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Correlation functions for the fractional generalized Langevin equation in the presence of internal and external noise

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We study generalized fractional Langevin equations in the presence of a harmonic potential. General expressions for the mean velocity and particle displacement, the mean squared displacement, position and velocity correlation functions, as well as normalized displacement correlation function are derived. We report exact results for the cases of internal and external friction, that is, when the driving noise is either internal and thus the fluctuation-dissipation relation is fulfilled or when the noise is external. The asymptotic behavior of the generalized stochastic oscillator is investigated, and the case of high viscous damping (overdamped limit) is considered. Additional behaviors of the normalized displacement correlation functions different from those for the regular damped harmonic oscillator are observed. In addition, the cases of a constant external force and the force free case are obtained. The validity of the generalized Einstein relation for this process is discussed. The considered fractional generalized Langevin equation may be used to model anomalous diffusive processes including single file-type diffusion. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4863478]

I. INTRODUCTION

Anomalous diffusion characterizes deviations from the linear scaling with time found for the mean squared displacement (MSD) $\langle x^2(t)\rangle \simeq t$ of normal Brownian motion. In particular, we are interested in power-law forms $\langle x^2(t)\rangle \simeq t^\alpha$ separating subdiffusion ($0 < \alpha < 1$) and superdiffusion ($\alpha > 1$).[44] Anomalous diffusion has been found in various physical systems.[20,26–28,33,56,57,61,67] Several stochastic approaches to anomalous diffusion exist. We mention the generalized Langevin equation (GLE),[4,22,33,39,48,53] that came to fame following Kubo's work[34] and the related fractional Brownian motion (FBM), originally introduced by Kolmogorov and popularized by Mandelbrot.[32,41] The fractional Langevin equation was further generalized in terms of more complex kernels.[14,17,37,51] Mainardi and Pironi[40] introduced a fractional Langevin equation as a particular case of a GLE, and for the first time represented the velocity and displacement correlation functions in terms of Mittag-Leffler (M-L) functions. Furthermore, fractional Langevin equation of distributed order[15] has been used to model single file diffusion and ultraslow diffusion. In our previous work,[51] we derived general expressions for variances and MSD for fractional GLEs (FGLEs) for a free particle driven by an arbitrary internal noise, we used a three parameter M-L frictional memory kernel and discussed its

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application to model anomalous diffusive processes in complex media including phenomena similar to single file diffusion or possible generalizations thereof. Recently, Eab and Lim[16] considered multi-fractional Langevin-type stochastic differential equations driven by single or multiple fractional Gaussian noise terms to describe retarded and accelerated anomalous diffusion. A different family of dynamic processes are described by the fractional Fokker-Planck equation[36,43,44,63] and the generalized Chapman-Kolmogorov equation.[42] The fractional Fokker-Planck equation corresponds to the diffusion limit of subdiffusive continuous time random walks,[56] which have a finite variance of jump lengths and a broad distribution of waiting times τ of the form $\psi(\tau) \simeq (\tau^*)^\alpha/\tau^{1+\alpha}$, with $0 < \alpha < 1$ such that no mean waiting time exists[43,44] (see also Ref. 50). While processes governed by the fractional Langevin equation and fractional Brownian motion reach ergodicity algebraically at sufficiently long times,[11,29] subdiffusive continuous time random walks and fractional Fokker-Planck equations lead to weak ergodicity breaking such that long time and ensemble averaged observables are no longer equivalent,[3,5,25] and the process ages.[2,46,58] Superdiffusive continuous time random walks, the Lévy walks, exhibit an ultraweak ergodicity breaking.[18,19] Furthermore, diffusion with space-dependent diffusion coefficients is weakly non-ergodic.[10]

Anomalous diffusion of a particle of unit mass $m = 1$ driven by a stationary random force $\xi(t)$ may be described in terms of the GLE:[34,40,69]

$$\ddot{x}(t) + \int_0^t \gamma(t-t')\dot{x}(t')\mathrm{d}t' + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = \xi(t), \quad \dot{x}(t) = v(t), \tag{1}$$

where $v(t)$ is the velocity at time $t > 0$, $x(t)$ is the particle displacement, and $\gamma(t)$ is the frictional memory kernel. $F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$ is an external force with potential $V(x)$ acting on the particle. The noise $\xi(t)$ is of a zero-mean ($\langle\xi(t)\rangle = 0$), and possesses the correlation function

$$\langle\xi(t)\xi(t')\rangle = C(t'-t). \tag{2}$$

In cases when the system reaches an equilibrium state, i.e., when the noise is internal, the correlation function C must be related to the frictional memory kernel via the second fluctuation-dissipation theorem[34,40,69] in the following way,

$$C(t) = k_B T\gamma(t), \tag{3}$$

where k_B is the Boltzmann constant and T is the absolute temperature of the heat bath. In this case, the fluctuation and dissipation relate to the same source. The frictional memory kernel satisfies the assumption $\lim_{t\to\infty}\gamma(t) = \lim_{s\to 0} s\hat{\gamma}(s) = 0$,[13] where $\hat{\gamma}(s) = \mathcal{L}[\gamma(t)](s)$ is the Laplace transform of $\gamma(t)$. In the case when the fluctuation and dissipation do not originate from a same source, the second fluctuation-dissipation theorem (3) does not hold. We then speak of external noise. We note that for a white Gaussian noise $\xi(t)$, the GLE (1) corresponds to the regular Langevin equation.[35,40,69] Different forms of the memory kernel have been used in order to model anomalous diffusive processes, such as power-law correlation functions,[4,13,39,40,64,66] and different forms of M-L correlation functions.[6,51,52,54,65]

In this paper, we investigate anomalous diffusion in terms of the correlation functions of the FGLE for a harmonic oscillator, i.e., for the external force field $F(x) = -\omega^2 x(t)$. These FGLEs generalize the form (1) by substituting the integer derivatives through by fractional order derivatives. In the work of Fa[17] a FGLE with nonlocal dissipative force was investigated. Such FGLEs were recently used by Lim and Teo,[37] Eab and Lim,[14] Sandev, Metzler, and Tomovski[51] to model a single file-type diffusion and generalized diffusion processes. In what follows, apart from the linear Hookean force we also obtain results for a free particle $V(x) = 0$ and a constant external force $F(x) = F\theta(t)$, where F is a constant and $\theta(t)$ is the Heaviside step function. In particular, we consider the role of the Einstein relation in such phenomena.

The paper is organized as follows. In Sec. II, we formulate the FGLE, determine its relaxation function, and derive the correlation functions. Detail analysis of the normalized displacement correlation function, which is a experimentally measured quantity, is done. Additional behaviors to the overdamped and underdamped motion of regular damped oscillator are observed. In Sec. III, we turn to a constant external force and investigate the generalized Einstein relation. Finally, we draw our conclusions in Sec. IV.

II. FGLE FOR A HARMONIC OSCILLATOR

We study the following FGLE,

$$ {}_{C}D_{0+}^{\mu}v(t)+\int_{0}^{t}\gamma(t-t')v(t')\mathrm{d}t'+\frac{\mathrm{d}V(x)}{\mathrm{d}x}=\xi(t),\quad {}_{C}D_{0+}^{\nu}x(t)=v(t), \tag{4} $$

for a harmonic oscillator with external potential $V(x)=\omega^2x^2/2$, where x is the particle position and v its velocity. The parameters are taken such that $0<\mu\leq 1$ and $0<\nu\leq 1$. We call $\gamma(t)$ the memory kernel. Finally, $\xi(t)$ is the (internal) noise with zero mean, for which relation (2) is satisfied. We note that $\mu+\nu>1$ needs to be fulfilled.[51]

The operator

$$ \left({}_{C}D_{0+}^{\gamma}f\right)(t)=\left(I_{0+}^{(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\right)(t),\quad n=[\Re(\gamma)]+1 \tag{5} $$

is the Caputo time fractional derivative[8] of order γ ($n-1<\gamma\leq n, n\in N$), with a Laplace transform

$$ \mathcal{L}\left[{}_{C}D_{0+}^{\gamma}f(t)\right](s)=s^{\gamma}\mathcal{L}\left[f(t)\right](s)-\sum_{k=0}^{n-1}f^{(k)}(0+)s^{\gamma-1-k} \tag{6} $$

and

$$ \left(I_{0+}^{\gamma}f\right)(t)=\frac{1}{\Gamma(\gamma)}\int_{0}^{t}\frac{f(t')}{(t-t')^{1-\gamma}}\mathrm{d}t',\quad t>0,\quad \Re(\gamma)>0 \tag{7} $$

is the Riemann-Liouville (R-L) fractional integral of order $\gamma>0$,[26,30] with properties $\left(I_{0+}^{0}f\right)(t)=f(t)$, and $I_{0+}^{\gamma}I_{0+}^{\delta}=I_{0+}^{\delta}I_{0+}^{\gamma}=I_{0+}^{\gamma+\delta}$, $\gamma>0$, $\delta>0$, and Laplace transform given by

$$ \mathcal{L}\left[I_{0+}^{\gamma}f(t)\right](s)=\frac{\mathcal{L}\left[f(t)\right](s)}{s^{\gamma}}. \tag{8} $$

For convenience, we here introduced the Riemann-Liouville fractional derivative

$$ \left({}_{RL}D_{0+}^{\gamma}f\right)(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(I_{0+}^{(n-\gamma)}f\right)(t) \tag{9} $$

of order γ,[26,30] which Laplace transform is given by

$$ \mathcal{L}\left[{}_{RL}D_{0+}^{\gamma}f(t)\right](s)=s^{\gamma}\mathcal{L}\left[f(t)\right](s)-\sum_{k=0}^{n-1}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(I_{0+}^{n-\gamma}f\right)(0+)s^{n-1-k}. \tag{10} $$

R-L fractional derivative is a left inverse of R-L fractional integral ${}_{RL}D_{0+}^{\gamma}I_{0+}^{\gamma}f(t)=f(t)$. Moreover, ${}_{RL}D_{0+}^{\delta}I_{0+}^{\gamma}f(t)=I_{0+}^{\gamma-\delta}f(t)$. Note that if we consider proper initial conditions (zero initial values) the R-L and Caputo fractional derivatives are equivalent since[26,30]

$$ \left({}_{RL}D_{0+}^{\gamma}f\right)(t)=\left({}_{C}D_{0+}^{\gamma}f\right)(t)+\sum_{k=0}^{n-1}\frac{t^{k-\gamma}}{\Gamma(k-\gamma+1)}f^{(k)}(0+). \tag{11} $$

Thus, in such cases the properties of R-L fractional derivative can be used for the Caputo fractional derivative. For the R-L fractional integral and Caputo fractional derivative the following relation holds true[30]

$$ I_{0+}^{\gamma}{}_{C}D_{0+}^{\gamma}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0+)\frac{t^k}{k!},\quad n-1<\gamma\leq n. \tag{12} $$

Particularly, for $0<\gamma\leq 1$ it is obtained $I_{0+}^{\gamma}{}_{C}D_{0+}^{\gamma}f(t)=f(t)-f(0+)$.

FGLE (4) contains a number of limiting cases. Thus, the case for a free particle $\omega=0$ was introduced by Lim and Teo[37] to model single file diffusion. The case $\nu=1$ is investigated by Fa,[17] and analyzed by Eab and Lim[14] in case of $\omega=0$ in the presence of an external force and different forms of the friction kernel. In Ref. 51, we derived the correlation function of the FGLE (4) for a free particle ($\omega = 0$). For $\mu = \nu = 1$ it yields the GLE for an arbitrary frictional memory kernel, which as a special case contains the fractional Langevin equation for anomalous diffusive processes, and classical Langevin equation for a standard Brownian motion. The particular case $\mu = \nu$, $\omega = 0$, and $\gamma(t) = \gamma\delta(t)$ was introduced by Kobelev and Romanov[31] to describe anomalous diffusion. Thus, our goal is to find general expressions of variances for this generalized model, which is not done elsewhere, for an arbitrary frictional memory kernel $\gamma(t)$ in case where the second fluctuation-dissipation theorem is satisfied. Additionally, we will investigate a FGLE in presence of an external constant force and we will discuss the validity of the generalized Einstein relation.

The fractional version of the generalized Langevin equation contains non-integer derivatives of physical quantities. Let us briefly motivate this choice. We start by noting that such fractional derivatives also occur, when we derive the dynamic equation for the continuous time random walk model with scale-free distribution of waiting times. While locally the process is well-defined, the generalized central limit theorem enforces the convergence to stable laws. In the context of the continuous time random walk subdiffusion, this means, for instance, that the Fokker-Planck equation acquires a fractional derivative (see, e.g., Metzler and Klafter[44]), and physical relations such as $\langle v(t)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)\rangle$ become fractional. Such equations are therefore valid on some mesoscopic level, on which averages over the physical observables already took place. Compared to experiments, the theory has been shown to represent an excellent quantitative description of the observed phenomena.

In the case of the generalized Langevin equation, we similarly view the variables to represent a mesoscopic description of the process. The expectation values of observables calculated from this theory then describe the dynamic behavior after averaging over the disorder of the system.

In addition to the numerous aforementioned special cases of FGLE (4), and the previous comment on the physical motivation to investigate FGLE (4), we further analyze and transform Eq. (4) in a more suitable form. If we apply the R-L fractional integral to Eq. (4), from relation (12), we obtain

$$v(t) - v_0 + I_{0+}^{\mu}\left[\int_0^t \gamma(t-t')v(t')\mathrm{d}t' + \omega^2 x(t)\right] = I_{0+}^{\mu}\xi(t), \tag{13}$$

where $v(0+) = v_0$. Thus, the first time derivative to (13) yields

$$\dot{v}(t) + {}_{RL}D_{0+}^{1-\mu}\left[\int_0^t \gamma(t-t')v(t')\mathrm{d}t'\right] + \omega^2\, {}_{RL}D_{0+}^{1-\mu}x(t) = {}_{RL}D_{0+}^{\mu}\xi(t), \tag{14}$$

where we use $\frac{\mathrm{d}}{\mathrm{d}t}I_{0+}^{\mu}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}I_{0+}^{1-(1-\mu)}f(t) = {}_{RL}D_{0+}^{1-\mu}f(t)$. So, Eq. (14) can be considered as a GLE for a particle on which acts generalized force $-\omega^2\, {}_{RL}D_{0+}^{1-\mu}x(t)$, and driven by noise $\bar{\xi}(t)$ which is a fractional derivative of the noise $\xi(t)$, i.e., $\bar{\xi}(t) = {}_{RL}D_{0+}^{\mu}\xi(t)$. The term ${}_{RL}D_{0+}^{1-\mu}\left[\int_0^t \gamma(t-t')v(t')\mathrm{d}t'\right]$ can be considered as a term which represents the memory effects of the complex environment on particle movement. In a same way, instead of defining velocity as a first time derivative of the displacement $\dot{x}(t) = v(t)$, it is used that ${}_C D_{0+}^{\nu}x(t) = v(t)$, thus, it is not clear what does it represent. By applying the R-L fractional integral as previously, we obtain

$$x(t) - x_0 = I_{0+}^{\nu}v(t) = \frac{1}{\Gamma(\nu)}\int_0^t \frac{v(t')}{(t-t')^{1-\nu}}\mathrm{d}t', \tag{15}$$

where $x(0+) = x_0$. Relation (15) means, as it was discussed by Kobelev and Romanov,[31] that the displacement is defined by the velocity only in the points within time interval of dimension ν, and is characteristic for a microscopic motion of a particle on a nondifferentiable curve. In such cases of fractal trajectories, some of the instant velocities and displacements do not contribute to the macroscopic motion, and thus anomalous diffusion occurs.[31] Application of fractional derivatives in the fractional Langevin equation is discussed by West[68] in the description of dynamics of fractal time series, to describe the evolution of a fractal statistical processes.

A. Relaxation functions

The Laplace transform of (4) becomes

$$\hat{X}(s) = \frac{x_0}{s}\left[1-\omega^2\hat{G}(s)\right] + v_0 s^{\mu+\nu-1}\hat{I}(s) + \hat{G}(s)\hat{F}(s), \tag{16a}$$

$$\hat{V}(s) = v_0 s^{\mu+\nu-1}\hat{G}(s) - \omega^2 x_0 s^{\nu-1}\hat{G}(s) + s^{\nu}\hat{G}(s)\hat{F}(s), \tag{16b}$$

where $\hat{X}(s) = \mathcal{L}[x(t)](s) = \int_0^\infty \exp(-st)x(t)\mathrm{d}t$ is the Laplace transform of $x(t)$, and analogously for the other quantities, $x_0 = x(0+)$ and $v_0 = v(0+)$. The Laplace transform of the abbreviation G is thereby given by

$$\hat{G}(s) = \frac{1}{s^{\mu+\nu} + s^{\nu}\hat{\gamma}(s) + \omega^2} \tag{17a}$$

and we introduce the following functions

$$\hat{g}(s) = s^{\nu}\hat{G}(s) = \frac{s^{\nu}}{s^{\mu+\nu} + s^{\nu}\hat{\gamma}(s) + \omega^2}, \tag{17b}$$

$$\hat{I}(s) = s^{-\nu}\hat{G}(s) = \frac{s^{-\nu}}{s^{\mu+\nu} + s^{\nu}\hat{\gamma}(s) + \omega^2}. \tag{17c}$$

From inverse Laplace transform of relations (16a) and (16b), we obtain the particle position and velocity,

$$x(t) = \langle x(t)\rangle + \int_0^t G(t-t')\xi(t')\mathrm{d}t', \tag{18a}$$

$$v(t) = \langle v(t)\rangle + \int_0^t g(t-t')\xi(t')\mathrm{d}t', \tag{18b}$$

where

$$\langle x(t)\rangle = x_0\left[1-\omega^2 I_{0+}^{1}G(t)\right] + v_0\cdot {}_{C}D_{0+}^{\mu+\nu-1}I(t),\quad \langle v(t)\rangle = v_0\cdot {}_{C}D_{0+}^{\mu+\nu-1}G(t) - \omega^2 x_0 I_{0+}^{1-\nu}G(t) \tag{19}$$

are the noise-averaged (deterministic) particle displacement and velocity. The functions $I(t)$, $G(t)$ (with $G(0) = 0$), and $g(t)$ are known as relaxation functions, and we will use them to analyze the correlation functions. Relaxation functions depend on frictional memory kernel $\gamma(t)$ and determine the relaxation law for particular process described by Eq. (4). As special cases one finds

$$\langle x(t)\rangle = x_0\left[1-\omega^2 I_{0+}^{1}G(t)\right] + v_0 G(t),\quad \langle v(t)\rangle = v_0 g(t) - \omega^2 x_0 I_{0+}^{1-\nu}G(t),\quad \mu = 1,\quad 0<\nu<1, \tag{20a}$$

$$\langle x(t)\rangle = x_0\left[1-\omega^2 I(t)\right] + v_0\cdot {}_{C}D_{0+}^{\mu}I(t),\quad \langle v(t)\rangle = v_0\cdot {}_{C}D_{0+}^{\mu}G(t) - \omega^2 x_0 G(t),\quad \nu = 1,\quad 0<\mu<1, \tag{20b}$$

$$\langle x(t)\rangle = x_0\left[1-\omega^2 I(t)\right] + v_0 G(t),\quad \langle v(t)\rangle = v_0 g(t) - \omega^2 x_0 G(t),\quad \mu = \nu = 1. \tag{20c}$$

B. Correlation functions

From relations (18a) and (18b), in case of an internal noise, we obtain the following general formulas for the position autocorrelation

$$\begin{aligned}\sigma_{xx} &= \langle x^2(t)\rangle - \langle x(t)\rangle^2 = 2\int_0^t \mathrm{d}t_1 G(t_1)\int_0^{t_1}\mathrm{d}t_2 G(t_2)C(t_1-t_2)\\ &= 2k_B T\left[\int_0^t \mathrm{d}\xi\, G(\xi)\frac{\xi^{\nu-1}}{\Gamma(\nu)} - \omega^2\int_0^t \mathrm{d}\xi\, I(\xi)\,{}_{C}D_{0+}^{\nu}I(\xi) - \int_0^t \mathrm{d}\xi\, G(\xi)\,{}_{C}D_{0+}^{\mu}G(\xi)\right],\end{aligned} \tag{21a}$$

the position-velocity cross-correlation

$$\begin{aligned}\sigma_{xv} &= \langle (v(t)-\langle v(t)\rangle)(x(t)-\langle x(t)\rangle)\rangle = \int_0^t \mathrm{d}t_1 g(t_1)\int_0^t \mathrm{d}t_2 G(t_2)C(t_1-t_2)\\ &= k_B T\left[\frac{1}{\Gamma(\nu)}\int_0^t \mathrm{d}\xi\, g(\xi)\xi^{\nu-1} - \int_0^t \mathrm{d}\xi\, g(\xi)\,{}_C D_{0+}^{\mu} G(\xi) - \int_0^t \mathrm{d}\xi\, G(\xi)\,{}_{RL} D_{0+}^{\mu} g(\xi)\right.\\ &\left. -\omega^2\int_0^t \mathrm{d}\xi\,\left(G^2(\xi)+g(\xi)I(\xi)\right)\right], \end{aligned}\tag{21b}$$

and the velocity autocorrelation

$$\begin{aligned}\sigma_{vv} &= \langle v^2(t)\rangle - \langle v(t)\rangle^2 = 2\int_0^t \mathrm{d}t_1 g(t_1)\int_0^{t_1} \mathrm{d}t_2 g(t_2)C(t_1-t_2)\\ &= -2k_B T\left[\int_0^t \mathrm{d}\xi\, g(\xi)\,{}_{RL} D_{0+}^{\mu} g(\xi) + \omega^2\int_0^t \mathrm{d}\xi\, G(\xi)\,{}_C D_{0+}^{\nu} G(\xi)\right], \end{aligned}\tag{21c}$$

where we apply the following formula[13]

$$\langle \hat{F}(s)\hat{F}(s')\rangle = k_B T\frac{\gamma(s)\gamma(s')}{s+s'}. \tag{22}$$

In the special case $\mu = 1$, $0 < \nu < 1$ we obtain

$$\sigma_{xx} = 2k_B T\left[\int_0^t \mathrm{d}\xi\, G(\xi)\frac{\xi^{\nu-1}}{\Gamma(\nu)} - \frac{1}{2}G^2(t) - \omega^2\int_0^t \mathrm{d}\xi\, I(\xi)\,{}_C D_{0+}^{\nu} I(\xi)\right], \tag{23a}$$

$$\begin{aligned}\sigma_{xv} &= \int_0^t \mathrm{d}t_1 g(t_1)\int_0^t \mathrm{d}t_2 G(t_2)C(t_1-t_2)\\ &= k_B T\left[\frac{1}{\Gamma(\nu)}\int_0^t \mathrm{d}\xi\, g(\xi)\xi^{\nu-1} - g(t)G(t) - \omega^2\int_0^t \mathrm{d}\xi\,\left(G^2(\xi)+g(\xi)I(\xi)\right)\right], \end{aligned}\tag{23b}$$

$$\sigma_{vv} = k_B T\left[1 - g^2(t) - 2\omega^2\int_0^t \mathrm{d}\xi\, G(\xi)\,{}_C D_{0+}^{\nu} G(\xi)\right]. \tag{23c}$$

Note that for $\nu = 1$, $0 < \mu < 1$ we recover the results obtained by Fa,[17]

$$\sigma_{xx} = 2k_B T\left[I(t) - \int_0^t \mathrm{d}\xi\, G(\xi)\,{}_C D_{0+}^{\mu} G(\xi) - \frac{\omega^2}{2}I^2(t)\right], \tag{24a}$$

$$\sigma_{xv} = \frac{1}{2}\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}t} = k_B T G(t)\left[1 - {}_C D_{0+}^{\mu} G(t) - \omega^2 I(t)\right], \tag{24b}$$

$$\sigma_{vv} = -2k_B T\left[\int_0^t \mathrm{d}\xi\, g(\xi)\,{}_{RL} D_{0+}^{\mu} g(\xi) + \frac{\omega^2}{2}G^2(t)\right]. \tag{24c}$$

Finally, the case $\mu = \nu = 1$ yields the well known result[13,64,66]

$$\sigma_{xx} = k_B T\left[2I(t) - G^2(t) - \omega^2 I^2(t)\right], \tag{25a}$$

$$\sigma_{xv} = k_B T G(t)\left[1 - g(t) - \omega^2 I(t)\right], \tag{25b}$$

$$\sigma_{vv} = k_B T\left[1 - g^2(t) - \omega^2 G^2(t)\right]. \tag{25c}$$

From the mean particle displacement (19) and variance (21a) we obtain the MSD and time-dependent diffusion coefficient $D(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle x^2(t)\rangle$,[40,48,54] respectively,

$$\begin{aligned}\langle x^2(t)\rangle &= x_0^2 + 2x_0v_0\,{}_CD_{0+}^{\mu+\nu-1}I(t) + v_0^2\left[{}_CD_{0+}^{\mu+\nu-1}I(t)\right]^2 - \omega^2 x_0 I_{0+}^1 G(t)\left[2x_0 - \left(\omega^2 x_0 - 2v_0\right) I_{0+}^1 G(t)\right] \\ &\quad + 2k_BT\left[\int_0^t \mathrm{d}\xi\, G(\xi)\frac{\xi^{\nu-1}}{\Gamma(\nu)} - \int_0^t \mathrm{d}\xi\, G(\xi)\,{}_CD_{0+}^{\mu}G(\xi) - \omega^2\int_0^t \mathrm{d}\xi\, I(\xi)\,{}_CD_{0+}^{\nu}I(\xi)\right],\end{aligned} \tag{26}$$

$$\begin{aligned}D(t) &= x_0v_0\,{}_CD_{0+}^{\mu+\nu}I(t) + v_0^2\,{}_CD_{0+}^{\mu+\nu-1}I(t)\,{}_CD_{0+}^{\mu+\nu}I(t) - \omega^2 x_0\,{}_CD_{0+}^{\nu}I(t)\left[x_0 - \left(\omega^2 x_0 - 2v_0\right) I_{0+}^1 G(t)\right] \\ &\quad + k_BTG(t)\left[\frac{t^{\nu-1}}{\Gamma(\nu)} - {}_CD_{0+}^{\mu}G(t) - \omega^2 I(t)\right].\end{aligned} \tag{27}$$

Note that for $\omega = 0$ we recover the results from Ref. 51.

C. Explicit results

We now consider explicit forms for the stochastic noise (internal and external), derive and analyze the mean particle displacement and velocity, as well as the correlation functions. The normalized displacement correlation function, defined through the two-point correlation function $\langle x(t)x_0\rangle$, is investigated. Behaviors different than those for the regular damped oscillator are observed.

1. Cases of internal noise

Here, we will consider different frictional memory kernels (Dirac delta, exponential, power-law, M-L type) and will analyze the MSD, in order to investigate the behavior of the oscillator, in cases when the second fluctuation-dissipation theorem (3) holds. First, we will use Dirac delta frictional memory kernel $\gamma(t) = 2\lambda\delta(t)$, $\lambda > 0$ (i.e., $\hat{\gamma}(s) = 2\lambda$). From relations (17b), (17a), and (17c) we obtain

$$\begin{aligned}g(t) &= \mathcal{L}^{-1}\left[\frac{s^{\nu}}{s^{\mu+\nu} + 2\lambda s^{\nu} + \omega^2}\right] = \mathcal{L}^{-1}\left[\frac{s^{\nu}}{s^{\mu+\nu} + 2\lambda s^{\nu}}\frac{1}{1 + \frac{\omega^2}{s^{\mu+\nu}+2\lambda s^{\nu}}}\right] \\ &= \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\left(-\omega^2\right)^n \frac{s^{-\nu n}}{(s^{\mu} + 2\lambda)^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)-\nu-1}E_{\mu,(\mu+\nu)(n+1)-\nu}^{n+1}\left(-2\lambda t^{\mu}\right),\end{aligned} \tag{28a}$$

$$G(t) = \sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)-1}E_{\mu,(\mu+\nu)(n+1)}^{n+1}\left(-2\lambda t^{\mu}\right), \tag{28b}$$

$$I(t) = \sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)+\nu-1}E_{\mu,(\mu+\nu)(n+1)+\nu}^{n+1}\left(-2\lambda t^{\mu}\right), \tag{28c}$$

where

$$E_{\alpha,\beta}^{\delta}(z) = \sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(\alpha k + \beta)}\frac{z^k}{k!}, \tag{29}$$

($\Re(\alpha) > 0$, $\beta, \delta, z \in \mathbb{C}$, $(\delta)_k$ is the Pochhammer symbol) is the three parameter M-L function,[49] which Laplace transform is given by[49,60]

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}^{\delta}(\omega t^{\alpha})\right](s) = \frac{s^{\alpha\delta-\beta}}{(s^{\alpha} - \omega)^{\delta}}. \tag{30}$$

The convergence of these series in three parameter M-L functions can be shown by following the procedure in Ref. 54. Moreover, detailed study of the convergence of series in three parameter M-L functions in the complex plane is presented in Ref. 47. By using asymptotic expansion formula[53,55]

$$E_{\alpha,\beta}^{\delta}(z)=\frac{(-z)^{-\delta}}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)}{\Gamma(\beta-\alpha(\delta+n))}\frac{z^{-n}}{n!},\quad |z|>1, \tag{31}$$

we find the relaxation functions in the long time limit $t\to\infty$

$$g(t)=\frac{1}{2\lambda t}E_{\nu,0}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right),\quad G(t)=\frac{t^{\nu}}{2\lambda t}E_{\nu,\nu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right),\quad I(t)=\frac{t^{2\nu}}{2\lambda t}E_{\nu,2\nu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right), \tag{32}$$

where $E_{\alpha,\beta}^{1}(z)=E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$ is the two parameter M-L function, and $E_{\alpha,1}(z)=E_{\alpha}(z)$ is the one parameter M-L function. Note that from relation (31) for $\delta=1$, one can obtain the asymptotic behavior of two parameter M-L function $E_{\alpha,\beta}(z)\simeq-\frac{z^{-1}}{\Gamma(\beta-\alpha)}$ for $z\to\infty$, since $E_{\alpha,\beta}(z)=-\sum_{n=0}^{\infty}\frac{z^{-(n+1)}}{\Gamma(\beta-\alpha(n+1))}=-\sum_{n=1}^{\infty}\frac{z^{-n}}{\Gamma(\beta-\alpha n)}$ for $|z|>1$, and for $\beta=\delta=1$—asymptotic behavior of one parameter M-L function $E_{\alpha}(z)\simeq-\frac{z^{-1}}{\Gamma(1-\alpha)}$ for $z\to\infty$, since $E_{\alpha}(z)=-\sum_{n=1}^{\infty}\frac{z^{-n}}{\Gamma(1-\alpha n)}$ for $|z|>1$.[53,55] In the short time limit ($t\to 0$), the relaxation functions are as follows

$$g(t)=t^{\mu-1}E_{\mu,\mu}\left(-2\lambda t^{\mu}\right),\quad G(t)=t^{\mu+\nu-1}E_{\mu,\mu+\nu}\left(-2\lambda t^{\mu}\right),\quad I(t)=t^{\mu+2\nu-1}E_{\mu,\mu+2\nu}\left(-2\lambda t^{\mu}\right). \tag{33}$$

From relations (28a)–(28c), also we can find the relaxation functions in case of a free particle ($\omega=0$), which are given by

$$g(t)=\lim_{\omega\to 0}\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)-\nu-1}E_{\mu,(\mu+\nu)(n+1)-\nu}^{n+1}\left(-2\lambda t^{\mu}\right)$$
$$=t^{\mu-1}E_{\mu,\mu}\left(-2\lambda t^{\mu}\right), \tag{34a}$$

$$G(t)=t^{\mu+\nu-1}E_{\mu,\mu+\nu}\left(-2\lambda t^{\mu}\right), \tag{34b}$$

$$I(t)=t^{\mu+2\nu-1}E_{\mu,\mu+2\nu}\left(-2\lambda t^{\mu}\right). \tag{34c}$$

From relations (19)–(28c), by using the following formulas for three parameter M-L function $I_{0+}^{\gamma}\left[t^{\beta}E_{\alpha,\beta+1}^{\delta}\left(-at^{\alpha}\right)\right]=t^{\beta+\gamma}E_{\alpha,\beta+\gamma+1}^{\delta}\left(-at^{\alpha}\right)$, and ${}_{C}D_{0+}^{\gamma}\left[t^{\beta}E_{\alpha,\beta+1}^{\delta}\left(-at^{\alpha}\right)\right]=t^{\beta-\gamma}E_{\alpha,\beta-\gamma+1}^{\delta}\left(-at^{\alpha}\right)$, $\alpha>0$, $\beta>0$, $\gamma>0$, $\delta>0$, a is a constant,[24,30] the average particle displacement and velocity become

$$\langle x(t)\rangle=x_0\left[1-\omega^2\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)}E_{\mu,(\mu+\nu)(n+1)+1}^{n+1}\left(-2\lambda t^{\mu}\right)\right]$$
$$+v_0\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)n+\nu}E_{\mu,(\mu+\nu)n+\nu+1}^{n+1}\left(-2\lambda t^{\mu}\right), \tag{35a}$$

$$\langle v(t)\rangle=v_0\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)n}E_{\mu,(\mu+\nu)n+1}^{n+1}\left(-2\lambda t^{\mu}\right)-\omega^2 x_0\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)n+\mu}E_{\mu,(\mu+\nu)n+\mu+1}^{n+1}\left(-2\lambda t^{\mu}\right), \tag{35b}$$

from where we find the asymptotic behaviors

$$\langle x(t)\rangle\simeq\begin{cases} x_0\left[1-\omega^2\frac{t^{\mu+\nu}}{\Gamma(1+\mu+\nu)}\right]+v_0\left[\frac{t^{\nu}}{\Gamma(1+\nu)}-2\lambda\frac{t^{\mu+\nu}}{\Gamma(1+\mu+\nu)}\right] & \text{for } t\to 0,\\ x_0E_{\nu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right)+\frac{v_0}{2\lambda}t^{\nu-\mu}E_{\nu,1+\nu-\mu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right)\simeq x_0\frac{2\lambda}{\omega^2}\frac{t^{-\nu}}{\Gamma(1-\nu)}+v_0\frac{t^{-\mu}}{\Gamma(1-\mu)} & \text{for } t\to\infty,\end{cases} \tag{36a}$$

$$\langle v(t)\rangle \simeq \begin{cases} v_0\left[1-2\lambda\frac{t^{\mu}}{\Gamma(1+\mu)}\right]-x_0\omega^2\left[\frac{t^{\mu}}{\Gamma(1+\mu)}-2\lambda\frac{t^{2\mu}}{\Gamma(1+2\mu)}\right] & \text{for } t\to 0,\\ \frac{v_0}{2\lambda}t^{-\mu}E_{\nu,1-\mu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right)-\frac{x_0\omega^2}{2\lambda}E_{\nu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right)\simeq\frac{v_0}{\omega^2}\frac{t^{-(\mu+\nu)}}{\Gamma(1-(\mu+\nu))}-x_0\frac{t^{-\nu}}{\Gamma(1-\nu)} & \text{for } t\to\infty. \end{cases} \tag{36b}$$

Here, we used the relation $E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}$.[24,30] Here, we note that same results as asymptotic behaviors (36a) and (36b) can be obtained by using Tauberian theorems.[21] For example, for the mean particle displacement $\langle x(t)\rangle$ in the long time limit $t \to \infty$, where according to the Tauberian theorem we analyze the behavior of the Laplace transform $\mathcal{L}[\langle x(t)\rangle] = \frac{x_0}{s}\left[1-\omega^2\hat{G}(s)\right] + v_0 s^{\mu-1}\hat{G}(s)$ for $s \to 0$, we obtain

$$\begin{aligned}\langle x(t)\rangle &= \mathcal{L}^{-1}\left[\frac{x_0}{s}\left(1-\frac{\omega^2}{s^{\mu+\nu}+2\lambda s^{\nu}+\omega^2}\right)+v_0\frac{s^{\mu-1}}{s^{\mu+\nu}+2\lambda s^{\nu}+\omega^2}\right]\\ &\simeq \mathcal{L}^{-1}\left[\frac{x_0}{s}\left(1-\frac{\omega^2}{2\lambda s^{\nu}+\omega^2}\right)+v_0\frac{s^{\mu-1}}{2\lambda s^{\nu}+\omega^2}\right]\\ &= \mathcal{L}^{-1}\left[x_0\frac{s^{\nu-1}}{s^{\nu}+\frac{\omega^2}{2\lambda}}+\frac{v_0}{2\lambda}\frac{s^{\mu-1}}{s^{\nu}+\frac{\omega^2}{2\lambda}}\right] = x_0E_{\nu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right)+\frac{v_0}{2\lambda}t^{\nu-\mu}E_{\nu,1+\nu-\mu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right).\end{aligned} \tag{37}$$

Similarly, for the short time limit $t \to 0$, i.e., according to the Tauberian theorem, when we analyze the behavior of $\mathcal{L}[\langle x(t)\rangle] = \frac{x_0}{s}\left[1-\omega^2\hat{G}(s)\right] + v_0 s^{\mu-1}\hat{G}(s)$ for $s \to \infty$, we obtain

$$\begin{aligned}\langle x(t)\rangle &\simeq \mathcal{L}^{-1}\left[\frac{x_0}{s}\left(1-\frac{\omega^2}{s^{\mu+\nu}+2\lambda s^{\nu}}\right)+v_0\frac{s^{\mu-1}}{s^{\mu+\nu}+2\lambda s^{\nu}}\right]\\ &= x_0\left[1-\omega^2 t^{\mu+\nu}E_{\mu,\mu+\nu+1}\left(-2\lambda t^{\mu}\right)\right]+v_0t^{\nu}E_{\mu,\nu+1}\left(-2\lambda t^{\mu}\right)\\ &\simeq x_0\left[1-\omega^2\frac{t^{\mu+\nu}}{\Gamma(1+\mu+\nu)}\right]+v_0\left[\frac{t^{\nu}}{\Gamma(1+\nu)}-2\lambda\frac{t^{\mu+\nu}}{\Gamma(1+\mu+\nu)}\right].\end{aligned} \tag{38}$$

Graphical representation of the mean particle displacement and velocity for $x_0 = 0$ and $v_0 = 1$, for different values of parameters, by using program package MATHEMATICA and series expansion of three parameter M-L function (29), is given in Figure 1. For the plots we use the first 100 terms from the series (35a) and (35b), and first 100 terms from the series expansion of three parameter M-L function, which was shown that are enough for graphical representation. The obtained plots were reproduced with much more terms from the series (35a) and (35b), and much more terms from the series expansion of three parameter M-L function. The force free case ($\omega = 0$) gives $\langle x(t)\rangle = v_0t^{\nu}E_{\mu,\nu+1}\left(-2\lambda t^{\mu}\right)$ and $\langle v(t)\rangle = v_0E_{\mu}\left(-2\lambda t^{\mu}\right)$, which for $\mu = \nu = 1$

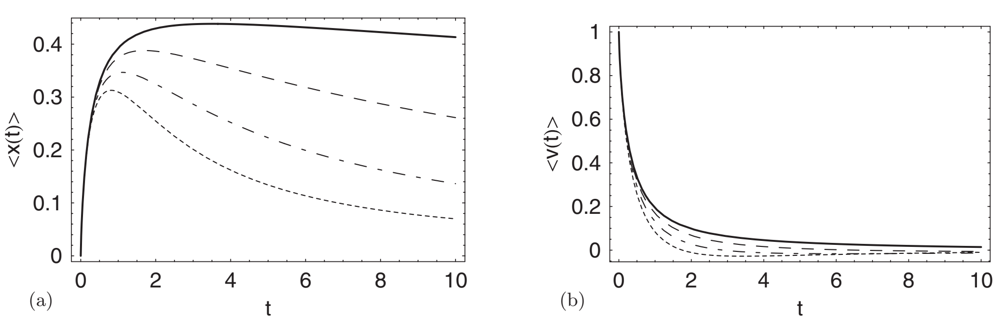

FIG. 1. Graphical representation of: (a) mean particle displacement (35a), (b) mean velocity (35b) for $v_0 = 1$, $x_0 = 0$, and Dirac delta frictional memory kernel. Parameters are as follows: $\mu = \nu = 3/4$, $\lambda = 1$; $\omega = 1/4$ (solid line), $\omega = 1/2$ (dashed line); $\omega = 3/4$ (dotted-dashed line); $\omega = 1$ (dotted line).

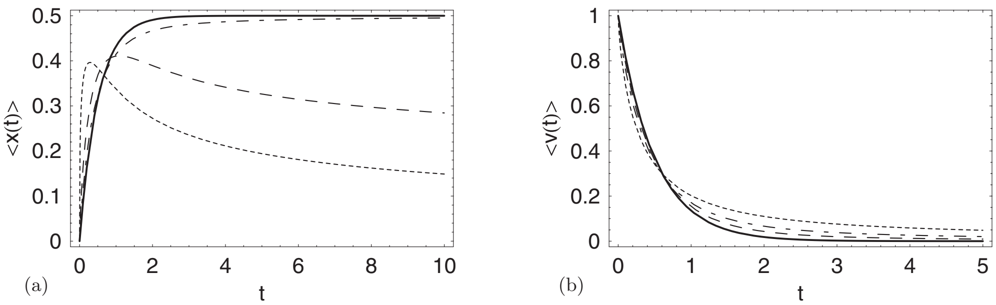

FIG. 2. Graphical representation of: (a) mean particle displacement (35a), (b) mean velocity (35b) for a free particle in case of $v_0 = 1, x_0 = 0$, and Dirac delta frictional memory kernel. Parameters are as follows: $\lambda = 1$; $\mu = \nu = 1$ (solid line)—normal Brownian motion, $\mu = 15/16$, $\nu = 3/4$ (dashed line); $\mu = \nu = 7/8$ (dotted-dashed line); $\mu = 3/4$, $\nu = 3/8$ (dotted line).

yields the known results for normal Brownian motion, $\langle x(t)\rangle = v_0 t E_{1,2}(-2\lambda t) = \frac{v_0}{2\lambda}\left(1 - e^{-2\lambda t}\right)$ and $\langle v(t)\rangle = v_0 E_1(-2\lambda t) = v_0 e^{-2\lambda t}$. We see that the mean velocity does not depend on the parameter ν in the force free case. The long time limit yields a power-law behavior, i.e., $\langle x(t)\rangle \simeq \frac{v_0}{2\lambda}\frac{t^{\nu-\mu}}{\Gamma(1+\nu-\mu)}$ and $\langle v(t)\rangle \simeq \frac{v_0}{2\lambda}\frac{t^{-\mu}}{\Gamma(1-\mu)}$. The case $\mu = \nu = \alpha > \frac{1}{2}$ for the mean particle displacement gives $\langle x(t)\rangle = v_0 t^{\alpha} E_{\alpha,\alpha+1}(-2\lambda t^{\alpha}) = \frac{v_0}{2\lambda}\left[1 - E_{\alpha}(-2\lambda t^{\alpha})\right]$, which in the long time limit $\langle x(t)\rangle \simeq \frac{v_0}{2\lambda}\left[1 - \frac{1}{2\lambda}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right]$ approaches the constant value $\frac{v_0}{2\lambda}$ following a power-law instead of the exponential approach in case of normal Brownian motion. The mean particle velocity shows slower (power-law) decay to zero instead of exponential decay in case of normal Brownian motion. These situations are represented in Figure 2. By substituting the relaxation functions in the general expressions for variances and MSD can be shown that anomalous diffusion occurs. For example, if we use $\mu = \nu$ and $\omega = 0$ we obtain that $\frac{\sigma_{xx}}{2k_BT} \simeq \frac{t^{2\nu-1}}{(2\lambda)(2\nu-1)[\Gamma(\nu)]^2}$, and thus $\sigma_{xx} \simeq t$ for $\mu = \nu = 1$, which are obtained in Ref. 31.

Let us now consider a power-law frictional memory kernel $\gamma(t) = C_\lambda \frac{t^{-\lambda}}{\Gamma(1-\lambda)}$, where $1 - \nu < \lambda < 1 + \mu$, and C_λ is a constant which depends on λ. We will investigate this special case, and the analysis for power-law memory kernel with different powers is straightforward. For example, another special case is when $\lambda \geq 1 + \mu$ or $\lambda \leq 1 - \nu$. By Laplace transform it follows $\hat{\gamma}(s) = C_\lambda s^{\lambda-1}$. From relations (17b), (17a), and (17c) we obtain

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left[\frac{s^{\nu}}{s^{\mu+\nu} + C_\lambda s^{\nu+\lambda-1} + \omega^2}\right] = \mathcal{L}^{-1}\left[\frac{s^{\nu}}{s^{\mu+\nu} + C_\lambda s^{\nu+\lambda-1}}\frac{1}{1 + \frac{\omega^2}{s^{\mu+\nu}+C_\lambda s^{\nu+\lambda-1}}}\right] \\ &= \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\left(-\omega^2\right)^n \frac{s^{\nu-(\nu+\lambda-1)(n+1)}}{\left(s^{\mu-\lambda+1} + C_\lambda\right)^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)-\nu-1} E^{n+1}_{\mu-\lambda+1,(\mu+\nu)(n+1)-\nu}\left(-C_\lambda t^{\mu-\lambda+1}\right), \end{aligned} \tag{39a}$$

$$G(t) = \sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)-1} E^{n+1}_{\mu-\lambda+1,(\mu+\nu)(n+1)}\left(-C_\lambda t^{\mu-\lambda+1}\right), \tag{39b}$$

$$I(t) = \sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)+\nu-1} E^{n+1}_{\mu-\lambda+1,(\mu+\nu)(n+1)+\nu}\left(-C_\lambda t^{\mu-\lambda+1}\right). \tag{39c}$$

Note that for $\lambda = 1 + \mu$ one obtains relaxation function $G(t) = \frac{t^{\mu+\nu-1}}{1+C_{1+\mu}} E_{\mu+\nu,\mu+\nu}\left(-\frac{\omega^2}{1+C_{1+\mu}} t^{\mu+\nu}\right)$ and for $\lambda = 1 - \nu$, $G(t) = t^{\mu+\nu-1} E_{\mu+\nu,\mu+\nu}(-(\omega^2 + C_{1-\nu}) t^{\mu+\nu})$.

Following same procedure as previous, by using the asymptotic expansion formula for three parameter M-L function for $t \to \infty$, relation (39c) yields

$$I(t) = \frac{t^{\lambda+2\nu-2}}{C_\lambda} E_{\lambda+\nu-1,\lambda+2\nu-1}\left(-\frac{\omega^2}{C_\lambda} t^{\lambda+\nu-1}\right). \tag{40}$$

For $t \to 0$, the relaxation function (39c) becomes

$$I(t) = t^{\mu+2\nu-1} E_{\mu-\lambda+1,\mu+2\nu}\left(-C_\mu t^{\mu-\lambda+1}\right). \tag{41}$$

From relations (39a)–(39c), in case of a free particle ($\omega = 0$), we obtain the following relaxation functions

$$\begin{aligned} g(t) &= \lim_{\omega\to 0}\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)-\nu-1} E_{\mu-\lambda+1,(\mu+\nu)(n+1)-\nu}^{n+1}\left(-C_\lambda t^{\mu-\lambda+1}\right) \\ &= t^{\mu-1} E_{\mu-\lambda+1,\mu}\left(-C_\lambda t^{\mu-\lambda+1}\right), \end{aligned} \tag{42a}$$

$$G(t) = t^{\mu+\nu-1} E_{\mu-\lambda+1,\mu+\nu}\left(-C_\lambda t^{\mu-\lambda+1}\right), \tag{42b}$$

$$I(t) = t^{\mu+2\nu-1} E_{\mu-\lambda+1,\mu+2\nu}\left(-C_\lambda t^{\mu-\lambda+1}\right), \tag{42c}$$

which will be used for investigation of the generalized Einstein relation.

For the average particle displacement and velocity (19), as previous, we obtain

$$\begin{aligned} \langle x(t)\rangle &= x_0\left[1-\omega^2\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)(n+1)} E_{\mu-\lambda+1,(\mu+\nu)(n+1)+1}^{n+1}\left(-C_\lambda t^{\mu-\lambda+1}\right)\right] \\ &\quad + v_0\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)n+\nu} E_{\mu-\lambda+1,(\mu+\nu)n+\nu+1}^{n+1}\left(-C_\lambda t^{\mu-\lambda+1}\right), \end{aligned} \tag{43a}$$

$$\begin{aligned} \langle v(t)\rangle &= v_0\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)n} E_{\mu-\lambda+1,(\mu+\nu)n+1}^{n+1}\left(-C_\lambda t^{\mu-\lambda+1}\right) \\ &\quad -\omega^2 x_0\sum_{n=0}^{\infty}(-\omega^2)^n t^{(\mu+\nu)n+\mu} E_{\mu-\lambda+1,(\mu+\nu)n+\mu+1}^{n+1}\left(-C_\lambda t^{\mu-\lambda+1}\right), \end{aligned} \tag{43b}$$

which yields the following asymptotic behaviors

$$\langle x(t)\rangle \simeq \begin{cases} x_0\left[1-\omega^2\frac{t^{\mu+\nu}}{\Gamma(1+\mu+\nu)}+\omega^2 C_\lambda\frac{t^{1+2\mu+\nu-\lambda}}{\Gamma(2+2\mu+\nu-\lambda)}\right]+v_0\left[\frac{t^{\nu}}{\Gamma(1+\nu)}-C_\lambda\frac{t^{1+\mu+\nu-\lambda}}{\Gamma(2+\mu+\nu-\lambda)}-\omega^2\frac{t^{\mu+2\nu}}{\Gamma(1+\mu+2\nu)}\right] & \text{for } t\to 0, \\ x_0 E_{\nu+\lambda-1}\left(-\frac{\omega^2}{C_\lambda}t^{\nu+\lambda-1}\right)+\frac{v_0 t^{-\mu}}{\omega^2}\left[\frac{1}{\Gamma(1-\mu)}-E_{\nu+\lambda-1,1-\mu}\left(-\frac{\omega^2}{C_\lambda}t^{\nu+\lambda-1}\right)\right] & \\ \simeq x_0\frac{C_\lambda}{\omega^2}\frac{t^{1-(\nu+\lambda)}}{\Gamma(2-(\nu+\lambda))}+\frac{v_0}{\omega^2}\frac{t^{-\mu}}{\Gamma(1-\mu)}-\frac{v_0 C_\lambda}{\omega^4}\frac{t^{1-(\mu+\nu+\lambda)}}{\Gamma(2-(\mu+\nu+\lambda))}, & \text{for } t\to\infty \end{cases} \tag{44a}$$

$$\langle v(t)\rangle \simeq \begin{cases} v_0\left[1-C_\lambda\frac{t^{1+\mu-\lambda}}{\Gamma(2+\mu-\lambda)}\right]-x_0\omega^2\left[\frac{t^{\mu}}{\Gamma(1+\mu)}-C_\lambda\frac{t^{1+2\mu-\lambda}}{\Gamma(2+2\mu-\lambda)}\right] & \text{for } t\to 0, \\ \frac{v_0 t^{-(\mu+\nu)}}{\omega^2}\left[\frac{1}{\Gamma(1-(\mu+\nu))}-E_{\nu+\lambda-1,1-\mu-\nu}\left(-\frac{\omega^2}{C_\lambda}t^{\nu+\lambda-1}\right)\right]-x_0 t^{-\nu}\left[\frac{1}{\Gamma(1-\nu)}-E_{\nu+\lambda-1,1-\nu}\left(-\frac{\omega^2}{C_\lambda}t^{\nu+\lambda-1}\right)\right] & \\ \simeq \frac{v_0}{\omega^2}\frac{t^{-(\mu+\nu)}}{\Gamma(1-(\mu+\nu))}-\frac{v_0 C_\lambda}{\omega^4}\frac{t^{1-(\mu+2\nu+\lambda)}}{\Gamma(2-(\mu+2\nu+\lambda))}-x_0\frac{t^{-\nu}}{\Gamma(1-\nu)}+\frac{x_0 C_\lambda}{\omega^2}\frac{t^{1-(2\nu+\lambda)}}{\Gamma(2-(2\nu+\lambda))} & \text{for } t\to\infty. \end{cases} \tag{44b}$$

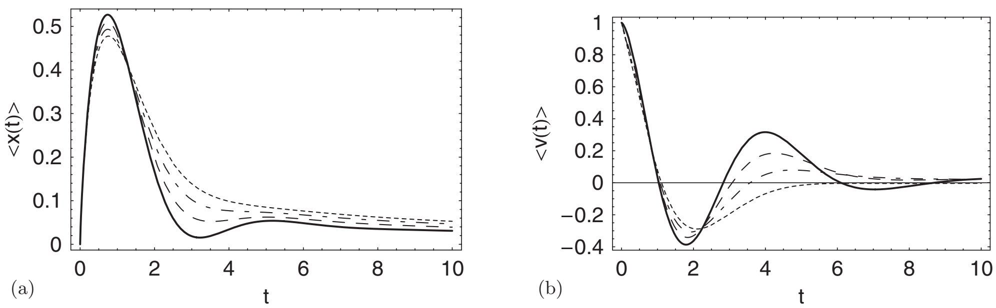

FIG. 3. Graphical representation of: (a) mean particle displacement (43a), (b) mean velocity (43b) for $v_0 = 1$, $x_0 = 0$, and power-law frictional memory kernel. Parameters are as follows: $C_\lambda = 1$, $\mu = \nu = 3/4$, $\omega = 1$; $\lambda = 3/9$ (solid line), $\lambda = 4/9$ (dashed line); $\lambda = 5/9$ (dotted-dashed line); $\lambda = 6/9$ (dotted line).

Graphical representation of the mean particle displacement and velocity for $x_0 = 0$ and $v_0 = 1$, for different values of parameters is given in Figure 3. The force free case ($\omega = 0$) for $x_0 = 0$ yields $\langle x(t)\rangle = v_0 t^{\nu} E_{\mu-\lambda+1,\nu+1}\left(-C_\lambda t^{\mu-\lambda+1}\right)$ and $\langle v(t)\rangle = v_0 E_{\mu-\lambda+1}\left(-C_\lambda t^{\mu-\lambda+1}\right)$, which in the long time limit turn to $\langle x(t)\rangle \simeq \frac{v_0}{C_\lambda}\frac{t^{\nu-(\mu-\lambda+1)}}{\Gamma(1+\nu-(\mu-\lambda+1))}$ and $\langle v(t)\rangle \simeq \frac{v_0}{C_\lambda}\frac{t^{-(\mu-\lambda+1)}}{\Gamma(1-(\mu-\lambda+1))}$. We see that the mean particle velocity does not depend on parameter ν for the force free case. Graphical representation of the force free case is given in Figure 4. Note that the case $\mu = \nu = 1$ (GLE with a power-law frictional memory kernel) yields the known results[64] for $t \to \infty$

$$\langle x(t)\rangle \simeq x_0\frac{C_\lambda}{\omega^2}\frac{t^{-\lambda}}{\Gamma(1-\lambda)} - \frac{v_0 C_\lambda}{\omega^4}\frac{t^{-(1+\lambda)}}{\Gamma(-\lambda)} = \frac{C_\lambda}{\omega^2}\frac{\sin(\lambda\pi)}{\pi}\left[x_0\frac{\Gamma(\lambda)}{t^\lambda} + \frac{v_0}{\omega^2}\frac{\Gamma(1+\lambda)}{t^{1+\lambda}}\right], \tag{45a}$$

$$\langle v(t)\rangle \simeq -\frac{v_0 C_\lambda}{\omega^4}\frac{t^{-(2+\lambda)}}{\Gamma(-(1+\lambda))} + \frac{x_0 C_\lambda}{\omega^2}\frac{t^{-(1+\lambda)}}{\Gamma(-\lambda)} = -\frac{C_\lambda}{\omega^2}\frac{\sin(\lambda\pi)}{\pi}\left[x_0\frac{\Gamma(1+\lambda)}{t^{1+\lambda}} + \frac{v_0}{\omega^2}\frac{\Gamma(2+\lambda)}{t^{2+\lambda}}\right], \tag{45b}$$

where we use $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}$, i.e., $\Gamma(-\alpha)\Gamma(1+\alpha) = -\frac{\pi}{\sin(\alpha\pi)}$. From Figures 3 and 4 we see that, by changing of parameters, various behaviors for the mean particle displacement and velocity occur. This frictional memory kernel also leads to anomalous diffusive processes which can be obtained from the variances and MSD.

2. Normalized displacement correlation function

One may consider the following conditions $x_0^2 = \frac{k_B T}{\omega^2}$, $\langle x_0 v_0\rangle = 0$ and $\langle \xi(t) x_0\rangle = 0$. Let us introduce the normalized displacement correlation function through the two-point correlation function

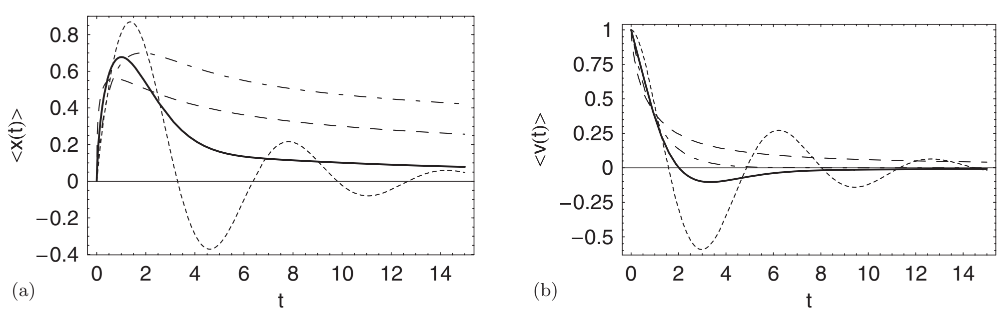

FIG. 4. Graphical representation of: (a) mean particle displacement (43a), (b) mean velocity (43b) for a free particle in case of $v_0 = 1$, $x_0 = 0$, and power-law frictional memory kernel. Parameters are as follows: $C_\lambda = 1$; $\mu = 3/4$, $\nu = 5/8$, $\lambda = 1/2$ (solid line), $\mu = 1/2$, $\nu = 3/8$, $\lambda = 3/4$ (dashed line), $\mu = 3/8$, $\nu = 3/4$, $\lambda = 3/8$ (dotted-dashed line); $\mu = 1$, $\nu = 7/8$, $\lambda = 1/4$ (dotted line).

$\langle x(t)x_0\rangle$, as $C_X(t)=\frac{\langle x(t)x_0\rangle}{\langle x_0^2\rangle}$.[4] It is an experimentally measured quantity, and was used, for example, in the analysis of experimental results for the fluctuations of the distance between fluorescein-tyrosine pair within a single protein.[45] From relation (16a), we obtain

$$\hat{C}_X(s)=\frac{s^{\mu+\nu-1}+s^{\nu-1}\hat{\gamma}(s)}{s^{\mu+\nu}+s^{\nu}\hat{\gamma}(s)+\omega^2}, \tag{46}$$

from where we find the following general expression

$$C_X(t)=1-\omega^2 I_{0+}^{1-\nu}I(t). \tag{47}$$

From relation (4) and by the definition of $C_X(t)$ it can be shown that the following fractional differential equation is satisfied

$${}_C D_{0+}^{\mu}\left[{}_C D_{0+}^{\nu}C_X(t)\right]+\int_0^t \gamma(t-t')\left[{}_C D_{0+}^{\nu}C_X(t')\right]\mathrm{d}t'+\omega^2 C_X(t)=0, \tag{48}$$

for initial conditions $C_X(0+)=1$ and ${}_C D_{0+}^{\nu}C_X(0+)=0$.

Let us first consider the Dirac delta noise $\gamma(t)=2\lambda\delta(t)$. Equation (48) becomes

$${}_C D_{0+}^{\mu}\left[{}_C D_{0+}^{\nu}C_X(t)\right]+2\lambda\left[{}_C D_{0+}^{\nu}C_X(t)\right]+\omega^2 C_X(t)=0. \tag{49}$$

By using Laplace transform of Caputo derivative (6) we obtain

$$s^{\mu}\left[s^{\nu}\hat{C}_X(s)-s^{\nu-1}C_X(0+)\right]-s^{\mu-1}{}_C D_{0+}^{\nu}C_X(0+)+2\lambda\left[s^{\nu}\hat{C}_X(s)-s^{\nu-1}C_X(0+)\right]+\omega^2\hat{C}_X(s)=0, \tag{50}$$

from where, by using initial conditions, it follows

$$\hat{C}_X(s)=\frac{s^{\mu+\nu-1}+2\lambda s^{\nu-1}}{s^{\mu+\nu}+2\lambda s^{\nu}+\omega^2}. \tag{51}$$

Inverse Laplace transform yields the solution of Eq. (49) given by

$$\begin{aligned} C_X(t) &= 1+\sum_{n=0}^{\infty}(-\omega^2)^{n+1}t^{(\mu+\nu)(n+1)}E_{\mu,(\mu+\nu)(n+1)+1}^{n+1}\left(-2\lambda t^{\mu}\right) \\ &= \sum_{n=0}^{\infty}(-\omega^2)^{n}t^{(\mu+\nu)n}E_{\mu,(\mu+\nu)n+1}^{n}\left(-2\lambda t^{\mu}\right), \end{aligned} \tag{52}$$

which satisfies the initial conditions, and which convergence is proven in Ref. 54. The long time limit yields $C_X(t)\simeq E_{\nu}\left(-\frac{\omega^2}{2\lambda}t^{\nu}\right)\simeq\frac{2\lambda}{\omega^2}\frac{t^{-\nu}}{\Gamma(1-\nu)}$, from where we see that $C_X(t)$ does not depend on parameter μ. In the short time limit $C_X(t)$ behaves as $C_X(t)\simeq 1-\omega^2\frac{t^{\mu+\nu}}{\Gamma(1+\mu+\nu)}$.

Remark 1 Note that if we use the procedure as in Sec. II, we can show that Eq. (49) can be transformed to more suitable form

$$\ddot{C}_X(t)+2\lambda\left[{}_C D_{0+}^{2-\mu}C_X(t)\right]+\omega^2\left[{}_C D_{0+}^{2-(\mu+\nu)}C_X(t)+\frac{t^{\mu+\nu-2}}{\Gamma(\mu+\nu-1)}\right]=0, \tag{53}$$

with initial conditions in classical form $\dot{C}_X(0+)=0$ *and* $C_X(0+)=1$. *Thus, the second term represents the memory effects of the environment, and the third term gives generalized force which acts on the particle.*

Remark 2 Here, we consider a special case $\mu=\nu=\alpha>\frac{1}{2}$ *which yields some interesting results for infinite series in three parameter M-L functions. As we will show, some infinite series in three parameter M-L functions can be simplified and represented through one parameter and two parameter M-L functions. For this special case, normalized displacement correlation function* *becomes*

$$C_X(t) = \sum_{n=0}^{\infty}(-\omega^2)^n t^{2\alpha n} E^n_{\alpha,2\alpha n+1}\left(-2\lambda t^{\alpha}\right). \tag{54}$$

From the other side we can write

$$\hat{C}_X(s) = \frac{s^{2\alpha-1}+2\lambda s^{\alpha-1}}{s^{2\alpha}+2\lambda s^{\alpha}+\omega^2} = \begin{cases} s^{-1} - \frac{r_1 r_2}{r_1-r_2}\left(\frac{s^{-1}}{s^{\alpha}-r_1} - \frac{s^{-1}}{s^{\alpha}-r_2}\right) & \text{if } \lambda \neq \omega, \\ \frac{s^{2\alpha-1}+2\omega s^{\alpha-1}}{(s^{\alpha}+\omega)^2} & \text{if } \lambda = \omega, \end{cases} \tag{55}$$

where $r_{1/2} = -\lambda \pm \sqrt{\lambda^2-\omega^2}$ *are roots of* $s^{2\alpha}+2\lambda s^{\alpha}+\omega^2 = (s^{\alpha}-r_1)(s^{\alpha}-r_2) = 0$, *and thus* $r_1 - r_2 = 2\sqrt{\lambda^2-\omega^2}$, $r_1 + r_2 = -2\lambda$, $r_1 r_2 = \omega^2$. *From relation* (55) for $C_X(t)$ *we obtain*

$$C_X(t) = \begin{cases} 1 - \frac{r_1 r_2 t^{\alpha}}{r_1-r_2}\left[E_{\alpha,\alpha+1}\left(r_1 t^{\alpha}\right) - E_{\alpha,\alpha+1}\left(r_2 t^{\alpha}\right)\right] = \frac{r_1 E_{\alpha}(r_2 t^{\alpha}) - r_2 E_{\alpha}(r_1 t^{\alpha})}{r_1-r_2} & \text{if } \lambda \neq \omega, \\ E^2_{\alpha,1}\left(-\omega t^{\alpha}\right) + 2\omega t^{\alpha} E^2_{\alpha,1+\alpha}\left(-\omega t^{\alpha}\right) = E_{\alpha}\left(-\omega t^{\alpha}\right) + \frac{\omega t^{\alpha}}{\alpha} E_{\alpha,\alpha}\left(-\omega t^{\alpha}\right) & \text{if } \lambda = \omega. \end{cases} \tag{56}$$

Thus, we obtain that the following relations holds true

$$\sum_{n=0}^{\infty}(-r_1 r_2 t^{2\alpha})^n E^n_{\alpha,2\alpha n+1}\left((r_1+r_2)\, t^{\alpha}\right) = \frac{r_1 E_{\alpha}\left(r_2 t^{\alpha}\right) - r_2 E_{\alpha}\left(r_1 t^{\alpha}\right)}{r_1 - r_2}, \tag{57}$$

i.e.,

$$\sum_{n=0}^{\infty}(-xy)^n E^n_{\alpha,2\alpha n+1}\left(x+y\right) = \frac{x E_{\alpha}\left(y\right) - y E_{\alpha}\left(x\right)}{x-y}, \tag{58}$$

where $x = r_1 t^{\alpha}$, $y = r_2 t^{\alpha}$, *and*

$$\sum_{n=0}^{\infty}(-\omega^2 t^{2\alpha})^n E^n_{\alpha,2\alpha n+1}\left(-2\omega t^{\alpha}\right) = E_{\alpha}\left(-\omega t^{\alpha}\right) + \frac{\omega t^{\alpha}}{\alpha} E_{\alpha,\alpha}\left(-\omega t^{\alpha}\right), \tag{59}$$

i.e.,

$$\sum_{n=0}^{\infty}(-x^2)^n E^n_{\alpha,2\alpha n+1}\left(2x\right) = E_{\alpha}\left(x\right) - \frac{x}{\alpha} E_{\alpha,\alpha}\left(x\right), \tag{60}$$

where $x = -\omega t^{\alpha}$. *Note that relations* (58) and (60) *can be obtained by using following formulas* $\sum_{n=0}^{\infty}(-xy)^n E^{n+1}_{\alpha,2\alpha n+\beta}\left(x+y\right) = \frac{x E_{\alpha,\beta}(x) - y E_{\alpha,\beta}(y)}{x-y}$ *for* $x \neq y$, *and* $\sum_{n=0}^{\infty}(-x^2)^n E^{n+1}_{\alpha,2\alpha n+\beta}\left(2x\right) = E_{\alpha,\beta}\left(x\right) + x\frac{\mathrm{d}}{\mathrm{d}x} E_{\alpha,\beta}\left(x\right)$, *consequently.*[59] *Let us show these. From relation* (52), *for the case of different roots we obtain*

$$\begin{aligned} 1 - xy\sum_{n=0}^{\infty}(-xy)^n E^{n+1}_{\alpha,2\alpha n+2\alpha+1}\left(x+y\right) &= 1 - \frac{xy}{x-y}\left[x E_{\alpha,2\alpha+1}(x) - y E_{\alpha,2\alpha+1}(y)\right] \\ &= \frac{x E_{\alpha}\left(y\right) - y E_{\alpha}\left(x\right)}{x-y} \end{aligned} \tag{61}$$

and

$$\begin{aligned} 1 - x^2\sum_{n=0}^{\infty}(-x^2)^n E^{n+1}_{\alpha,2\alpha n+2\alpha+1}\left(2x\right) &= 1 - x^2\left[E_{\alpha,2\alpha+1}(x) + x\frac{\mathrm{d}}{\mathrm{d}x} E_{\alpha,2\alpha+1}\left(x\right)\right] \\ &= E_{\alpha}(x) - \frac{x}{\alpha} E_{\alpha,\alpha}\left(x\right), \end{aligned} \tag{62}$$

for the case of equal roots, where we used some basic relations for the M-L functions.[24]

Graphical representation of the normalized displacement correlation function is given in Figure 5. From the plots of Figs. 5(a)–5(c) we see that for different values of frequency ω and

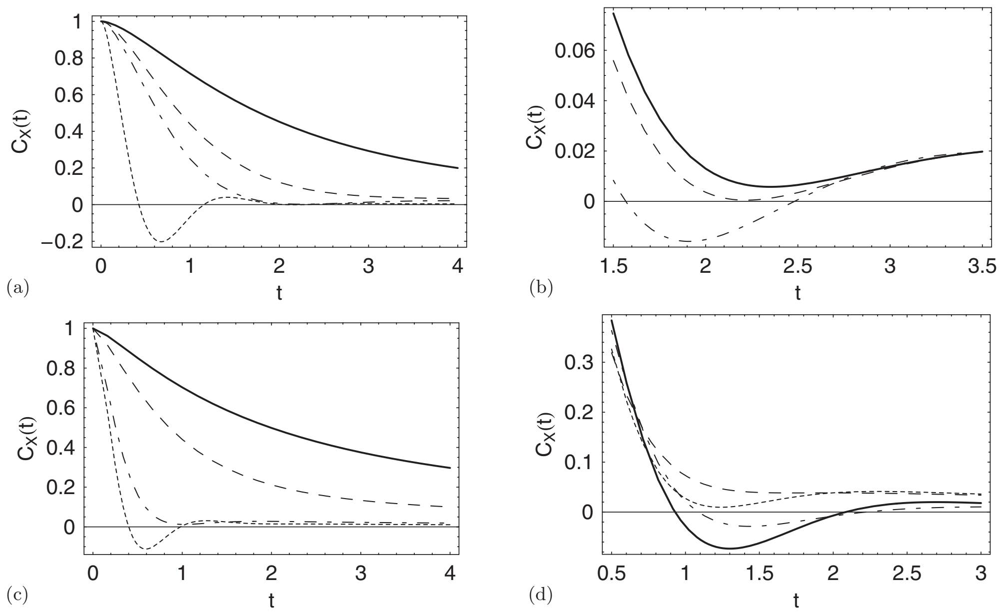

FIG. 5. Graphical representation of normalized displacement correlation function (52) in case of Dirac delta frictional memory kernel; $\lambda = 1$; (a) $\mu = \nu = 7/8$, $\omega = 1$ (solid line), $\omega = 3/2$ (dashed line), $\omega = 59/32$ (dotted-dashed line); $\omega = 4$ (dotted line), (b) $\mu = \nu = 7/8$, $\omega = 29/16$ (solid line), $\omega = 119/64$ (dashed line), $\omega = 2$ (dotted-dashed line), (c) $\mu = \nu = 3/4$, $\omega = 1$ (solid line), $\omega = 3/2$ (dashed line), $\omega = 3$ (dotted-dashed line); $\omega = 4$ (dotted line), (d) $\omega = 5/2$; $\mu = \nu = 7/8$ (solid line), $\mu = \nu = 3/4$ (dashed line), $\mu = 3/4$, $\nu = 7/8$ (dotted-dashed line), $\mu = 7/8$, $\nu = 3/4$ (dotted line).

given values of μ and ν, the normalized displacement correlation function has different behaviors, from monotonic decay without crossing the zero line, oscillation-like behavior crossing the zero line, as well as non-monotonic decay approaching the zero line without crossing it. These results are different than the one for classical harmonic oscillator, where only two different behaviors may occur: overdamped motion for which $\langle x(t)\rangle > 0$ for any time t when $\langle x_0\rangle > 0$ and there are no oscillations, and underdamped motion, where $\langle x(t)\rangle$ crosses the zero line and oscillates.[4] Frequency on which transition from overdamped to underdamped motion appears is so-called critical frequency. In fractional models, there are additional definitions of critical frequencies.[4] These are frequencies on which the oscillator changes its behavior, for example, from monotonic to non-monotonic decay without crossings of zero line, or the frequency on which crossings of zero line appear. From the plot of Fig. 5(d) we see that for different values of μ and ν and constant frequency ω the oscillator may have different behavior. Note that in the long time limit the normalized displacement correlation function $C_X(t) \simeq E_\nu\left(-\frac{\omega^2}{2\lambda}t^\nu\right) \simeq \frac{2\lambda}{\omega^2}\frac{t^{-\nu}}{\Gamma(1-\nu)}$ is a completely monotone function, due to the fact that the one parameter M-L function $E_\alpha(-t^\alpha)$ is a completely monotone function for $0 < \alpha < 1$ (see, for example, Ref. 7).

For the power-law frictional memory kernel $\gamma(t) = C_\lambda \frac{t^{-\lambda}}{\Gamma(1-\lambda)}$ Eq. (48) becomes

$$ {}_C D_{0+}^{\mu}\left[{}_C D_{0+}^{\nu} C_X(t)\right] + C_\lambda I_{0+}^{1-\lambda}\left[{}_C D_{0+}^{\nu} C_X(t)\right] + \omega^2 C_X(t) = 0. \tag{63} $$

From the Laplace transform of Caputo derivative (6), by using the initial conditions $C_X(0+) = 1$ and ${}_C D_{0+}^{\nu} C_X(0+) = 0$, we obtain

$$ s^\mu\left[s^\nu \hat{C}_X(s) - s^{\nu-1}\right] + C_\lambda s^{\lambda-1}\left[s^\nu \hat{C}_X(s) - s^{\nu-1}\right] + \omega^2 \hat{C}_X(s) = 0, \tag{64} $$

from where it follows

$$ \hat{C}_X(s) = \frac{s^{\mu+\nu-1} + C_\lambda s^{\nu+\lambda-2}}{s^{\mu+\nu} + C_\lambda s^{\nu+\lambda-1} + \omega^2}. \tag{65} $$

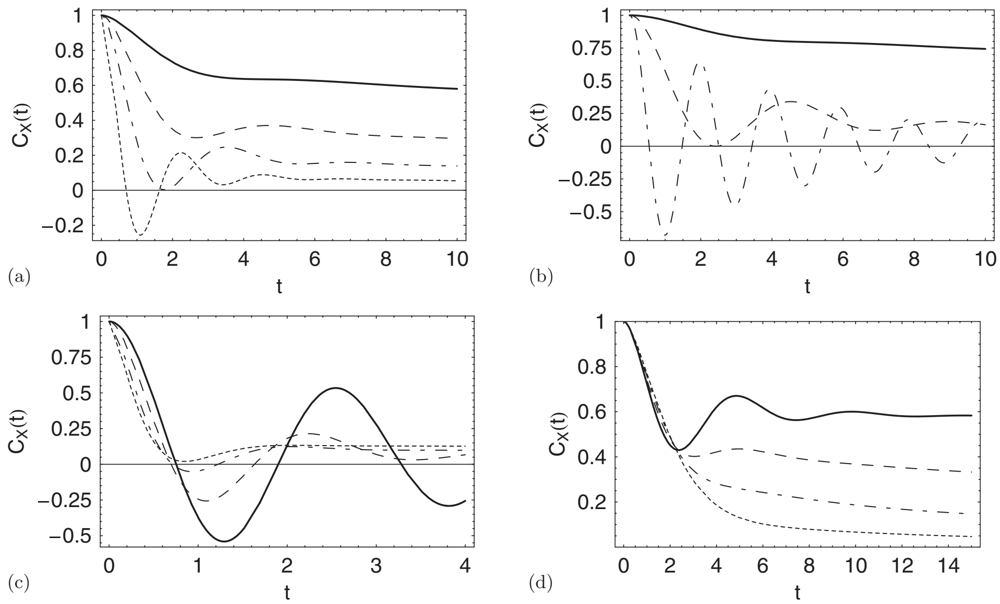

FIG. 6. Graphical representation of normalized displacement correlation function (66) in case of power-law frictional memory kernel; $C_\lambda = 1$; (a) $\mu = \nu = 7/8$, $\lambda = 1/2$, $\omega = 1/2$ (solid line), $\omega = 7/8$ (dashed line), $\omega = 11/8$ (dot-dashed line); $\omega = 9/4$ (dotted line), (b)[4] $\mu = \nu = 1$, $\omega = 0.3$ (solid line), $\omega = 1.053$ (dashed line), $\omega = 3$ (dot-dashed line), (c) $\omega = 9/4$, $\lambda = 1/2$, $\mu = \nu = 1$ (solid line), $\mu = \nu = 7/8$ (dashed line), $\mu = \nu = 3/4$ (dot-dashed line), $\mu = 3/4$, $\nu = 5/8$ (dotted line), (d) $\mu = \nu = 7/8$, $\omega = 1$, $\lambda = 1/5$ (solid line), $\lambda = 1/2$ (dashed line), $\lambda = 3/4$ (dot-dashed line), $\lambda = 15/16$ (dotted line).

The inverse Laplace transform of (65) yields the solution

$$\begin{aligned} C_X(t) &= 1 + \sum_{n=0}^{\infty}(-\omega^2)^{n+1} t^{(\mu+\nu)(n+1)} E^{n+1}_{\mu-\lambda+1,(\mu+\nu)(n+1)+1}\left(-C_\lambda t^{\mu-\lambda+1}\right) \\ &= \sum_{n=0}^{\infty}(-\omega^2)^{n} t^{(\mu+\nu)n} E^{n}_{\mu-\lambda+1,(\mu+\nu)n+1}\left(-C_\lambda t^{\mu-\lambda+1}\right), \end{aligned} \tag{66}$$

which satisfies the initial conditions, and which convergence is proven in Ref. 54. The long time limit yields power-law decay of normalized displacement correlation function $C_X(t) \simeq E_{\nu+\lambda-1}\left(-\frac{\omega^2}{C_\lambda} t^{\nu+\lambda-1}\right) \simeq \frac{C_\lambda}{\omega^2}\frac{t^{-(\nu+\lambda-1)}}{\Gamma(1-(\nu+\lambda-1))}$, and for the short time limit it is obtained $C_X(t) \simeq 1 - \omega^2\frac{t^{\mu+\nu}}{\Gamma(1+\mu+\nu)} + C_\lambda\omega^2\frac{t^{1+2\mu+\nu-\lambda}}{\Gamma(2+2\mu+\nu-\lambda)}$. The case $\mu = \nu = 1$ recovers the results obtained in Ref. 4, i.e., $C_X(t) \simeq \frac{C_\lambda}{\omega^2}\frac{t^{-\lambda}}{\Gamma(1-\lambda)}$ for $t \to \infty$ and $C_X(t) \simeq 1 - \omega^2\frac{t^2}{2} + C_\lambda\omega^2\frac{t^{4-\lambda}}{\Gamma(5-\lambda)}$ for $t \to 0$. As a special case, one can consider $\lambda = 1 - \frac{\nu-\mu}{2}$. In this case $C_X(t)$ is given by (56) where $\alpha \to \frac{\mu+\nu}{2}$ and $\lambda \to \frac{C_\lambda}{2}$. The case $\mu = \nu = 1$, i.e., $\ddot{C}_X(t) + C_{\lambda\,C}D^{\lambda}_{0+}C_X(t) + \omega^2 C_X(t) = 0$ corresponds to that considered by Burov and Barkai,[4] where they investigated the overdamped, underdamped, and critical behaviors of such fractional Langevin equation by using the method of complementary polynomials. Graphical representation of the normalized displacement correlation function is given in Figure 6. From plot (a) of Fig. 6 we see that there are different type behaviors of $C_X(t)$, such as monotonic decay, non-monotonic decay, oscillation-like behavior without and with crossings of the zero line. In plot (b) of Fig. 6 we present the results obtained by Burov and Barkai[4] for fractional Langevin equation ($\mu = \nu = 1$). Plot (c) of Fig. 6 shows that by decreasing parameters μ and ν, for fixed λ and frequency ω, the normalized displacement correlation function from behavior with zero crossings may turn to behavior without zero crossings. Changes in behavior of $C_X(t)$, from oscillation-like behavior without zero crossings to non-monotonic and monotonic decay, for fixed μ, ν, and ω, by increasing parameter λ, are shown in plot (d) of Fig. 6.

Such oscillations of the $C_X(t)$, as shown in Figures 5 and 6, were observed in the molecular dynamic simulations of fluctuations of donor-acceptor distance for a single protein,[38] and a power-law decay of $C_X(t)$ in the distance between fluorescein-tyrosine pair within a single protein.[45]

Remark 3 Same as in Remark 1, Eq. (63) can be transformed to

$$\ddot{C}_X(t) + C_\lambda \left[{}_C D_{0+}^{1+\lambda-\mu} C_X(t) \right] + \omega^2 \left[{}_C D_{0+}^{2-(\mu+\nu)} C_X(t) + \frac{t^{\mu+\nu-2}}{\Gamma(\mu+\nu-1)} \right] = 0, \tag{67}$$

with initial conditions $\dot{C}_X(0+) = 0$ *and* $C_X(0+) = 1$. *Again, second term represents the memory effects of the environment, and the third term gives the generalized force acting on the particle. Thus, normalized displacement correlation function can be considered in a same way as in the classical case, taking into account the memory effect of the complex environment on particle movement, and more general form of the potential energy function (different from the harmonic potential approximation), which gives the confined movement of the particle.*

3. External noise case

In the case of external noise, the fluctuation-dissipation theorem (3) does not hold and we cannot use relations (21a)–(21c). In this case, we proceed as follows. Let us consider a power-law correlation function $C(t) = C_\theta t^{-\theta}/\Gamma(1-\theta)$, where $0 < \theta < 1$, and a power-law friction kernel $\gamma(t) = C_\lambda t^{-\lambda}/\Gamma(1-\lambda)$, where $1-\nu < \lambda < 1+\mu$. For the correlations we obtain

$$\sigma_{xx} = 2\int_0^t \mathrm{d}t_1 G(t_1) \int_0^{t_1} \mathrm{d}t_2 G(t_2) C(t_1 - t_2) = 2C_\theta \int_0^t \mathrm{d}\xi\, G(\xi) I_{0+}^{1-\theta} G(\xi), \tag{68a}$$

$$\sigma_{xv} = \int_0^t \mathrm{d}t_1 g(t_1) \int_0^t \mathrm{d}t_2 G(t_2) C(t_1 - t_2) = C_\theta \int_0^t \mathrm{d}\xi \left(G(\xi) I_{0+}^{1-\theta} g(\xi) + g(\xi) I_{0+}^{1-\theta} G(\xi) \right), \tag{68b}$$

$$\sigma_{vv} = 2\int_0^t \mathrm{d}t_1 g(t_1) \int_0^{t_1} \mathrm{d}t_2 g(t_2) C(t_1 - t_2) = -2C_\theta \int_0^t \mathrm{d}\xi\, g(\xi) I_{0+}^{1-\theta} g(\xi), \tag{68c}$$

where $g(t)$, $G(t)$, and $I(t)$ are given by (39a)–(39c), respectively. For the fractional integrals of relaxation functions, needed to calculate correlations, we find

$$I_{0+}^{1-\theta} g(t) = \sum_{n=0}^{\infty} (-\omega^2)^n t^{(\mu+\nu)(n+1)-\nu-\theta} E_{\mu-\lambda+1,(\mu+\nu)(n+1)-\nu-\theta+1}^{n+1} \left(-C_\lambda t^{\mu-\lambda+1} \right), \tag{69a}$$

$$I_{0+}^{1-\theta} G(t) = \sum_{n=0}^{\infty} (-\omega^2)^n t^{(\mu+\nu)(n+1)-\theta} E_{\mu-\lambda+1,(\mu+\nu)(n+1)-\theta+1}^{n+1} \left(-C_\lambda t^{\mu-\lambda+1} \right), \tag{69b}$$

where we use the formula $I_{0+}^{\gamma} \left[t^{\beta} E_{\alpha,\beta+1}^{\delta}(-at^{\alpha}) \right] = t^{\beta+\gamma} E_{\alpha,\beta+\gamma+1}^{\delta}(-at^{\alpha})$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, a is a constant.[24] To find analytical expressions for the correlations from these results is a nontrivial problem, and can be achieved by using the formula for a product of two M-L functions.[60] By using the asymptotic expansion formula for the three parameter M-L function, in the long time limit we obtain

$$I_{0+}^{1-\theta} G(t) = \frac{t^{\nu+\lambda-1-\theta}}{C_\lambda} E_{\nu+\lambda-1,\nu+\lambda-\theta} \left(-\frac{\omega^2}{C_\lambda} t^{\nu+\lambda-1} \right) \simeq \frac{1}{\omega^2} \frac{t^{-\theta}}{\Gamma(1-\theta)}. \tag{69c}$$

From relations (69c) and (68a) it follows that $\sigma_{xx} \simeq t^{1-(\nu+\lambda+\theta)}$ for $t \to \infty$. For the short time limit, we obtain

$$I_{0+}^{1-\theta} G(t) = t^{\mu+\nu-\theta} E_{\mu-\lambda+1,\mu+\nu-\theta+1} \left(-C_\lambda t^{\mu-\lambda+1} \right) \simeq \frac{t^{\mu+\nu-\theta}}{\Gamma(1+\mu+\nu-\theta)}, \tag{69d}$$

so relations (69d) and (68a) yield $\sigma_{xx} \simeq t^{2(\mu+\nu)-\theta}$ for $t \to 0$. Note that the case of a free particle ($\omega = 0$) yields

$$I_{0+}^{1-\theta} g(t) = t^{\mu-\theta} E_{\mu-\lambda+1,\mu-\theta+1}\left(-C_\lambda t^{\mu-\lambda+1}\right), \tag{69e}$$

$$I_{0+}^{1-\theta} G(t) = t^{\mu+\nu-\theta} E_{\mu-\lambda+1,\mu+\nu-\theta+1}\left(-C_\lambda t^{\mu-\lambda+1}\right), \tag{69f}$$

which for $\nu = 1$ are equivalent to those obtained in Ref. 17.

From relations (68a) and (69f), and by help of the asymptotic expansion formula of the M-L function, in the long time limit $t \to \infty$ we recover the following results by Lim and Teo[37]

$$\sigma_{xx} \simeq t^{2\lambda-\theta+2\nu-2}, \quad 2\lambda - \theta + 2\nu - 2 > 0, \tag{70a}$$

$$\sigma_{xx} \simeq \ln(t), \quad 2\lambda - \theta + 2\nu - 2 = 0, \tag{70b}$$

$$\sigma_{xx} \simeq \text{const}, \quad 2\lambda - \theta + 2\nu - 2 < 0. \tag{70c}$$

Note that the variance σ_{xx} in the long time limit does not depend on μ, but it depends on ν. The case $\nu = 1$ corresponds to the results obtained by Fa,[17]

$$\sigma_{xx} \simeq t^{2\lambda-\theta}, \quad 2\lambda - \theta > 0, \tag{71a}$$

$$\sigma_{xx} \simeq \ln(t), \quad 2\lambda - \theta = 0, \tag{71b}$$

$$\sigma_{xx} \simeq \text{const}, \quad 2\lambda - \theta < 0. \tag{71c}$$

Thus, for given values of parameters a logarithmic dependence of the variance on time is obtained. Such processes are known as ultraslow processes.[12]

D. Overdamped limit

Let us assume that our harmonic oscillator moves in the high friction limit, i.e., is subject to strong viscous damping. This means that the inertial term ${}_C D_{0+}^{\mu} v(t)$ can be neglected, and we are concerned with

$$\int_0^t \gamma(t-t')v(t')\mathrm{d}t' + \omega^2 x(t) = \xi(t), \quad {}_C D_{0+}^{\nu} x(t) = v(t). \tag{72}$$

Such overdamped behavior appears naturally on microscopic scales in water-like environments, for instance, when we consider slow protein conformation fluctuations.[9] Following the same procedure as above, by help of the Laplace transform method we obtain

$$x(t) = \langle x(t)\rangle + \int_0^t G_0(t-t')\xi(t')\mathrm{d}t', \tag{73a}$$

$$v(t) = \langle v(t)\rangle + \int_0^t g_0(t-t')\xi(t')\mathrm{d}t', \tag{73b}$$

where

$$\langle x(t)\rangle = x_0\left[1 - \omega^2 I_{0+}^{1} G_0(t)\right], \quad \langle v(t)\rangle = -\omega^2 x_0 I_{0+}^{1-\nu} G_0(t). \tag{73c}$$

Here, we introduced

$$G_0(t) = \mathcal{L}^{-1}\left[\hat{G}_0(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{\nu}\hat{\gamma}(s) + \omega^2}\right], \tag{73d}$$

such that $G_0(0) = 0$, $I_0(t) = \mathcal{L}^{-1}\left[s^{-\nu}\hat{G}_0(s)\right]$, i.e., ${}_C D_{0+}^{\nu} I_0(t) = G_0(t)$, and ${}_C D_{0+}^{\nu} G_0(t) = g_0(t)$. In a same way as in (21a)–(21c), for the correlators we find

$$\sigma_{xx} = 2k_B T \int_0^t \mathrm{d}\xi\, G_0(\xi) \left[\frac{\xi^{\nu-1}}{\Gamma(\nu)} - \omega^2 I_0(\xi)\right], \tag{74a}$$

$$\sigma_{xv} = k_B T \left[\frac{1}{\Gamma(\nu)} \int_0^t \mathrm{d}\xi\, g_0(\xi)\xi^{\nu-1} - \omega^2 \int_0^t \mathrm{d}\xi \left(G_0^2(\xi) + g_0(\xi) I_0(\xi)\right)\right], \tag{74b}$$

$$\sigma_{vv} = \langle v^2(t)\rangle - \langle v(t)\rangle^2 = -2k_B T \omega^2 \int_0^t \mathrm{d}\xi\, G_0(\xi) g_0(\xi). \tag{74c}$$

Note that if we consider δ-shaped friction kernel, $\gamma(t) = 2\lambda\delta(t)$, we obtain for the relaxation functions

$$g_0(t) = \frac{1}{2\lambda t} E_{\nu,0}\left(-\frac{\omega^2}{2\lambda} t^{\nu}\right), \quad G_0(t) = \frac{t^{\nu}}{2\lambda t} E_{\nu,\nu}\left(-\frac{\omega^2}{2\lambda} t^{\nu}\right), \quad I_0(t) = \frac{t^{2\nu}}{2\lambda t} E_{\nu,2\nu}\left(-\frac{\omega^2}{2\lambda} t^{\nu}\right). \tag{75}$$

These results are equivalent to those obtained from the full model (Eq. (32)) in the long time limit when inertial effects become negligible. Thus, we can conclude that in the long time limit instead of the FGLE (4) we can investigate the corresponding overdamped motion (72), in analogy to ordinary diffusion. The same situations occur in case of power-law friction kernel, i.e., the relaxation functions in the overdamped limit are equivalent to Eq. (40).

Let us now consider the case when we use the three parameter M-L friction kernel[51,54]

$$C(t) = \frac{C_{\alpha,\beta,\delta}}{\tau^{\alpha\delta}} t^{\beta-1} E_{\alpha,\beta}^{\delta}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right), \tag{76}$$

where τ is a characteristic time scale, $C_{\alpha,\,\beta,\,\delta}$ is a proportionality coefficient independent of time, and we observe the restrictions $\alpha > 0$, $\beta > 0$, $\delta > 0$. The noise correlator (76) satisfies the condition $\lim_{t\to\infty}\gamma(t) = \lim_{s\to 0} s\hat{\gamma}(s) = 0$[54] for $\beta < 1 + \alpha\delta$. This memory kernel is a very useful tool to generate various different behaviors of the particle, sub-diffusion, super-diffusion, or normal diffusion.[51,54] The complete monotonicity of function of form (76) is discussed by Capelas de Oliveira *et al.*[7]

For the relaxation function $G_0(t)$, we obtain

$$G_0(t) = \frac{1}{\omega^2} \sum_{k=0}^{\infty} \left(-\frac{\gamma_{\alpha,\beta,\delta}}{\omega^2}\right)^k t^{(\beta-\nu)k-1} E_{\alpha,(\beta-\nu)k}^{\delta k}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right), \tag{77}$$

where $\gamma_{\alpha,\,\beta,\,\delta} = C_{\alpha,\,\beta,\,\delta}/[k_B T \tau^{\alpha\delta}]$. The convergence of series in three parameter M-L functions of form (77) is proven in Ref. 54. Here, we use the Laplace transform of the three-parameter M-L function (30) and the following Laplace transform formula[62]

$$\frac{s^{\mu(\alpha-1)}}{s^{\alpha} + \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}+\nu)^{\gamma}}\right]} = \mathcal{L}\left[\sum_{k=0}^{\infty} (-\lambda)^k t^{2\alpha k+\alpha+\mu-\mu\alpha-1} E_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}^{\gamma k}\left(-\nu t^{\rho}\right)\right](s). \tag{78}$$

The long time limit produces

$$\begin{aligned} G_0(t) &= \frac{t^{-1}}{\omega^2} E_{\beta-\nu-\alpha\delta,0}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T \omega^2} t^{\beta-\nu-\alpha\delta}\right) \\ &= \frac{k_B T t^{\nu+\alpha\delta-\beta-1}}{C_{\alpha,\beta,\delta}} E_{\nu+\alpha\delta-\beta,\nu+\alpha\delta-\beta}\left(-\frac{k_B T \omega^2}{C_{\alpha,\beta,\delta}} t^{\nu+\alpha\delta-\beta}\right), \end{aligned} \tag{79}$$

where $\nu + \alpha\delta - \beta > 0$, and we use the relation $E_{-\alpha,\,\beta}(z) = -z^{-1}E_{\alpha,\,\alpha+\beta}(1/z)$ for $\alpha > 0$.[23] Thus, the behavior of the relaxation function $G(t)$ of a harmonic oscillator in the long time limit is same as that of $G_0(t)$. This can be shown following the procedure in Ref. 54, if in relations (17a) and (76), we apply the Tauberian theorem.[21] Indeed, for $\alpha\delta - \beta < \mu$ we obtain

$$\hat{G}(s) = \frac{1}{s^{\mu+\nu} + \frac{C_{\alpha,\beta,\delta}}{k_B T} s^{\nu+\alpha\delta-\beta} + \omega^2} \approx \frac{k_B T}{C_{\alpha,\beta,\delta}} \cdot \frac{1}{s^{\nu+\alpha\delta-\beta} + \frac{k_B T \omega^2}{C_{\alpha,\beta,\delta}}}, \quad s \to 0. \tag{80}$$

By inverse Laplace transform, for $t \to \infty$ follows relation (79).

III. FGLE WITH EXTERNAL CONSTANT FORCE: GENERALIZED EINSTEIN RELATION

We now turn to the case of a constant external force, $F(x) = F\theta(t)$, for which we find that

$${}_C D_{0+}^{\mu} v(t) + \int_0^t \gamma(t-t')v(t')\mathrm{d}t' - F = \xi(t), \quad {}_C D_{0+}^{\nu} x(t) = v(t). \tag{81}$$

For vanishing initial conditions ($x_0 = 0$, $v_0 = 0$), from the Laplace transform of Eq. (81) we obtain

$$x(t) = \langle x(t)\rangle_F + \int_0^t G(t-t')\xi(t')\mathrm{d}t' \tag{82}$$

with

$$\langle x(t)\rangle_F = F\mathcal{L}^{-1}\left[s^{-1}\hat{G}(s)\right], \quad \hat{G}(s) = \frac{1}{s^{\mu+\nu} + s^{\nu}\hat{\gamma}(s)}. \tag{83}$$

Thus, we conclude that the mean displacement is given by

$$\langle x(t)\rangle_F = F\int_0^t G(\xi)d\xi. \tag{84}$$

Conversely, if we consider the force free case $F(x) = 0$, from relation (21a) it follows that

$$\left[\langle x^2(t)\rangle - \langle x(t)\rangle^2\right]_0 = 2k_B T \int_0^t \mathrm{d}\xi\, G(\xi)\left[\frac{\xi^{\nu-1}}{\Gamma(\nu)} - {}_C D_{0+}^{\mu} G(\xi)\right]. \tag{85}$$

From Eq. (34b) in case of Dirac δ-noise and Eq. (42b) in case of a power-law noise, in the long time limit we obtain

$$\left[\langle x^2(t)\rangle - \langle x(t)\rangle^2\right]_0 = 2k_B T \int_0^t \mathrm{d}\xi\, G(\xi)\frac{\xi^{\nu-1}}{\Gamma(\nu)}. \tag{86}$$

Thus, from relations (84) and (86), we conclude that the generalized Einstein relation[1,43]

$$\langle x(t)\rangle_F = \frac{F}{2k_B T}\left[\langle x^2(t)\rangle - \langle x(t)\rangle^2\right]_0 \tag{87}$$

does not hold for the considered FGLE. Only if we use $\nu = 1$ we see that the generalized Einstein relation holds in the long time limit. The validity of the generalized Einstein relation (87) was shown for the fractional Langevin equation,[39] which can be obtained from the results in this paper if we use $\mu = \nu = 1$, and $\gamma(t) = C_\lambda t^{-\lambda}/\Gamma(1-\lambda)$, for $0 < \lambda < 2$. Furthermore, we show that for $\nu = 1$ (the case considered in Ref. 17, and investigated in case of a free particle in Ref. 14), in case of Dirac-δ and power-law noises, generalized Einstein relation holds true in the long time limit. Detail investigation of violation of generalized Einstein relation and description of its nature for a specific model, describing electronic transport in disordered system, is given by Barkai and Fleurov.[1] As we discussed in Sec. II, in the FGLE we view the variables to represent a mesoscopic description of the process, and thus the expectation values of observables calculated from this theory then describe the dynamic behavior after averaging over the disorder of the system.

IV. CONCLUSIONS

In this paper, we derived general formulas for the correlations of the FGLE for a harmonic oscillator with two time fractional derivatives in case of an internal noise. It is shown that they are different than the one of GLE, which cannot be used for FGLE models in order to investigate anomalous diffusion. Friction memory kernels of Dirac δ, power-law, and M-L forms are considered.

Different cases for modeling anomalous diffusive processes are investigated. As special case, we recovered the results for a free particle. The normalized displacement correlation function $C_X(t)$, which is experimentally measured quantity, is analyzed and some interesting behaviors as those obtained in Ref. 4 are observed. We obtained that by changing parameters the normalized displacement correlation function may have monotonic decay and non-monotonic decay without crossings of zero line, as well as oscillation-like behavior with and without crossings of zero line. As an addition to these, relations for series in three-parameter M-L function are derived. We showed that some infinite series in three parameter M-L functions can be simplified and represented in terms of one parameter and two parameter M-L functions. The correlations for the FGLE in case of external noise of power-law form are obtained as well. As an addition, results in the overdamped case are obtained. These are shown to be equivalent to the long time results for the FGLE with inertia terms. Thus, we are allowed to take the long time limit in analogy to regular diffusion. We showed that with only few parameters many different behaviors of the particle dynamics occur and relative complex data may be described. The validity of the generalized Einstein relation for such FGLEs dynamics is discussed.

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