Correlated continuous time random walks: combining scale-invariance with long-range memory for spatial and temporal dynamics

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Abstract

Standard continuous time random walk (CTRW) models are renewal processes in the sense that at each jump a new, independent pair of jump length and waiting time are chosen. Globally, anomalous diffusion emerges through scale-free forms of the jump length and/or waiting time distributions by virtue of the generalized central limit theorem. Here we present a modified version of recently proposed correlated CTRW processes, where we incorporate a power-law correlated noise on the level of both jump length and waiting time dynamics. We obtain a very general stochastic model, that encompasses key features of several paradigmatic models of anomalous diffusion: discontinuous, scale-free displacements as in Lévy flights, scale-free waiting times as in subdiffusive CTRWs, and the long-range temporal correlations of fractional Brownian motion (FBM). We derive the exact solutions for the single-time probability density functions and extract the scaling behaviours. Interestingly, we find that different combinations of the model parameters lead to indistinguishable shapes of the emerging probability density functions and identical scaling laws. Our model will be useful for describing recent experimental single particle tracking data that feature a combination of CTRW and FBM properties.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Anomalous diffusion arises in a wide range of systems across disciplines and is usually characterized in terms of the mean squared displacement (MSD)

$$\langle [X(t) - X(0)]^2 \rangle \simeq t^{2H} \tag{1}$$

of a random variable $X(t)$, where the anomalous diffusion or Hurst exponent H distinguishes subdiffusion ($0 < H < \frac{1}{2}$) from superdiffusion ($H > \frac{1}{2}$) [1, 2]. Normal ($H = \frac{1}{2}$) and ballistic ($H = 1$) motion are contained as limiting cases. In general, stochastic modelling is the approach of choice when the extent and complexity of a deterministic, multidimensional system prohibits analytical, first principles treatment. The time evolution of a small subsystem (for instance, the dispersion of a tracer particle in aquifers, a labelled molecule in a biological cell or the price of an individual stock on the market) is described in terms of a stochastic process $X(t)$. Examples for anomalous diffusion of the form (1) range from the motion of charge carriers in amorphous semiconductors [3] over the diffusion of submicron tracers in living biological cells [4] or the dynamics of small particles in weakly chaotic flows [5] to the dispersion of chemical tracers in the groundwater [6] or the dynamics of stock markets [7], just to name a few [1, 2, 8, 9].

In general anomalous diffusion processes are not universal and thus their definition through equation (1) is not unique. Instead, the form (1) may be caused by multiple physical mechanisms, some of which are very distinct conceptually. Several pathways to anomalous diffusion have been discussed. Among others, these include (i) trapping mechanisms leading to long sojourn times, (ii) long-ranged temporal correlations induced by interaction with a complex surrounding and (iii) long-distance displacements. A prominent approach to mathematically model these effects is via stochastic processes such as continuous time random walks (CTRWs) [3, 10], fractional Brownian motion (FBM) [11], or Lévy flights and walks [12]. They are paradigmatic in the sense that they are designed to tackle one specific key property (i)–(iii). Thus CTRWs were proposed as a model for charge carrier transport in amorphous semiconductors [3], where individual charges reside on specific, sparsely distributed acceptor sites for long, effectively random time spans, before hopping to a neighbour site. Moreover, crosslinked polymer filament networks as found in mammalian cells can cause similar caging effects for micron-sized objects [13], similarly to multiscale trapping times of particles on sticky surfaces [14]. FBM addresses the problem of highly correlated stock market decisions [15] and telecommunications [16], and the associated fractional Gaussian noise fuels the diffusion of a single tracer particle in viscoelastic or crowded environments [17–21]. Lévy flights and walks [12] provide a statistical description for the motion of tracers in weakly chaotic systems [5] or of the linear particle diffusion along a fast-folding polymer chain [22].

However, in complex, disordered environments we would expect that more than one of the patterns (i) to (iii) emerge, compete and collude to generate anomalous diffusion patterns, and it remains an open challenge to identify and differentiate them. Thus, the global properties of dispersion in amorphous media can be related to the microscale flow dynamics by adding a memory component to the standard CTRW description [23]. Modern single particle tracking techniques in experiment and simulations indeed corroborate the co-existence of different diffusion mechanisms [4, 8, 24–26]. For instance, for the motion of individual granules in the intracellular fluid of living cells characteristics of CTRW-style trapping and FBM-like anticorrelations were observed [4]. We here introduce a stochastic process, namely the correlated CTRW (CCTRW), that merges and extends the classical paradigmatic models of CTRW, FBM, and Lévy flights. We study in particular two quantities, which are typically
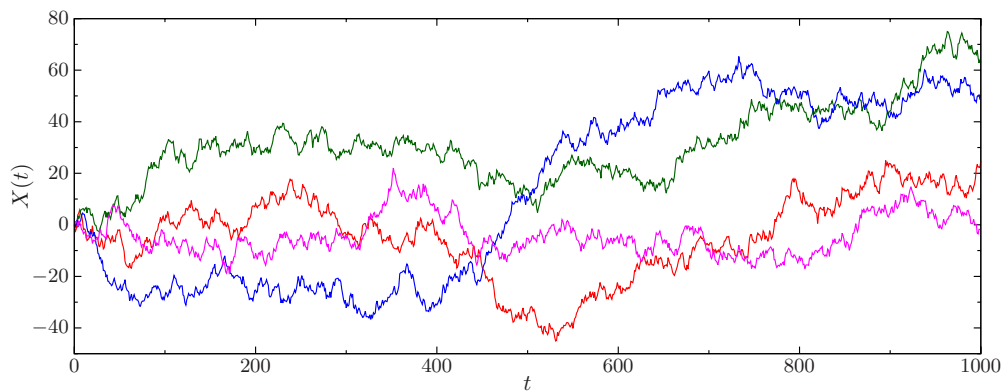
Figure 1. Trajectories $X(t)$ of ordinary *Brownian motion*. The sample paths are erratic but continuous and at no time a favoured direction can be determined. In the context of CCTRWs as defined in section 2.3, the parameters are: $\mu = 2$, $K = 1/2$, $\alpha = 1$, and $G = 1$.

accessible experimentally: the scaling of the particle position X with time t, and the shape of the probability density function (PDF) $p(x;t)$ of the particle displacement x at some instant of time t. The result is a very flexible stochastic model, that will be of use for the data analysis of stochastic processes in complex systems. One immediate lesson is the interplay of the underlying stochastic modes, the blend of which may lead to indistinguishable forms for the PDF $p(x;t)$ for different sets of model parameters.

The paper is organized as follows. In section 2 we define the ingredients of the CCTRW model. First, subsections 2.1 and 2.2 recapitulate the essential definitions and properties of Lévy flights and CTRW-trapping theory. Second, we define a correlated version of these models by means of stochastic integration in subsection 2.3. The stationarity (closely related to the physical concept of equilibrium) of the CCTRW is briefly discussed in section 3. We study extensively the scaling behaviour and the PDF of the position coordinate for CCTRWs in section 4. The parallels and differences with other CCTRW models in the literature are outlined in section 5. A brief overview on our main results and potential extensions and further studies of the model are summarized in the conclusion, section 6.

2. Model definition

The most commonly used theoretical model for (normal) diffusion dynamics is the celebrated Brownian motion. In its standard form, this random process describes the dynamics of a point-like particle as an unbiased, continuous but erratic motion in an unbounded embedding space. In the mathematics literature, such a process $X(t)$ with positive time $t \in \mathbb{R}_0^+$, is usually referred to as the Wiener process, and it is uniquely defined by the following three properties: (i) $X(0) = 0$; (ii) a sample trajectory $X(t)$ is almost surely continuous everywhere; (iii) increments $X(t_2) - X(t_1)$ have a Gaussian distribution with mean 0 and variance $|t_2 - t_1|$, and they are mutually independent for any non-overlapping time intervals. Typical sample realizations of Brownian motion are shown in figure 1. Individual trajectories are indeed characterized by a continuous but non-smooth behaviour. There is no notable global drift and neither a specific point in time nor some spatial region stands out from the rest. In ensemble measurements, Brownian motion features a normal diffusive behaviour ($H = \frac{1}{2}$). More generally, the position

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coordinate scales with time as $X(t) \sim t^{1/2}$.[6] The independence of increments is reflected by the correlation function, $\langle X(t_1)X(t_2)\rangle = \min\{t_1, t_2\}$ for any t_1, t_2. In this sense, we call $X(t)$ an uncorrelated process[7].

Thus, Brownian motion is an ideal candidate to model the diffusive motion in an environment where the bombardment by small particles from a surrounding heat bath induces vivid but short displacements of a relatively inert, point-like test particle. The typical example is a micron-scale molecule in a dilute water solution at room temperature, as discussed in Albert Einstein's groundbreaking studies [27] and monitored in the seminal works by Perrin [28]. There, the momentum transfer from the surrounding water molecules occurs much faster than the average large particle motion observed under a microscope. Displacements of the test particle thus indeed appear to be random and independent, yet small on an observational scale.

However, when we want to describe diffusive dynamics in complex environments, we are forced to drop several of the idealizing assumptions (i) to (iii). For instance, in chaotic systems [29] or in highly disordered optical materials [30] large scale displacements occur almost instantaneously, resulting in highly non-continuous sample trajectories. Conversely, in disordered environments such as the above-mentioned amorphous semiconductors or the densely crowded intracellular fluid of biological cells, the assumption of a steady time evolution is questionable, since charge carriers or tracer molecules can become trapped in microenvironments or stick to reactive surfaces for long time periods [13]. Finally, the independence of increments cannot be taken for granted when the particle motion is strongly coupled to its environment, for instance, in single file diffusion [21] or in viscoelastic media [18], leading to an effective memory in the history of the motion.

In the following sections, we generalize the standard Brownian motion to a larger class of one-dimensional, random motions of a point particle to accommodate the above-mentioned effects. We proceed stepwise: first, in section 2.1, we introduce the concept of Lévy flights (allowing discontinuities in sample paths). In section 2.2 we consider CTRWs (with long sojourn times). Finally in section 2.3, we generalize to processes defined in terms of stochastic integrals (to account for FBM-style memory effects). On each level of generalization, we focus our analytical discussion on the study of scaling laws in general, and the evolution of the PDF of the particle position x with respect to time t, in particular.

2.1. Lévy flights

The first assumption that we drop in order to extend ordinary Brownian motion is the continuity of sample paths. A widely used stochastic approach to model such type of anomalous diffusion property are Lévy flights. This type of idealized random motion is not continuous in space, but instead consists of a series of random jumps; see figure 2. The jump lengths δX are characterized by heavy-tailed jump statistics, that is, by PDFs with power-law tails, $\lambda(\delta x) \simeq |\delta x|^{-1-\mu}$ with $0 < \mu < 2$ for large distances $|\delta x|$. The key feature of such stable PDFs is the diverging second moment, $\langle (\delta X)^2 \rangle = \int_{-\infty}^{\infty} (\delta x)^2 \lambda(\delta x)\, d(\delta x) = \infty$: discontinuous jumps occur on arbitrary large spatial scales.

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[6] Let $X(t)$ be a stochastic process and $H > 0$. We use the notation $X(t) \sim t^H$ to indicate a time scaling relation: the random variable $X(t)$ has the same distribution as $t^H X(1)$. We note however, that for most cases we consider in this work, an actually stronger statement holds: the motion is self-similar with index H, that is, for any $c > 0$, the processes $X(ct)$ and $c^H X(t)$ have the same finite-dimensional distributions.

[7] Mathematically, two random variables A and B are called uncorrelated if $\langle AB \rangle = 0$. Hence, calling Brownian motion an uncorrelated process is a slight abuse of language. To be precise, its increments are independent and thus uncorrelated. Still, we stick to the common habit of referring to a (continuous time) random walk or diffusion process $X(t)$ as an (un)correlated process, whenever its increments have this property, that is $\langle [X(t_1 + t) - X(t_1)][X(t_2 + t) - X(t_2)] \rangle = 0$ for non-overlapping time intervals.
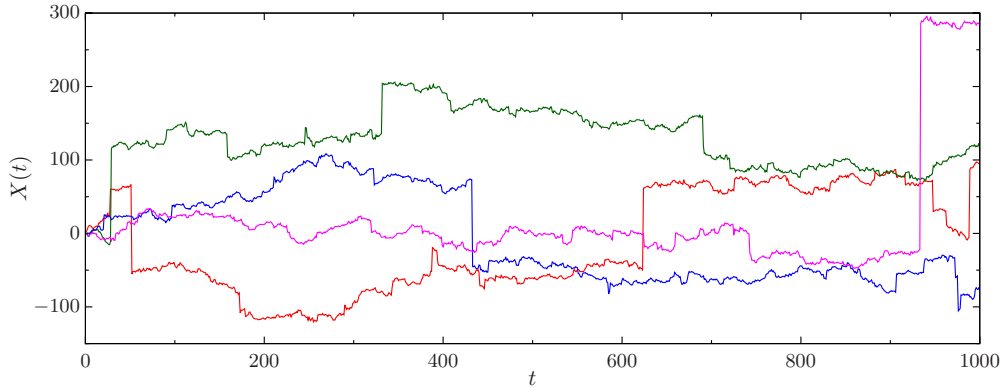
Figure 2. Sample trajectories $X(t)$ of *Lévy flights*. The motion is uncorrelated and unbiased, but characterized by large-scale, discontinuous jumps. In the context of CCTRWs as defined in section 2.3, the parameters are: $\mu = 3/2$, $K = 2/3$, $\alpha = 1$, and $G = 1$.

To be more precise, we here connect Lévy flights with the mathematical concept of a symmetric Lévy μ-stable process, $X(t) = L_\mu(t)$. Apart from the discontinuity of individual trajectories, also ensemble properties differ significantly from the Brownian case: the capability of covering large distances by single, instantaneous jumps implies a superdiffusive scaling of the position coordinate with time, $X(t) \sim t^{1/\mu}$. In particular, the PDF $p_\mu(x; t)$ for the position X at time t assumes the scaling form

$$p_\mu(x; t) = t^{-1/\mu} \ell_\mu(xt^{-1/\mu}). \tag{2}$$

where the scaling function ℓ_μ is a symmetric μ-stable law. The latter is uniquely defined in terms of the characteristic function

$$\langle \exp\{ikL_\mu(1)\} \rangle = \int_{-\infty}^{\infty} e^{ikx} \ell_\mu(x) \, dx = \exp(-|k|^\mu), \tag{3}$$

including Gaussian statistics in the limit $\mu = 2$. Indeed, in a distributional sense, the limiting case $X(t) = L_2(t)$ is an ordinary Brownian motion. The scaling function ℓ_μ (and thus the PDF p_μ) has the same heavy-tail property as the PDF of individual jumps, $\ell_\mu(x) \simeq |x|^{-1-\mu}$ for large $|x|$. Consequently, second (and higher) order moments diverge, $\langle X^2(t) \rangle = \int_{-\infty}^{\infty} x^2 p_\mu(x; t) \, dx = \infty$.

Yet, in terms of correlations, Lévy flights are on the same level as ordinary Brownian motion. Indeed, both processes are Markovian. The jump lengths δX are mutually independent and identical in a distributional sense. Consequently, the increments of Lévy stable motion $X(t_2) - X(t_1)$, are characterized by mutual independence and distributional equality.

2.2. Continuous time random walks

Despite their spatial discontinuity, Lévy flights are evolving continuously in time, in the sense that the particle remains at any specific position for only an infinitesimal amount of time. To account for the possibility to encounter deep traps on some random energy landscapes, a common generalized stochastic model is used, namely, subdiffusive CTRWs. In contrast to ordinary Brownian motion or Lévy flights, these processes include random long-time trapping periods δT, usually referred to as waiting times. These are distributed according to heavy-tailed waiting time statistics, $\psi(\delta t) \simeq \delta t^{-1-\alpha}$ with $0 < \alpha < 1$ for large δt. In close analogy to the effects of scale-free jump lengths for Lévy flights, the infinite first moment of the waiting times amounts to immobilization periods on all time scales; see figure 3.
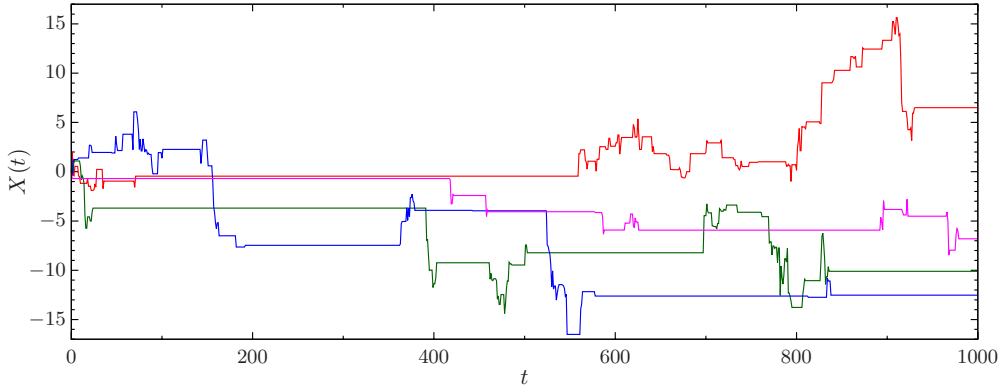
Figure 3. Sample trajectories $X(t)$ of *CTRWs*. The spatially continuous motion is paused for scale-free waiting periods. This considerably slows down the exploration of space as compared to ordinary Brownian motion. Waiting times are heavy-tailed (see text) and thus assume values on all time scales, but they are also mutually independent. In the context of CCTRWs as defined in section 2.3, the parameters are: $\mu = 2$, $H = 1/2$, $\alpha = 1/2$, and $G = 2$.

Mathematically, one can model such systems by use of the subordination method [31, 32]. An internal time parameter s is introduced, which plays the role of the number of jumps performed along the trajectory, that is successively delayed by trapping events. As this 'internal time' s increases, on the one hand, the spatial exploration evolves according to a process $Y(s)$. The appropriate choice for $Y(s)$ depends on the characteristics and features of the physical system we intend to describe (external force fields, drift, friction, etc). Typically, $Y(s)$ is assumed to be Markovian and thus, in the above sense, is continuously evolving in time. As a paradigmatic example, we define $Y(s) = L_\mu(s)$, i.e., the spatial dynamics are modelled in terms of an unbiased and unconfined Lévy flight with stable index $0 < \mu < 2$, as defined in the previous section 2.1.

On the other hand, we model the punctuated progression of real laboratory time as measured by the observer in terms of a separate random process $T(s)$: as the internal time s increases, consecutive waiting times accumulate to the laboratory time $T(s)$. In order to model waiting times which are distributed by heavy-tailed statistics, one can simply choose $T(s) = L_\alpha^+(s)$, where $0 < \alpha < 1$. The latter is a special type of Lévy flight itself, namely, a one-sided (or totally skewed) Lévy α-stable motion. It is a positive, strictly increasing process and thus an appropriate representation of the random time progression of the particle motion. Moreover, it has the typical Lévy flight property of scale-free discontinuous, jump-like evolution. Statistically, it is characterized by the time scaling $T(s) \sim s^{1/\alpha}$ and one-sided α-stable distributions. In particular, the PDF $g_\alpha(t; s)$ for the laboratory time T at given internal time s reads

$$g_\alpha(t; s) = s^{-1/\alpha} \ell_\alpha^+(ts^{-1/\alpha}). \tag{4}$$

Here, ℓ_α^+ is a one-sided α-stable law. Its most natural representation is via its Laplace transform

$$\langle \exp\{-\theta L_\alpha^+(1)\}\rangle = \int_0^\infty e^{-\theta t} \ell_\alpha^+(t)\, dt = \exp(-\theta^\alpha). \tag{5}$$

The distribution is heavy-tailed, $\ell_\alpha^+(t) \simeq t^{-1-\alpha}$, so that the expectation value (and higher order moments) of the laboratory time diverges, $\langle T(s)\rangle = \int_0^\infty t g_\alpha(t; s)\, dt = \infty$. Note that in the limit $\alpha \to 1$ the PDF in equation (4) becomes a Dirac δ-distribution, $g_1(t; s) = \delta(t - s)$. Thus, this limiting case restores the equivalence of internal and laboratory time, such that the particle motion is no longer paused for random waiting time periods.

To complete the definition of this type of CTRWs we introduce an inverse process $S(t)$ which measures the evolution of internal time as function of laboratory time t,

$$S(t) = \inf\{s > 0 : T(s) > t\}, \tag{6}$$

which is also sometimes referred to as first-hitting time or counting process. The particle motion as seen by the observer is now given by a process $X(t) = Y(S(t))$, i.e. the random, unsteady progression of internal time is modelled by $S(t)$, while independently the spatial displacements during times of dynamic activity are governed by the process $Y(s)$. Individual paths of $X(t)$ most notably feature discontinuities in both their spatial and temporal evolution. Meanwhile, ensemble statistics combine the distributional properties of both independent random processes $Y(s)$ and $S(t)$. For instance, let $h_\alpha(s; t)$ denote the PDF for internal time S at laboratory time t. Recall that $p_\mu(y; s)$ is the PDF of the position Y when the internal time s has passed. Then the PDF $p_{\mu,\alpha}(x; t)$ for the particle position X at time t can by computed as

$$p_{\mu,\alpha}(x; t) = \int_0^\infty p_\mu(x; s) h_\alpha(s; t)\, \mathrm{d}s. \tag{7}$$

We conclude this section with a remark on correlations in this type of CTRW. Both the displacement dynamics $Y(s)$ and the laboratory time evolution $T(s)$ with respect to internal time s belong to the class of Lévy flight processes. As such, their respective increments are stationary and mutually independent for non-overlapping time intervals. In the language of individual jump distances δX or waiting times δT this means that the latter form sequences of mutually independent, identically distributed (iid) random variables. As mentioned above, this renewal property might be considered as a severe simplification when we want to model real physical systems. We will therefore drop this property in the following section and define a CTRW where successive jump lengths or waiting times are correlated.

2.3. Correlated continuous time random walks

In standard CTRW models, individual jump lengths and waiting times, respectively, are independent of each other. Our goal is to extend this theory to systems where highly complex environments induce long-range correlations. In that we build on previous results; we discuss the parallels with akin CCTRWs in section 5. In the present paper, we follow an idea proposed in [34] to introduce a process close in spirit to FBM and its heavy-tailed generalization, the linear fractional μ-stable motion [35]. The basic theoretical approach is to derive a correlated process from an uncorrelated one in terms of a (stable) stochastic integral. By this method, correlations are introduced without altering the distributional properties of the process itself.

Without going into the details of stochastic integrals [35] we here provide an exemplary calculation as a motivation for our method. Consider first a discrete time random walk in continuous space, $Y_n \in \mathbb{R}$ with $n \in \mathbb{N}$. At the nth step, the random walker covers a random jump distance $\delta Y_n = Y_{n+1} - Y_n$. If we assume the δY_n are mutually independent and iid, we call the random walk Y_n an uncorrelated process. For example, define $\delta Y_n = \xi_n$, where the ξ_n are Gaussian iid random variables with zero mean and variance σ^2. Then the distribution of the position variable Y_n follows as (we assume $Y_0 = 0$)

$$Y_n = \sum_{j=1}^n \delta Y_j = \sum_{j=1}^n \xi_j \stackrel{d}{=} n^{1/2} \cdot \xi_1. \tag{8}$$

Here, $\stackrel{d}{=}$ denotes an equality in distribution, and thus the position Y_n after the nth step also has a Gaussian distribution, scaling as $Y_n \sim n^{1/2}$. Consequently the 'diffusion' law, $\langle Y_n^2 \rangle = \sigma^2 n$, is normal for this simple, uncorrelated random walk. Note that this result is a generic one, since, by virtue of the central limit theorem, any series of iid random displacements ξ_n with zero

mean and finite variance produces asymptotically Gaussian behaviour on sufficiently large scales.

Now, how can we add correlations to this simple random walk process, without altering its Gaussian nature? One method is by means of a linear transformation, as proposed in [34]. Let ξ_n be iid Gaussian random variables as above. We introduce a nonrandom function M_k, which we will refer to as correlation kernel, and define the correlated jump lengths $\delta\widetilde{Y}_n = \sum_{k=1}^{n} M_{n-k+1}\xi_k$. The latter have strong similarities in distribution with their uncorrelated counterparts δY_n: both are sequences of Gaussian random variables centred at zero. However, the correlated sequence is not necessarily stationary, since $\langle\delta\widetilde{Y}_n^2\rangle = \tilde{\sigma}_n^2 = \sum_{k=1}^{n} M_k^2$. More severely, the δY_n are by definition mutually independent, while for the correlated sequence we have, for any $n, m \in \mathbb{N}$,

$$\langle\delta\tilde{Y}_n\delta\tilde{Y}_{n+m}\rangle = \sum_{k=1}^{n}\sum_{l=1}^{n+m} M_{n-k+1} M_{n+m-l+1}\langle\xi_k\xi_l\rangle = \sigma^2\sum_{k=1}^{n} M_k M_{k+m}. \tag{9}$$

Depending on the exact behaviour of the correlation kernel M_k, this covariance function can have either a negative or positive sign, where a positive (negative) covariance function indicates a tendency for any two jumps to go in the same (opposite) direction. We can therefore say that the jump lengths are either *persistent* or *antipersistent*, respectively. Only by choosing $M_k = \delta_{1k}$, δ_{ik} denoting the Kronecker symbol, the $\delta\tilde{Y}_n$ become mutually independent.

The random walk process $\widetilde{Y}_n$ associated with such correlated jump lengths has the following properties (again, assume $\widetilde{Y}_0 = 0$):

$$\widetilde{Y}_n = \sum_{j=1}^{n}\delta\tilde{Y}_j = \sum_{k=1}^{n}\widetilde{M}_{n-k+1}\xi_k \stackrel{d}{=} \xi_1 \cdot \left(\sum_{k=1}^{n}\widetilde{M}_k^2\right)^{1/2}$$

$$\widetilde{M}_k = \sum_{l=1}^{k} M_l. \tag{10}$$

Thus, the correlations indeed preserve the Gaussian nature of the process. $\widetilde{Y}_n$ can itself be written as a linear transformation of the iid Gaussian variables ξ_k in terms of the correlation kernel $\widetilde{M}_k$. Note that we altered the scaling behaviour with the introduction of correlations: in contrast to the normal scaling $Y_n \sim n^{1/2}$, the scaling of the process $\widetilde{Y}_n$ is more complex in general and depends on the exact form of the kernel $\widetilde{M}_k$.

This method of correlating a Gaussian random walk can be readily transferred to a time-continuous process such as the Lévy flights and CTRWs as defined in the previous sections[8]. We start from the linear process representation in equation (10), namely $Y_n = \sum_{k=1}^{n} M_{n-k+1}\xi_k$ (dropping the tilde here and in the following). Now we perform the following, purely formal substitutions. The discrete step number n is replaced by a continuous (internal) time variable s. The sum over the Gaussian iid random variables is rewritten in terms of a stochastic integration with respect to the Lévy stable noise $dL_\mu(s)$. Here, $L_\mu(s)$ is a symmetric Lévy flight with stable index $0 < \mu < 2$. Finally, for the correlation kernel $M(s)$ we choose a power-law, so that correlations are potentially of a long-ranged kind. In summary, we define the stochastic process $Y(s)$ in terms of a stable integral through

$$Y(s) := (\mu K)^{1/\mu} \int_0^s (s - s')^{K-1/\mu}\, dL_\mu(s'). \tag{11}$$

Here $0 < \mu < 2$, and we will refer to $K > 0$ as the Hurst exponent. With this definition of the process Y we stay in the domain of μ-stable processes. In particular, at given time s, its PDF $p_{\mu,K}(y; s)$ is of the stable form (2), albeit with an altered time scaling $Y(s) \sim s^K$,

$$p_{\mu,K}(y; s) = s^{-K}\ell_\mu(ys^{-K}). \tag{12}$$

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[8] How and when a time-discrete correlated random walk *converges* to a time-continuous correlated motion is taken up in [33, 36].

The scaling prefactor $(\mu K)^{1/\mu}$ in equation (11) makes sure that the scaling function ℓ_μ is again exactly represented by the characteristic function in equation (3).

What really sets the process $Y(s)$ apart from the ordinary Lévy motion $L_\mu(s)$ is the stochastic dependence of increments. We may also say the noise related to $Y(s)$ is strongly correlated, or coloured. However, to assess the nature of interdependence here, we cannot use the covariance function like in equation (9). While the latter is a meaningful and precise measure of dependence for Gaussian processes, $\mu = 2$, it is ill-defined for stable processes $\mu < 2$. In reference [35], several alternative concepts to deal with the stable cases are introduced and discussed, such as covariation or codifference functions. In short, applying these analytical tools to the correlated process $Y(s)$, we find positive, long-range dependence when $K > 1/\mu$, and negative, short-range dependence when $K < 1/\mu$. (Compare this to the analogous discussion on linear fractional stable motion in [35]. An extensive discussion of the notion of long-range dependence can be found in [37]).

We can also supplement these considerations by spectral analysis arguments, compare also [36]. Consider a sample path of a Lévy flight $L_\mu(s)$ and denote its Fourier transform by $\widehat{L}_\mu(\omega)$. Now since the stable stochastic integral (11) is of a convolution form, there is a simple relation in Fourier space between the correlated noise $dY(s)$ and the Lévy stable noise $dL_\mu(s)$, namely $d\widehat{Y}(\omega) \propto d\widehat{L}_\mu(\omega)/(-i\omega)^{K-1/\mu}$. When comparing the two noise types in the case $K > 1/\mu$ we thus find that the correlation kernel in the stable integral (11) emphasizes the low frequency components of the correlated noise. In a sample path of $Y(s)$, we may conceive this as a comparatively steady motion, even in the form of long-term periodic cycles. Conversely, when $K < 1/\mu$, high frequencies are amplified. A sample path $Y(s)$ is then fluctuating violently as compared to an ordinary Lévy flight.

Hence, both the analysis in terms of covariation/codifference functions and the spectral analysis support the idea of an either *persistent* or *antipersistent* motion $Y(s)$. (Throughout the rest of this work, such statements are equivalent to saying that the respective increment/noise process is persistent or antipersistent.) For $K > 1/\mu$, persistence effects long cycles of seemingly steady, biased motion. If $K < 1/\mu$, antipersistent motion is observed as being wildly fluctuating, since strong, short range, negative memory leads to a quick succession of directional turns. The special case $K = 1/\mu$ recovers ordinary Lévy flights with mutually independent jump lengths.

The last step in the definition of our CCTRW model is the introduction of correlations of waiting times. Analogously to the above, we define

$$T(s) := (\alpha G)^{1/\alpha} \int_0^s (s - s')^{G-1/\alpha}\, dL_\alpha^+(s'), \tag{13}$$

in terms of a stable integral with respect to one-sided Lévy α-stable noise $dL_\alpha^+(s)$. Here, $0 < \alpha < 1$ and $G \geqslant 1/\alpha$. The corresponding PDF $g_{\alpha,G}(t; s)$ at given internal time s in this case reads

$$g_{\alpha,G}(t; s) = s^{-G}\ell_\alpha^+(ts^{-G}), \tag{14}$$

where the basic shape is still provided by a one-sided α-stable law ℓ_α^+ as defined in equation (5). The scaling with internal time s in this case reads $T(s) \sim s^G$. While $T(s)$ is still an α-stable motion, waiting times are no longer independent. Note that for $T(s)$ to be an increasing process, we need to require that $G \geqslant 1/\alpha$. Thus, correlations in waiting times are necessarily of the persistent type, and have a tendency to increase with s. The only exception to this rule is $G = 1/\alpha$, a parameter setting which brings us back to heavy-tailed but uncorrelated waiting times.
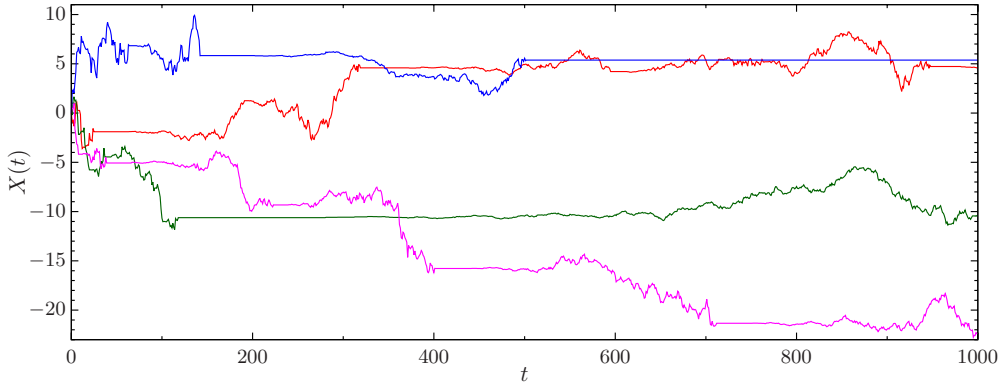
Figure 4. Sample trajectories $X(t)$ of *CCTRWs*. The spatially continuous motion is paused for large-scale waiting periods, which appear on all time scales. Waiting times are *not* independent but persistent here: long rests are directly followed by periods of reduced dynamic activity, but then slowly turn into vivid almost Brownian-like motion. Note however that also spatial displacements are persistent. The parameters are chosen such that the resulting scaling with time and the shape of the PDF are the same as for the uncorrelated CTRWs in figure 3 (as explained in section 4). In the context of CCTRWs as defined in this section, the parameters are: $\mu = 2$, $K = 1.1/2$, $\alpha = 1/2$, and $G = 2.2$.

In complete analogy to the uncorrelated case we now introduce the inverse process $S(t)$ according to equation (6) and combine it with a correlated stable motion, $X(t) = Y(S(t))$. The PDF for the particle position X at real time t is then given by

$$p_{\mu,\alpha,K,G}(x; t) = \int_0^\infty p_{\mu,K}(x; s) h_{\alpha,G}(s; t) \, \mathrm{d}s, \tag{15}$$

where $h_{\alpha,G}(s; t)$ denotes the PDF of internal time S at real time t. We will extensively discuss this PDF in section 4. To study this process on a trajectory basis, see figure 4.[9]

This completes the definition of the CCTRW model that we discuss in the present paper. A discontinuous progression of spatial displacements and laboratory time is modelled in terms of the stable noises $\mathrm{d}L_\mu(s)$ and $\mathrm{d}L_\alpha^+(s)$. Correlations are separately introduced by power-law correlation kernels to both the spatial dynamics $Y(s)$ and the time evolution $T(s)$. The full model is defined in terms of four parameters: $0 < \mu < 2$ and $0 < \alpha < 1$ determine the respective distributional properties of individual jump lengths δX and waiting times δT. In particular, they define the heavy tails $\lambda(\delta x) \simeq |\delta x|^{-1-\mu}$ and $\psi(\delta t) \simeq \delta t^{-1-\alpha}$. The special cases of continuous spatial and/or temporal evolution are included in the CCTRW model on a distribution level as the limits $\mu \to 2$ and $\alpha \to 1$. The parameters $K > 0$ and $G \geqslant 1/\alpha$ directly measure the scaling exponents with respect to internal time, $Y(s) \sim s^K$ and $T(s) \sim s^G$. Finally, the nature of the correlations can be assessed by comparing respective parameter pairs: jump distances (waiting times) are persistent if $K > 1/\mu$ ($G > 1/\alpha$), uncorrelated if $K = 1/\mu$ ($G = 1/\alpha$), or antipersistent if $K < 1/\mu$ (impossible for waiting times).

---

[9] All trajectories in figures 1–4 were generated by simulating long random walk trajectories and rescaling temporal and spatial coordinate appropriately. Thus we approximate the time-continuous motion defined in the respective sections. Heavy-tailed jump lengths or waiting times, respectively, are realized by drawing stable random variables [39]. To introduce correlations as in equations (11) and (13), we use the method described in [35], section 7.11.

3. Stationarity

We defined the stable processes $Y(s)$ and $T(s)$ directly in terms of their distributional, scaling and correlation properties, as characterized through the parameters μ, α, K, and G, respectively. We will now further study this quite large class of processes through their stationarity properties. For this, we apply the preliminary definition of the *n*th order increments of a stochastic process $Y(s)$,

$$
\begin{aligned}
\Delta^{(1)}Y(s; \tau) &= Y(s + \tau) - Y(s) \\
\Delta^{(2)}Y(s; \tau_1, \tau_2) &= \Delta^{(1)}Y(s + \tau_2; \tau_1) - \Delta^{(1)}Y(s; \tau_1) \\
&\;\;\vdots \\
\Delta^{(n)}Y(s; \tau_1, \ldots, \tau_n) &= \Delta^{(n-1)}Y(s + \tau_n; \tau_1, \ldots, \tau_{n-1}) \\
&\quad - \Delta^{(n-1)}Y(s; \tau_1, \ldots, \tau_{n-1}).
\end{aligned}
\tag{16}
$$

Thus, $\Delta^{(1)}Y$ is the usual process increment while $\Delta^{(2)}Y$ is an increment of increments, etc. If $Y(s)$ is meant to stand for the position of a particle at time s, then the ratio $\Delta^{(1)}Y(s; \tau)/\tau$ can be viewed as the average velocity (bearing in mind that the one-time velocity, i.e., the limit $\tau \to 0$, in general does not exist for the processes discussed here). Likewise, $\Delta^{(2)}Y(s; \tau_1, \tau_2)/(\tau_1 \tau_2)$ corresponds to the intuitive notion of an acceleration, and higher order increments represent higher levels of temporal evolution.

We now say the *n*th order increments of $Y(s)$ are asymptotically stationary in distribution (ASD), if the random variable $\Delta^{(n)}Y(s; \tau_1, \ldots, \tau_n)$ has a nontrivial limiting distribution for large times, $s \to \infty$. In the following we will determine such degrees of stationarity for the stable processes $Y(s)$ and $T(s)$ as defined in the previous section. Note that this classification is not a purely academic one. For the application and interpretation of a stochastic process as a real world model system, stationarity properties are highly relevant. Let, for instance, $Y(s)$ model an animal foraging process. Then stationarity of first order increments is an indication for a time-independent search strategy: the distance $\Delta^{(1)}Y$ travelled during, say, $\tau = 1$ day is statistically indistinguishable from one day to the next. Conversely, nonstationary statistics of travel distances can be a signature for an adaptive search strategy, an ageing animal, or changes in the environment. In this case, we could further ask whether or not such internal or external variations are stationary. This relates to second order increments. On smaller scales, $Y(s)$ could be a model for particle diffusion in a heat bath. There, nonstationarity of first order increments is the fingerprint either of an inhomogeneous environment (i.e., the particle displacement statistics changes as the particle explores various spatial regions) or a non-equilibrated environment (i.e., the noise imposed by interaction with the surrounding heat bath is itself nonstationary). Then, analysis of second and higher order increments yields information on the precise nature of the spatial or temporal variations in the surroundings.

The displacement process $Y(s)$ as defined through the stable integral (11) is a nonstationary process, as indicated by the time scaling $Y(s) \sim s^K$. As the particle explores its surrounding space, the probability to find it in any region of fixed size around the origin of motion is decaying with time. Now, the integral representation of first order increments reads

$$
\begin{aligned}
\Delta^{(1)}Y(s; \tau) = (\mu K)^{1/\mu} \Bigg\{ &\int_0^s [(s + \tau - s')^{K-1/\mu} - (s - s')^{K-1/\mu}]\, \mathrm{d}L_\mu(s') \\
&+ \int_s^{s+\tau} (s + \tau - s')^{K-1/\mu}\, \mathrm{d}L_\mu(s') \Bigg\},
\end{aligned}
\tag{17}
$$

so that its distribution is given in terms of the characteristic function

$\langle \exp(\mathrm{i}k\Delta^{(1)}Y(s;\tau)) \rangle$

$$= \exp\left[-\mu K|k|^{\mu} \int_0^s |(s+\tau-s')^{K-1/\mu} - (s-s')^{K-1/\mu}|^{\mu}\,\mathrm{d}s' \right.$$
$$\left. -\mu K|k|^{\mu} \int_s^{s+\tau} |(s+\tau-s')^{K-1/\mu}|^{\mu}\,\mathrm{d}s' \right]$$
$$= \exp\left[-|k|^{\mu} I_{\mu,K}^{(1)}(s) - |k\tau^K|^{\mu} \right], \tag{18}$$

where we used the abbreviation

$$I_{\mu,K}^{(1)}(s) = \mu K \int_0^s |(s'+\tau)^{K-1/\mu} - (s')^{K-1/\mu}|^{\mu}\,\mathrm{d}s'. \tag{19}$$

Nonstationarity is indicated by the explicit s-dependence of the integral $I_{\mu,K}^{(1)}$. The latter vanishes identically if $K = 1/\mu$. This is natural, since these cases are the symmetric Lévy stable motions, $Y(s) = L_{\mu}(s)$, which have stationary increments by definition. Conversely, for any $K \neq 1/\mu$, the integral differs from zero, so in general the first order increments of the stable motion $Y(s)$ are nonstationary. However, they can still be asymptotically stationary, depending on the parameters. The expression in the integral $I_{\mu,K}^{(1)}$ behaves, for large s', like $(s')^{\mu K-\mu-1}$. The asymptotics at large times $s \gg \tau$ are therefore given through

$$I_{\mu,K}^{(1)}(s) \simeq \begin{cases} \text{const}, & \text{for } 0 < K < 1, \\ \log(s), & \text{for } K = 1, \\ \tau^{\mu} s^{\mu(K-1)}, & \text{for } K > 1. \end{cases} \tag{20}$$

First order increments are hence ASD whenever $0 < K < 1$, while spreading indefinitely when $K \geqslant 1$. We can readily extend the procedure to the study of increments of arbitrary order, see the appendix. In general, we have to distinguish two classes of parameter settings.

If we can find a nonnegative integer m such that $K = 1/\mu + m$, then all increments of order $n > m$ are stationary in distribution, and lower order increments, $n \leqslant m$ on average broaden. This includes the Lévy stable motions, $m = 0$, $K = 1/\mu$, with stationary increments of all orders. To understand this, recall that correlated and Lévy stable noises are related in Fourier space through $\mathrm{d}\widehat{Y}(\omega) = \mathrm{d}\widehat{L}_{\mu}(\omega)/(-\mathrm{i}\omega)^{K-1/\mu}$. Now for $K = 1/\mu + m$, this suggests we can interpret $Y(s)$ as an m-fold repeated integration of a Lévy stable noise. In other words, for $m = 0$, $Y(s)$ is a Lévy flight, so increments are stationary; for $m = 1$, the noise generating $Y(s)$ is already a Lévy flight, therefore only second and higher order increments of $Y(s)$ are stationary; for $m = 2$, the noise generating the noise of $Y(s)$ is a Lévy flight, so we find stationary third order increments; etc.

The opposite case is $K \neq 1/\mu + m$ for all nonnegative integers m. Interestingly, here the result is μ-independent: all increments of order $n > K$ are ASD, while lower order increments, $n \leqslant K$, are spreading indefinitely. An extensive and mathematically rigorous treatment of stochastic processes with stationary nth order increments can be found in [40].

The one-sided α-stable process (13), which describes the evolution of laboratory time with respect to the internal time has completely analogous properties. Increments of any order are stationary if $G = 1/\alpha$, since then $T(s) = L_{\alpha}^{+}(s)$ is a one-sided Lévy stable motion. If there is a nonnegative integer m such that $G = 1/\alpha + m$, then only increments of order $n > m$ are stationary in distribution. If there is no such m, increments of orders $n > G$ are ASD. Lower order increments are nonstationary at all times. Note, however, that for the waiting time process we are forced to require $G \geqslant 1/\alpha$ for the following reason. Writing out the integral representation for the first order increments,

$$\Delta^{(1)}T(s;\tau) = (\alpha G)^{1/\alpha} \left\{ \int_0^s [(s+\tau-s')^{G-1/\alpha} - (s-s')^{G-1/\alpha}] \, dL_\alpha^+(s') \right.$$

$$\left. + \int_s^{s+\tau} (s+\tau-s')^{G-1/\alpha} \, dL_\alpha^+(s') \right\}, \tag{21}$$

we see that the first integral could potentially give a negative contribution when $G < 1/\alpha$. This is clearly unacceptable in terms of causality: negative increments in laboratory time $T(s)$ would correspond to waiting times finishing earlier than they began. We therefore consider only $G \geqslant 1/\alpha$, which has two implications. On the one hand, as mentioned above, correlated motions are necessarily persistent. On the other hand, first order increments—reflecting waiting time statistics—are nonstationary. More precisely, they are, in a statistical sense, increasing beyond all bonds, as their (one-sided!) distribution continuously broadens with internal time s.

We conclude this section with a general remark on stationarity in CTRW models. The inverse process $S(t)$ measuring the internal time at fixed laboratory time t, equation (6), is a highly nonstationary process, as are all of its increments. This holds even when $G = 1/\alpha$, i.e. when waiting times are not correlated. This phenomenon has been discussed extensively in the CTRW literature, where it is commonly referred to as ageing [41–43] and is closely related to other peculiar effects such as weak ergodicity breaking [43, 44]. The deeper reason behind this nonstationarity are scale-free characteristics of waiting times. In the context of diffusion dynamics, for instance, this absence of a typical time scale is motivated by an immense heterogeneity of the environment. In effect, the particle encounters an indefinitely broad range of waiting times and falls into deeper and deeper traps while exploring the environment. Thus, CTRW models are by definition highly nonstationary stochastic processes, and it is indeed natural to extend the common model candidates ($K = 1/\mu$ for uncorrelated, stationary jump distances and $G = 1/\alpha$ for uncorrelated, stationary waiting times) to the larger class of stable, but correlated and potentially nonstationary motions considered here.

4. Time scaling analysis and probability density function

For ordinary Lévy flights or CTRWs, the tail parameters μ and α determine both the distributional and the scaling properties of the process. The present correlated model is slightly more complex in this respect. While the shape of the PDF depends on all four parameters, only the Hurst parameters K and G determine the time scaling. To see this, recall that for the Lévy stable motions we have the characteristic scalings $L_\mu(s) \sim s^{1/\mu}$ and $L_\alpha^+(s) \sim s^{1/\alpha}$. From equations (11) and (13) it follows that $Y(s) \sim s^K$ and $T(s) \sim s^G$. Consequently, the internal time scales as $S(t) \sim t^{1/G}$, and for the correlated motion we get

$$X(t) = Y(S(t)) \sim t^H, \quad \text{where } H = K/G. \tag{22}$$

We therefore call H the scaling or Hurst exponent of the correlated motion $X(t)$. Interestingly, from the point of view of time scaling, persistence in waiting times competes with persistence in jump distances, and the process can turn out to be either sub ($H < 1/2$), or superdiffusive ($H > 1/2$), or exhibit a normal diffusive scaling ($H = 1/2$). Conversely, measuring the Hurst exponent H alone does not reveal specific information on the time scaling of correlated waiting times (G) and correlated jumps (K), but only on their ratio.

This ambiguity actually goes beyond a simple time scaling analysis and extends to the analysis of the PDF, as we show now. Let $h_{\alpha,G}(s;t)$ denote the probability density for the internal time S at given laboratory time t. Recall that $T(s) \sim s^G$ is a monotonically increasing process. This implies [32] $S(t) \stackrel{d}{=} (t/T(1))^{1/G}$ for any fixed laboratory time t. Therefore,

$$h_{\alpha,G}(s;t) = Gts^{-G-1}\ell_\alpha^+(ts^{-G}). \tag{23}$$

We can now combine equations (11), (6) and (23) to write the PDF $p_{\mu,\alpha,K,G}(x;t)$ for the CCTRW $X(t) = Y(S(t))$ at time t in terms of stable densities[10],

$$
\begin{aligned}
p_{\mu,\alpha,K,G}(x;t) &= \int_0^\infty p_{\mu,K}(x;s)h_{\alpha,G}(s;t)\,\mathrm{d}s \\
&= \int_0^\infty \frac{1}{s^K}\ell_\mu\left(\frac{x}{s^K}\right)\frac{Gt}{s^{G+1}}\ell_\alpha^+\left(\frac{t}{s^G}\right)\mathrm{d}s \\
&= \frac{1}{t^H}\int_0^\infty \ell_\mu\left(\frac{x}{(st)^H}\right)\frac{1}{s^{H+2}}\ell_\alpha^+\left(\frac{1}{s}\right)\mathrm{d}s \\
&=: \frac{1}{t^H}q_{\mu,\alpha,H}\left(\frac{x}{t^H}\right).
\end{aligned}
\tag{24}
$$

This representation demonstrates that the qualitative shape of the PDF can be classified in terms of only three parameters: the tail parameters μ and α and the scaling exponent $H = K/G$. This means that two processes may seemingly be the same when only studying their PDF and time scaling behaviour, although they are inherently different with respect to their correlations. Apparently, persistence in jump distances can balance persistence in waiting times, similar to the previously observed twin paradox [45]. Consider, for instance, the stochastic process $X(t)$ defined by $\mu = 2$, $K = 1/2$, $\alpha = 1/2$ and $G = 2$. This special case has been studied extensively in the literature, as it represents the simplest type of a CTRW process and is bare of correlations both in jump distances and waiting times. For comparison, now define $X'(t)$ by choosing $\mu' = 2$, $K' = 1.1/2$, $\alpha' = 1/2$ and $G' = 2.2$. Obviously, $X'(t)$ is different from the ordinary CTRW $X(t)$, since both its jump distances and its waiting times are persistent. This is clearly visible when investigating a few sample trajectories, as provided in figures 3 and 4.[11] However, on the level of time scaling analysis, equation (22), and PDF, equation (24), the random motions are indistinguishable, since $H = H'$.

To study the PDF $p(x;t)$ analytically (we drop subscript parameters from here on), it is natural to first study equation (24) in Fourier–Laplace domain. Making direct use of equations (2) and (4), we find

$$
\begin{aligned}
p(k;u) &\equiv \int_{-\infty}^\infty \int_0^\infty \mathrm{e}^{\mathrm{i}kx-ut}p(x;t)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^\infty \exp(-|k|^\mu s^{H\mu/\alpha})u^{\alpha-1}\exp(-u^\alpha s)\,\mathrm{d}s.
\end{aligned}
\tag{25}
$$

We now interpret the integral as a Laplace transform with respect to internal time s, while expressing the exponential in terms of a Fox H-function [47],

$$
\exp(-z) = H_{0,1}^{1,0}\left[z\;\middle|\;\overline{}\atop(0,1)\right].
\tag{26}
$$

After some straightforward manipulations of the H-function [47] we arrive at the following representation in Fourier–Laplace space,

$$
p(k;u) = \frac{\alpha}{u\mu H}H_{1,2}^{1,1}\left[\frac{u^\alpha}{|k|^{\alpha/H}}\;\middle|\;\begin{matrix}(1,\alpha/(\mu H))\\(1,1)\end{matrix}\right].
\tag{27}
$$

Inverting to laboratory time t and real space x, we find [47]

$$
p(x;t) = \frac{t^{-H}}{2\mu\sqrt{\pi}}H_{2,3}^{2,1}\left[\frac{|x|}{2t^H}\;\middle|\;\begin{matrix}(1-1/\mu,1/\mu);\,(1-H,H)\\(0,1/2),\,(1-H/\alpha,H/\alpha);\,(1/2,1/2)\end{matrix}\right].
\tag{28}
$$

---

[10] The subordination integral (24) is evaluated numerically to generate the PDF plots in figure 5. For the associated stable densities ℓ_μ (equation (2)) and ℓ_α^+ (equation (4)), numerical evaluation tools are available for computer programs such as Mathematica or MATLAB.

[11] Methods to estimate such parameters from empirical CCTRW trajectory data are discussed in [46].

Since for H-functions, series representations for small and large arguments are known, we can now analyse in detail the behaviour around the origin and in the tails. Series expansions can in principle be evaluated up to any order, see [47, 48]. Here, we discuss the leading order contributions to the PDF, or equivalently, to the scaling function $q(z) = p(z; 1)$.

In the vicinity of the starting position, $z \approx 0$, we find that the qualitative shape depends highly on the ratio α/H, if waiting times are heavy-tailed, $\alpha < 1$:

$$q(z \approx 0) \sim \begin{cases} \text{const} \cdot |z|^{-1+\alpha/H}, & \alpha/H < 1, \\ \text{const} \cdot \log|z/2|, & \alpha/H = 1, \\ q(0) - \text{const} \cdot |z|^{-1+\alpha/H}, & 1 < \alpha/H < 3, \\ q(0) - \text{const} \cdot z^2 \log|z/2|, & \alpha/H = 3, \\ q(0) - \text{const} \cdot z^2, & \alpha/H > 3. \end{cases} \tag{29}$$

The constants depend on the parameters μ, α, H, but not on the scaling variable z. Thus, the behaviour around the origin can be divergent ($\alpha/H \leqslant 1$), continuous with divergent derivative ($1 < \alpha/H < 2$), continuous with discontinuous first derivative ($2 \leqslant \alpha/H \leqslant 3$), and continuous with vanishing first derivative ($\alpha/H > 3$). While the cusp-like shape for low values of α/H is reminiscent of CTRW propagators, the increasingly smoother shape for higher values of α/H is imitating Gaussian distributions. Also note that in the absence of heavy-tailed waiting times, corresponding to $\alpha \to 1$, the scaling function returns to the class of stable laws, which are completely smooth (i.e., infinitely differentiable) everywhere. Example plots are given in figure 5.

In contrast, if $\mu < 2$, we find that heavy tails are directly inherited from the underlying jump length distribution,

$$q(z \to \infty) \simeq |z|^{-1-\mu}, \quad \text{for } \mu < 2. \tag{30}$$

This holds regardless of which type of correlations or waiting time distributions characterize the motion, see also figure 5. In the special case of Gaussian jump lengths, the tails of the PDF are of exponential type, $\log[q(z \to \infty)] \simeq -|z|^{1/2+H(1-\alpha)/\alpha}$.

Finally, let us point out an interesting, but maybe not intuitively expected property of the scaling function $q(z)$. From equation (28) one can derive [47]

$$q(z)|_{\alpha \to 1} = q(z)|_{H \to 0}. \tag{31}$$

The limit $\alpha \to 1$ leads back to a steady time progression, ultimately rendering internal time and laboratory time equivalent. Interestingly, when we study the shape of the PDFs, this is effectively the same as choosing H very small. Thus, if either waiting times are sufficiently persistent, or jump distances are sufficiently antipersistent, then the shape of the propagator indicates dynamics devoid of any stalling or trapping mechanisms.

5. Comparison with other models of correlated motions

We now briefly compare, contrast and connect the CCTRW model discussed in the previous sections to other existing models of correlated motion.

First, we stress that CCTRWs are distinct from the correlated (persistent) random walk models as discussed in [49–52]. The latter are two-dimensional random walk models, aiming at describing animal foraging and movements patterns. Angular correlations are introduced by means of nonuniform angular distributions, governing the directional evolution of the random walk at each step. Angular and step length distribution define characteristic correlation scales, beyond which the dynamics are essentially Brownian.
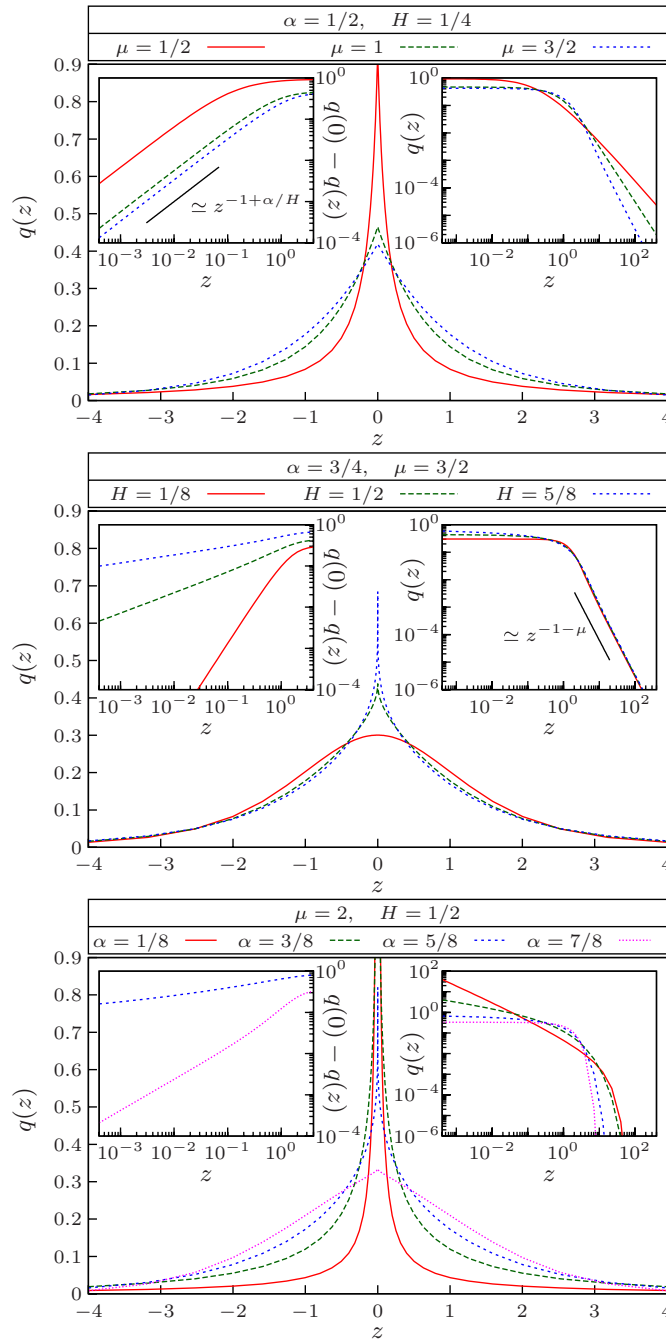
Figure 5. Scaling function $q(z)$ for the propagator $p(x;t) = t^{-H}q(xt^{-H})$, numerically evaluated through equation (24). The insets detail the behaviour around the origin (*left*) and in the tails (*right*). Top: while the exponent of power-law tails varies with μ, the qualitative behaviour at the origin is universally given by $q(0) - q(z) \simeq |z|^{-1-\alpha/H}$. Centre: conversely, a fixed stable exponent $\mu < 2$ defines the tail properties, $q(z) \simeq |z|^{-1-\mu}$. By variation of the ratio α/H, the shape of the maximum turns from a distinct cusp to a smooth Gaussian-like bell. Bottom: with $\mu = 2$, the tails are stretched exponentials. When $\alpha < H$, the scaling function diverges at the origin, $q(0) = \infty$. With $H = 1/2$, an analysis of the MSD universally indicates normal diffusion, since $X^2(t) \sim t$.

The present CCTRW model is a direct continuation of the CTRW with correlated waiting times presented in [34]. The authors discuss a laboratory time process (see equations (23) and (24) in [34], we slightly adopt the notation to our needs)

$$T(s) = \int_0^s m(s - s') \, dL_\alpha^+(s'), \qquad \text{with} \qquad m(s) = \int_0^s M(s') \, ds'. \qquad (32)$$

While the correlation kernel $m(s)$ defines the integral representation of laboratory time $T(s)$, the function $M(s) = dm/ds$ can be interpreted as a correlation kernel for the noise or waiting time process 'dT/ds' (see equation (20) of [34]). Two different types of correlation kernels are taken into consideration. First, power-law correlated waiting times, $M(s) \propto s^{-\beta}$, $\beta < 1$, lead to a power-law correlated laboratory time process, $m(s) \propto s^{1-\beta}$. By identifying $G = 1 - \beta + 1/\alpha$, $G > 1/\alpha$, we exactly arrive at the process definition used here, equation (13). Since jump lengths in the model of [34] are Gaussian and independent (which, in our language, means $\mu = 2$, $K = 1/2$) we expect a scaling relation $X(t) \sim t^H = t^{K/G} = t^{\alpha/[2\alpha(1-\beta)+2]}$. This is fully consistent with the MSD analysis in equation (A.6) of [34]. A second interesting choice for the kernel behaviours is an exponentially decaying one, i.e. $M(s) \propto \exp(-\Delta s)$, $\Delta > 0$, corresponding to $m(s) \propto 1 - \exp(-\Delta s)$. While the full scaling behaviour is difficult to calculate explicitly, we can look at the limiting cases $t \ll \Delta$ and $t \gg \Delta$. By virtue of the monotonic increase of the process $T(s)$, this is equivalent to studying approximations with respect to internal time s. For small s, we have $m(s) \simeq s = s^{1-0}$, while for large s, we get $m(s) \simeq 1 = s^{1-1}$. Hence, we expect a turnover from the scaling $X(t) \sim t^{\alpha/[2\alpha+2]}$ at $t \ll \Delta$ to $X(t) \sim t^{\alpha/2}$ at $t \gg \Delta$. This is in perfect agreement with the MSD results equations (36) and (38) in [34].

The authors of [46] discuss a subordinated process close in spirit to the one presented here. In their case, the model ingredients are scale-free, correlated jump statistics as in equation (11). Waiting times are not correlated, yet have an interesting distributional property: tail statistics are intermediately power-law distributed, but an exponential cut-off introduces an intrinsic time scale and ensures finiteness of all moments. Consequently, the diffusion process $X(t)$, defined via subordination, behaves very differently during different temporal regimes, as separated by the average waiting time. We recommend in particular the interesting discussion on the estimation of parameters from sample trajectory data.

While correlations in CTRW waiting times are not discussed explicitly in [38], the author establishes an intimate principal connection between correlated waiting times, fluctuating waiting time distributions, and rate fluctuations in underlying higher dimensional Markovian dynamics. The methods and concepts from this work may thus provide a useful approach to build a microscopic foundation of CCTRWs on the one hand, and to find reasonable model extensions on the other.

Finally, we wish to draw the connection to the CCTRW introduced in [53]. On the discrete random walk level, the basic idea is to define a nonstationary and correlated sequence of jump lengths or waiting times in terms of separate random walk processes. For instance, correlated jump lengths δY_n are derived from a Lévy flight in jump length space. In other words,

$$\delta Y_n = \sum_{j=1}^n \xi_j,$$

$$Y_n = \sum_{j=1}^n \delta Y_j = \sum_{j=1}^n \sum_{k=1}^j \xi_k, \qquad (33)$$

where the ξ_j are independent, symmetric μ-stable random variables. The intuitive way of guessing a long time limit approximation of this process can be found by replacing sums with

integrals:

$$Y(s) = \int_0^s \int_0^{s'} \mathrm{d}L_\mu(s'') \, \mathrm{d}s' = \int_0^s L_\mu(s') \, \mathrm{d}s'. \tag{34}$$

Indeed the convergence in distribution of the discrete random walk Y_n to the continuous process $Y(s)$ was proved in [54]. The latter can be thought of, according to above equation, as an integrated symmetric Lévy flight. Now according to the spectral analysis discussion we brought up in section 3, such process should actually be included in the class of correlated motions discussed in the present paper. Indeed, we could also rewrite the double sum in equation (33) as

$$Y_n = \sum_{j=1}^n (n-j)\,\xi_j. \tag{35}$$

The analogous step for continuous time is formal integration by parts of equation (34):

$$Y(s) = \int_0^s (s-s')\,\mathrm{d}L_\mu(s'). \tag{36}$$

Up to a constant prefactor, this exactly corresponds to our definition (11) with $K = 1/\mu + 1$. We thus find, in complete accordance with [53, 54], that the integrated Lévy flight is a μ-stable process with superdiffusive scaling $Y(s) \sim s^K = s^{1/\mu+1}$.

Correlated waiting times however, are defined in [53, 54] in a slightly different manner. There, consecutive waiting times δT_n are taken from a symmetric Lévy flight, subject to a reflecting boundary condition at $\delta T_n = 0$. In short,

$$\delta T_n = \left| \sum_{j=1}^n \zeta_j \right|,$$

$$T_n = \sum_{j=1}^n \delta T_j = \sum_{j=1}^n \left| \sum_{k=1}^j \zeta_k \right|, \tag{37}$$

where the ζ_j are independent, symmetric α-stable random variables with $0 < \alpha \leqslant 2$. The continuous version is

$$T(s) = \int_0^s \left| \int_0^{s'} \mathrm{d}L_\alpha(s'') \right| \mathrm{d}s' = \int_0^s |L_\alpha(s')| \, \mathrm{d}s', \tag{38}$$

which is *not* a stable process [54], and hence cannot be represented by any of our correlated laboratory time processes (13). Still, there is a formal analogy in scaling behaviours. It is easy to show that the integrated Lévy flight on the positive half-line, equation (38), is self-similar with $T(s) \sim s^{1/\alpha+1}$. The present model yields the same scaling for $G = 1/\alpha + 1$; interestingly, this corresponds to a single integration of a one-sided α-stable motion. In the case of independent Gaussian jump lengths, $\mu = 2$ and $K = 1/2$, such scaling produces subdiffusive dynamics $X(t) \sim t^{K/G} = t^{\alpha/[2(1+\alpha)]}$, as previously found in [53].

6. Conclusions

The correlated continuous time random walk (CCTRW) we introduced here combines the effects of displacements with infinite variance, sojourn times with infinite mean and long-range temporal correlations. It is thus applicable to a wide range of complex, heterogeneous systems. We found that the PDF is very distinct from an ordinary Gaussian distribution. We studied its shape extensively, revealing information contained in the tail properties and the

detailed behaviour around the origin. However, care must be taken when assessing the effects of correlations: processes with contrasting jump length and waiting time correlations can be indiscernible on the level of scaling and propagator analysis.

Moreover, we classified CCTRWs in the context of processes with stationary increments of higher order. Such considerations indicate an intimate connection between strong correlations and higher, possibly fractional-order integrals of stochastic noise processes.

Further studies of this process should include an in-depth discussion on the actual correlations within the correlated model. This question is particularly intricate, on the one hand, for scale-free displacements ($\mu < 2$), since the ordinary correlation function $\langle X(t_1)X(t_2)\rangle$ is ill-defined. But even for CTRW dynamics with $\mu = 2$, on the other hand, the issue of inter-dependences is a subtle one. By means of simple subordination arguments, one can show that the increments—and by this any notion of 'velocities'—of an unbounded, possibly CCTRW are uncorrelated. Yet, they are not stochastically independent (see e.g. [32]). It thus remains an open question how (or if) correlation functions can be used to determine and assess dependences, especially within CCTRW data[12].

It would also be interesting to study a process where correlations within displacements and waiting times are coupled, such as Lévy walks. Finally, the physical properties of such processes such as ageing or (non-)ergodicity will be of interest.

Acknowledgments

Appendix. Asymptotic distributional stationarity of higher order increments

We establish here the asymptotic behaviour of increments of arbitrary order for the correlated stable motion $Y(s)$ as defined through equation (11). Recall that by the term ASD we designate the asymptotic long-time stationarity of a single-time distribution.

First, rewrite the correlated motion $Y(s)$, equation (11), as

$$Y(s) = (\mu K)^{1/\mu} \int_0^\infty M(s - s') \, \mathrm{d}L_\mu(s'), \qquad M(s) = \theta(s) \, s^{K-1/\mu}, \qquad (A.1)$$

with θ denoting the Heaviside step function, i.e. $\theta(s \geqslant 0) = 1$ and $\theta(s < 0) = 0$.

The nth order increments, equation (16), have a similar stochastic integral representation, namely

$$\Delta^{(n)}Y(s; \tau_1, \ldots, \tau_n) = (\mu K)^{1/\mu} \int_0^\infty M^{(n)}(s - s') \, \mathrm{d}L_\mu(s') \qquad (A.2)$$

with associated integration kernels

$$
\begin{aligned}
M^{(1)}(s; \tau) &= M(s + \tau) - M(s) \\
M^{(2)}(s; \tau_1, \tau_2) &= M^{(1)}(s + \tau_2; \tau_1) - M^{(1)}(s; \tau_1) \\
&\;\;\vdots \\
M^{(n)}(s; \tau_1, \ldots, \tau_n) &= M^{(n-1)}(s + \tau_n; \tau_1, \ldots, \tau_{n-1}) - M^{(n-1)}(s; \tau_1, \ldots, \tau_{n-1}).
\end{aligned}
$$
$$(A.3)$$

---

[12] In this context it is interesting to note that even for a standard renewal subdiffusive CTRW with $\mu = 2$, the presence of spatial confinement may introduce anticorrelations in the velocity correlation function that are practically indistinguishable from the velocity correlator of FBM [55].

The characteristic function of the distribution of nth order increments is related through

$$\log\{\langle\exp[ik\Delta^{(n)}Y(s;\tau_1,\ldots,\tau_n)]\rangle\}$$
$$= -\mu K|k|^\mu \int_0^\infty |M^{(n)}(s-s';\tau_1,\ldots,\tau_n)|^\mu\,\mathrm{d}s'$$
$$= -|k|^\mu\left(I_{\mu,K}^{(n)}(s) - I_{\mu,K}^{(n)}(-\infty)\right) \tag{A.4}$$

with

$$I_{\mu,K}^{(n)}(s) = \mu K \int_0^s |M^{(n)}(s';\tau_1,\ldots,\tau_n)|^\mu\,\mathrm{d}s'. \tag{A.5}$$

The question of whether or not such distribution has a nontrivial limit for $s \to \infty$ is determined by the integral $I_{\mu,K}^{(n)}(s)$ and hence by the tail asymptotics of the integration kernel $M^{(n)}(s)$. Notice that the step function $\theta(s)$ in the process kernel (A.1) passes on to the increment kernels (A.3) and contributes in the form $\theta(s+\tau_1), \theta(s+\tau_2), \ldots, \theta(s+\tau_1+\tau_2), \theta(s+\tau_1+\tau_3),\ldots$, etc. It thus defines several *lower* bounds for the integral $I_{\mu,K}^{(n)}(-\infty)$. Conversely, for $s \geqslant 0$, all step functions entering the integral $I_{\mu,K}^{(n)}(s)$ are identically equal unity. At this point, we have to distinguish two parameter classes.

First, we can have $K = 1/\mu+m$ for some nonnegative integer m. Then the process kernel is $M(s \geqslant 0) = s^m$, and we can use standard polynomial calculus. One can show that for all $n \leqslant m$ increment kernels $M^{(n)}$ are polynomials of degree $m-n$ and thus the nonstationary contribution $I_{\mu,K}^{(n)}(s)$ grows indefinitely for large s. Conversely, for $n \geqslant m+1$, $M^{(n)}$ vanishes identically and so does the nonstationary contribution to the characteristic function. In particular, we have the Lévy stable motions, $m = 0$, $K = 1/\mu$, with stationary increments of all orders $n \geqslant 1$.

Now consider the second class of parameter pairs, i.e. $K \neq 1/\mu + m$ for all nonnegative integers m. In this case, the process kernel is a noninteger power-law, and we recursively find tail asymptotics

$$M^{(n)}(s;\tau_1,\ldots,\tau_n) \sim \tau_n\left(\frac{\partial M^{(n-1)}}{\partial s}\right)(s;\tau_1,\ldots,\tau_{n-1}) \tag{A.6}$$

for $s \gg \tau_1 + \cdots + \tau_n$. This implies the explicit tail behaviour,

$$M^{(n)}(s;\tau_1,\ldots,\tau_n) \sim \tau_1\cdots\tau_n\left(\frac{\partial^n M}{\partial s^n}\right)(s)$$
$$= \frac{\Gamma(K-1/\mu)}{\Gamma(K-1/\mu-n)}\,\tau_1\cdots\tau_n\,s^{K-1/\mu-n}, \tag{A.7}$$

so for the ultimate time dependence of the nonstationary part of the characteristic function, we get

$$I_{\mu,K}^{(n)}(s) \sim \mu K \left|\frac{\Gamma(K-1/\mu)}{\Gamma(K-1/\mu-n)}\,\tau_1\cdots\tau_n\right|^\mu$$
$$\times \begin{cases} \text{const}, & \text{for } n > K, \\ \log(s), & \text{for } n = K, \\ [\mu(K-n)]^{-1}\,s^{\mu(K-n)}, & \text{for } n < K. \end{cases} \tag{A.8}$$

Hence, only increments of order $n > K$ are ASD, while all lower order increments are spreading indefinitely with time.

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