delta^2(\Delta)} \rangle$ drops substantially (Fig. 4(a)). The reason is that for a small domain size, there are almost no regions of fast diffusivity. For small values Brownian asymptote, see the dashed-dotted line in the right graph in Fig. 4(a). In such cases of smaller domain size, the ergodicity breaking parameter attains relatively small values, as shown in Fig. 5(a), indicating a more ergodic behavior. This effect of the noise is similar to that for noisy continuous time random walks.²² As δx increases, the ergodicity breaking parameter approaches values close to those of the standard HDP, $EB(\Delta = 1) \approx 0.34$ for $\alpha_0 = -2$ and $EB(\Delta = 1) \approx 1.1$ for $\alpha_0 = 1$, with $T = 10^4$.^{33,35} This is indicated by the dotted lines in Fig. 5(a). Thus, frequent hopping events between individual domains destroy the characteristic of the noise-free HDP scaling and cause the diffusion to be more ergodic. This is our second important conclusion. We note that homogenization theories provide an effective description of the heterogeneities of the environment on small scales via an effective conductivity on larger length scales, applicable for various random media and quasi-periodic environments.¹⁷ It is not surprising that in the periodic case, if the number of periods is sufficiently large, the system falls into the domain of attraction of homogenization theorems, and more ergodic behaviour emerges. In our analysis here, obviously the case of larger domain sizes is the more interesting one.

For larger σ values, the MSD stops following HDP scaling law (1) and instead two nearly Brownian regimes are detected for short and long diffusion times, see the left panel in Fig. 4(b). Similar to noisy continuous time random walks,^{7,22} for noisy HDPs, we observe a superposition of anomalous scaling for the MSD inherent to HDPs with the linear MSD increase due to particle jumping between the stratified domains. The latter term contributes stronger for smaller δx values: after a given number of steps performed, the particle visits more $D(x)$ domains and its diffusion on the length scale $\gg \delta x$ becomes effectively more normal and ergodic.

The time averaged MSD is an approximately linear function of the lag time Δ . For smaller domain size δx , we observe a more confined amplitude spread of the time averaged MSD traces $\overline{\delta^2}$, see the evolution from left to right in Figs. 4(a) and 4(b). A similar behavior occurs for subdiffusive noisy HDPs, as demonstrated in Figs. 4(d) and 4(e) consistent with smaller values of the ergodicity breaking parameter. This is our third main result. For superdiffusive noisy HDPs, given large domain sizes δx , we observe more distant particle excursions and thus a broader amplitude spread of individual time traces

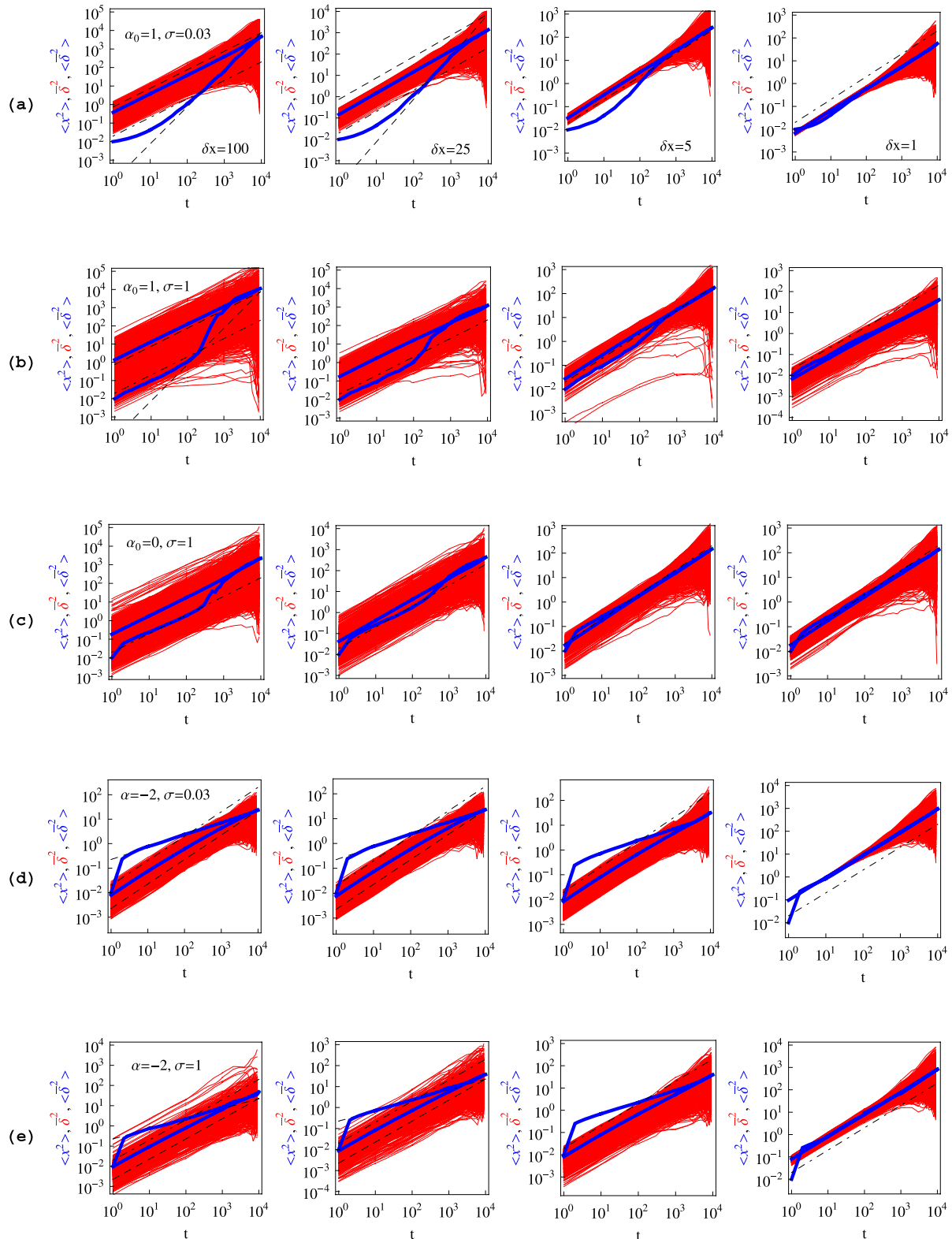


FIG. 4. Ensemble and time averaged MSDs and amplitude scatter of individual traces $\overline{\delta^2}$ for noisy HDPs with quenched disorder. The values of α , σ , and domain size δx are indicated in the plots. The panels (a) and (b) are for superdiffusive noisy HDPs, panel (c) stands for noisy Brownian motion, and panels (d) and (e) represent subdiffusive noisy HDPs. The MSD is computed with respect to the position of the center of the first domain, $\langle (x(t) - x_c, 1)^2 \rangle$. Parameters: $T = 10^4$, $N = 10^3$, and δx values are the same in each column. The notations for the curves and asymptotes are the same as in Fig. 2.

$\overline{\delta^2}(\Delta)$, particularly for large values σ^2 of the noise strength (Fig. 4(b)): at larger σ^2 , we correspondingly obtain larger values of the ergodicity breaking parameter, compare panels (a) and (b) in Fig. 5.

We find that the distribution $\phi(\xi)$ of the amplitude scatter features a skewed form, which is characterized by its second, third, and fourth moments corresponding to the ergodicity breaking parameter, the skewness S , and the kurtosis

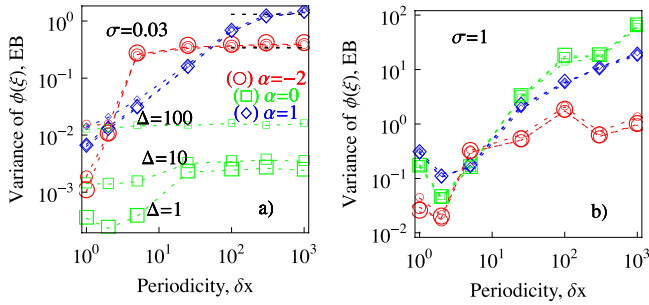


FIG. 5. Ergodicity breaking parameter of noisy HDPs with quenched disorder of the medium. The parameters are the same as in Fig. 4, the values for α and σ being indicated in the plots. The black dotted lines represent the ergodicity breaking parameter EB for the standard HDPs. The large, medium, and small symbols correspond to the lag times $\Delta = 1, 10$, and 100 , respectively.

K , respectively. For larger values of σ^2 , $S(\xi)$ and $K(\xi)$ grow with the domain size δx and are more irregular than the distribution $\phi(\xi)$ itself, due to worsening statistics for higher order moments (not shown). Note that for short lag times $\Delta/T \ll 1$, the ergodicity breaking parameter for large domain sizes δx approaches the values of the corresponding normal HDPs,³³ compare Fig. 5(a). At small domain size δx , the non-ergodic properties of the standard HDPs, in contrast, are masked by the noise in the stratified spatial domains.

For superdiffusive HDPs, the particles tend to localize in the center of each domain, while for subdiffusive values $\alpha_0 < 0$, they tend to spread towards regions of low diffusivity near the domain borders. In the long time limit, the particles spread over many domains, establishing the shape of the probability density function $P(x, t)$ presented in Fig. 6. The local minima and maxima of $P(x, t)$ correspond to the regions of low and

fast diffusivity $D(x)$, respectively, see Figs. 1(c), 1(d), and 6. For relatively large domain size δx , the probability density function of the noisy HDPs becomes dominated by the contribution from the seed domain. The spreading of particles over superdiffusive HDP domains in the long time limit is symmetric and nearly Gaussian,

$$P(x, t) = \frac{1}{\sqrt{4\pi D_{\text{eff}} t}} \exp\left(-\frac{(x - x_{c,1})^2}{4D_{\text{eff}} t}\right), \quad (20)$$

with the effective diffusivity D_{eff} . The mean particle displacement with respect to the center of the seed domain vanishes, $\langle x(t \rightarrow \infty) \rangle \rightarrow 0$. To compute D_{eff} analytically, a homogenization procedure and generic concepts of diffusion in random and highly heterogeneous media would need to be applied.^{17,50}

C. Subdiffusive noisy HDPs, $\alpha_0 = -2$

Subdiffusive noisy HDPs in the quenched scenario share a number of trends with the above descriptions of the cases $\alpha = 0$ and $\alpha = 1$. In particular, as the domain size δx decreases, the amplitude spread of individual time averaged MSD traces $\overline{\delta^2}$ decreases (Figs. 4(d) and 4(e)). Because of the sublinear scaling of the ensemble MSD of the normal subdiffusive HDPs ($\alpha_0 < 0$), the ensemble averaged MSD approaches the time averaged MSD $\langle \overline{\delta^2(\Delta)} \rangle$ from above. Moreover, the scaling of the ensemble averaged MSD of subdiffusive noisy HDPs with quenched disorder turns from subdiffusive to Brownian as the domain size δx decreases. The physical reason for this crossover behavior is the random character of hops between domains with a varying local exponent α . We find that, similarly to superdiffusive noisy HDPs, the ensemble averaged MSD initially follows scaling (1) of normal HDPs while at

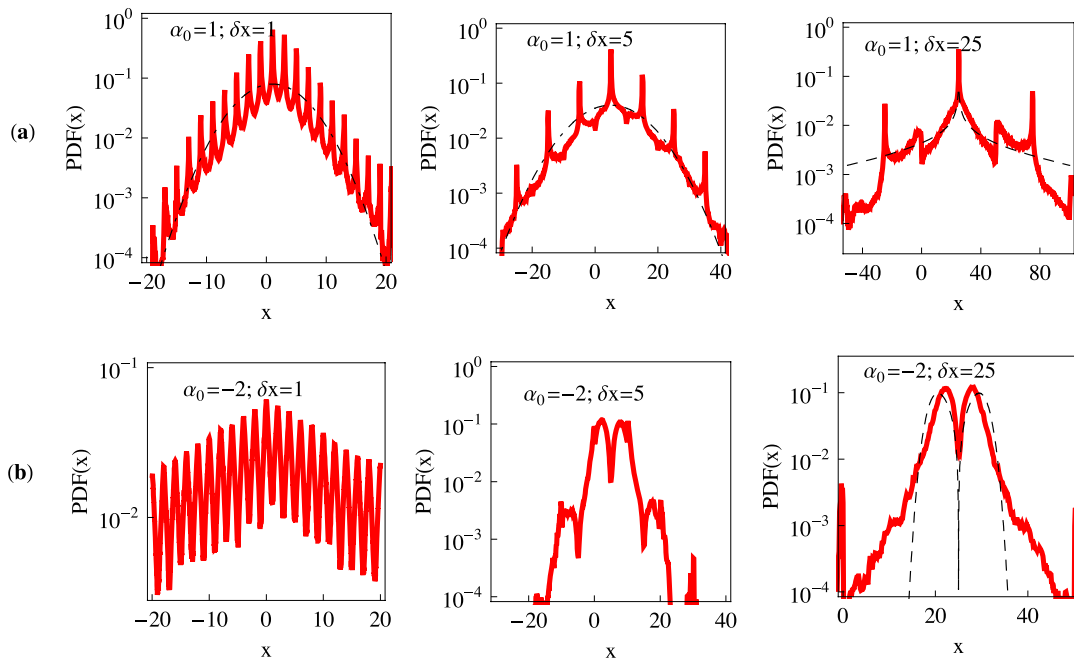


FIG. 6. Probability density functions of noisy HDPs with quenched disorder for noise strength $\sigma^2 = 1$ and varying domain size δx . The other parameters are the same as in Fig. 4. Panel (a) represents superdiffusive and panel (b) stands for subdiffusive noisy HDPs. The horizontal shift of the peak positions with respect to those of the standard HDPs³³ is due to the shift (δx) of the domain center positions, see Eq. (16). The dashed curves represent the probability density functions of standard HDPs, given by Eq. (12), while the dashed-dotted curves are result (20).

later times, an approximately linear scaling is observed. For smaller periodicities δx , the linear scaling becomes dominant, as demonstrated for large σ^2 values in Fig. 4(e) from left to right.

The probability density function of quenched noisy HDPs in the long time limit is a combination of local probability densities of the standard HDPs. For large periodicities δx , the probability density function is again dominated by the contribution from the seed domain, as can be seen in the right panel of Fig. 6(b). Similar to superdiffusive noisy HDPs, we find that the time averaged MSD is linear in the lag time, $\langle \overline{\delta^2(\Delta)} \rangle \sim \Delta$, while the amplitude spread of individual time averaged MSDs grows with the noise strength σ^2 and becomes diminished for smaller medium periodicities δx . We also see that for subdiffusive noisy HDPs, the saturation of the ergodicity breaking parameter to the values of normal HDPs occurs at much smaller values of δx as compared to superdiffusive noisy HDPs (Fig. 5(a)).

VI. CONCLUSIONS

We studied a stochastic process based on a combination of HDPs with multiplicative noise and additional disorder of the environment, distinguishing annealed and quenched scenarios. The scaling exponent α_0 was taken to vary in time, or the environment was assumed to be structured into periodic domains of given periodicity. We investigated the diffusive and ergodic properties of these noisy HDPs. The superposition of the additional stochasticity onto the standard HDP with its deterministic variation of the diffusivity revealed a variety of new features: the scaling relations for the ensemble and time averaged MSD of the noisy HDPs being dramatically altered as compared to the normal HDP behavior.

For annealed disorder, the scaling exponent α of the diffusivity profile switches in time and the gradient field of the particle diffusivity has a single origin at $x = 0$. We demonstrated how the Gaussian spread $p(\alpha)$ of the scaling exponent gives rise to a strongly asymmetric scatter of individual time averaged MSD traces. Rapidly switching diffusivity profiles in such an annealed environment cause transient particle trapping in low-diffusivity regions. For superdiffusive motion, the effects of the α spread are more pronounced.

In the case of a quenched environment, a spatially stratified medium is modeled in terms of domains of width $2\delta x$ with a normal distribution of the local HDP exponent. Upon particle diffusion, the averaging is thus performed over ensembles of particle trajectories generated for different spatial distributions of the scaling exponents α in the domains. One of the key findings is that for small periodicity δx , the sub- and superdiffusive scaling of normal HDPs cross over to a linear growth of the ensemble averaged MSD as function of time. External noise thus progressively masks the statistics of the underlying HDP, the main conclusion of this study.

We note that the noisy HDPs discussed here represent an independent class of stochastic processes. They are specifically different from space and/or time fractional diffusion equations (both monofractional and distributed orders).^{5,7,51} The latter

are highly non-local in space and/or time and are related to continuous time random walk processes with power-law forms for the jump length and/or waiting time distributions. HDPs are truly Markovian yet multiplicative processes. Any disorder in space or time thus leads to a significantly different behaviour of the system between HDPs and processes described by fractional diffusion equations.

What could be the physical phenomena captured by the noisy HDP discussed here? From a biological perspective, the diffusion of small molecules in assemblies of non-identical, interconnected cells is a relevant example. The cell-to-cell variations of the diffusivity are inherent to biological tissues, while every individual cell features a space dependent diffusivity in its cytoplasm.⁸ At cell-to-cell boundaries, the diffusivity likely varies with a jump, as captured by our stratified model of the quenched disorder, with possibly discontinuous diffusivity across the system. We note that heterogeneous diffusivities can, for instance, play a role in the formation of gradients of morphogen molecules in a developing cell tissue,⁵² a process known to involve features of anomalous diffusion. It also features a division of fluxes of the molecules into fluxes through cells, across the outer cell membranes, and transport in extracellular spaces.⁵³ Heterogeneous diffusion of water molecules in brain tissues⁵⁴ and strongly heterogeneous structures of cardiac muscle tissue with nontrivial cell-cell coupling⁵⁵ could be another example. Similarly, the domains in the noisy HDP could represent internal compartments in a single cell. The quenched case would correspond to static environments whereas the annealed scenario would stand for environments, which change rapidly compared to the typical crossing times between domains.

Our results for noisy HDPs could also be useful for the description of nano-objects trapped in dynamical temperature fields⁵⁶ and of particles in strong temperature gradients.⁵⁷ Another field of relevance is the tracer diffusion in heterogeneous assemblies of distributed obstacles⁵⁸ mimicking features of the cell cytoplasm⁸ and diffusion on chemically and mesoscopically periodically patterned solid-liquid interfaces.⁵⁹ On a macroscopic scale, water diffusion in subsurface hydrology applications is to be mentioned,¹² as well as tracer motion in porous heterogeneous media.^{60,61} For the latter, there likely exists a distance-dependent diffusivity within each pore, constructing a network governing, i.e., the diffusion of water and contaminants in soil specimen.^{12,62} Finally, in statistical models of financial stock price variations,⁶³ the terms stochastic versus correlated volatility widely occur, representing the diffusivity in random walk models.⁶⁴ Some patterns of correlated or clustered volatility observed in financial data thus correspond to a systematically varying diffusivity in our model of quenched noisy HDPs. Some repeats of non-Brownian up-and-down trends in stock price fluctuations⁶⁴ can thus be considered as HDPs repeatedly occurring in time.

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