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Encounter distribution of two random walkers on a finite one-dimensional interval

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Abstract
We analyse the first-passage properties of two random walkers confined to a finite one-dimensional domain. For the case of absorbing boundaries at the endpoints of the interval, we derive the probability that the two particles meet before either one of them becomes absorbed at one of the boundaries. For the case of reflecting boundaries, we obtain the mean first encounter time of the two particles. Our approach leads to closed-form expressions that are more easily tractable than a previously derived solution in terms of the Weierstrass' elliptic function.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The properties of the first passage, i.e. the crossing of a given threshold value in a stochastic process, are key clues to the quantitative understanding of diverse dynamic systems [1]. For instance, the distribution of first-passage times and, if it exists, the associated mean first-passage time are relevant quantities for the outflux of tracer substances from a catchment in subsurface hydrology [2], the one-dimensional diffusion of proteins along a DNA molecule [3] or the combined one- and three-dimensional search in facilitated diffusion in gene regulation [4], up to the crossing of preset prices in economic contexts [5]. The theory of first passage has recently experienced some important advances, in particular, concerning the behaviour of a single random walker in a confined environment [6, 7]. We note that the concept of first passage can be extended to generalized diffusion processes [8], e.g Lévy flights [9] or subdiffusion [10]. For subdiffusive continuous time random walks the results obtained for the Markovian case can be extended by subordination arguments [11].

A classical example for the application of first-passage concepts for encounter processes is the Smoluchowski picture of the diffusion regulation of bimolecular chemical reactions upon mutual diffusional encounter of the two molecular reactants in a three-dimensional liquid environment [12]. However, there are many cases in which the diffusional encounter of particles in a *one-dimensional* (or pseudo one-dimensional) environment becomes relevant. To name but a few examples, we mention the diffusional sliding motion of proteins or enzymes on DNA [3, 13], the diffusion of chemical reactants in the nanoconfinement of fluidic channels [14] and the relative motion of two aminoacids of a protein along the one-dimensional reaction coordinate [15].

In an unconfined environment, the encounter problem of two random walkers reduces to the consideration of the relative coordinate of the two walkers, with a diffusivity that equals the sum of the two individual diffusion constants. Similar to this unconfined case, for the diffusion of two particles on a finite domain with *periodic* boundary conditions, we may assume that one of the two walkers is fixed, and that the other diffuses with diffusivity $D = 2D_1$, where D_1 is the diffusion constant of a single walker. The problem is therefore equivalent to the first-passage problem for a single random walker, such that the mean first encounter time becomes

$$\langle T\rangle = \frac{1}{2D}d(L-d), \tag{1}$$

where d denotes the initial distance between the two random walkers and L the interval size.

If, however, we consider reflective or absorbing boundaries at the interval endpoints, the problem becomes more involved despite the seeming simplicity of this process [16]. Indeed, we can no longer reduce the two-walker problem to an effective one-walker scenario, because we now need to consider two free parameters to characterize the system, for instance, the position of one walker and its distance to the second walker, instead of only the mutual distance in the unbounded or periodic case. We here consider two cases of continuous time Brownian motion: first, the probability P_M that the random walkers meet *before* one of them is removed at the absorbing interval boundaries; and second, the typical encounter time of the two walkers in the presence of reflective boundaries. Somewhat surprisingly, these two problems are quite hard to solve. An analytic solution for the former problem (continuous time) has only recently been presented [17]:

$$P_M(x_1,x_2) = -\frac{2}{\pi}\Im\left\{\log\left[\wp\left(\frac{\omega(x_2+\mathrm{i}x_1)}{L\sqrt{8}}\right)\right]\right\}. \tag{2}$$

Here, $\omega = \int_1^\infty (x(x-1))^{-3/4}dx \approx 5.244$, $\mathrm{i} = \sqrt{-1}$, and the initial positions of the two walkers are x_1 and x_2 [17]. In equation (2), $\wp$ represents the Weierstrass elliptic function satisfying the differential equation

$$\wp'(x)^2 = 4\wp^3(x) - \wp(x). \tag{3}$$

The computation of the imaginary part of the logarithm of a complex number may become difficult and quite time consuming numerically, and elliptic functions are often cumbersome to deal with in analytic calculations. It would therefore be desirable to find a simpler expression for this problem. In the following we show that weak approximations lead to closed-form expressions for the relevant quantities in terms of trigonometric, hyperbolic and logarithmic functions. The high accuracy of these rather simple analytical results is corroborated by numerical simulations.

The paper is structured as follows. In the next section, we calculate the encounter probability of the two walkers before being annihilated by hitting the absorbing walls. We then proceed to the scenario of first encounter in the opposite case of reflecting boundaries, before concluding in section 4.

2. Encounter probability

We first consider the case of two absorbing boundaries and calculate the probability for encounter of the two walkers before either one of them becomes removed on hitting a boundary. To this end we rephrase the problem of two walkers in a one-dimensional domain by a single walker in a finite two-dimensional domain of size $L \times L$. We then seek the probability that, after starting from the point (x_1, x_2), this single random walker on the two-dimensional domain crosses the diagonal $x = y$, before touching the boundaries for the two coordinates, $x \in \{0, L\}$ or $y \in \{0, L\}$. Without loss of generality, we assume that $x_1 > x_2$.

As the process is terminated when the two-dimensional random walker crosses the diagonal $x = y$, we use the method of images to determine the associated probability. First, we compute the probability that a two-dimensional random walker with diffusion coefficient $D = 2D_1$, where D_1 is the diffusion constant of a single one-dimensional walker, hits a given wall in an $L \times L$ square, whose boundaries are absorbing. The probability $P(x, y, t|x_1, x_2)$ to be at position (x, y) at time t starting from (x_1, x_2) at $t = 0$ becomes

$$P(x, y, t|x_1, x_2) = \frac{4}{L^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sin\left(\frac{kx\pi}{L}\right)\sin\left(\frac{kx_1\pi}{L}\right)\sin\left(\frac{ly\pi}{L}\right)\sin\left(\frac{lx_2\pi}{L}\right) \times\exp\left(-\frac{(k^2+l^2)D\pi^2 t}{L^2}\right) \tag{4}$$

in terms of the eigenmode expansion. The probability $\mathrm{Pr}_{\mathrm{wall}}(x = L|x_1, x_2)$ to hit the wall at $x = L$ starting from (x_1, x_2) at $t = 0$ is then calculated as

$$\begin{aligned}\mathrm{Pr}_{\mathrm{wall}}(x = L|x_1, x_2) &= -\int_0^{\infty}\int_{y=0}^{L} D\left.\frac{\partial P(x, y, t|x_1, x_2)}{\partial x}\right|_{x=L}\mathrm{d}y\,\mathrm{d}t\\ &= -\frac{4}{\pi^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(-1)^k\frac{k}{l}(1-(-1)^l)\sin\left(\frac{kx_1\pi}{L}\right)\sin\left(\frac{lx_2\pi}{L}\right)\frac{1}{k^2+l^2}\\ &= -\frac{8}{\pi^2}\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)x_2\pi}{L}\right)}{2l+1}\sum_{k=1}^{\infty}\frac{(-1)^k k}{k^2+(2l+1)^2}\sin\left(\frac{kx_1\pi}{L}\right).\end{aligned} \tag{5}$$

The term in $-D(\partial P/\partial x)(x, y, t)$ is the probability flux in the $+x$ direction, at point (x, y) and time t (Fick's first law). We then exploit the Fourier expansion of the sinh function (more precisely, the periodic function equal to sinh between $-\pi$ and π),

$$\frac{\sinh(ax)}{\sinh(a\pi)} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{a^2+n^2}\sin(nx), \tag{6}$$

to rewrite the probability $\mathrm{Pr}_{\mathrm{wall}}$ to hit the wall at $x = L$:

$$\mathrm{Pr}_{\mathrm{wall}}(x = L|x_1, x_2) = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)x_2\pi}{L}\right)}{2l+1}\frac{\sinh\left(\frac{(2l+1)x_1\pi}{L}\right)}{\sinh((2l+1)\pi)}. \tag{7}$$

This expression simplifies when we introduce the approximation $\sinh((2l+1)\pi) \approx \exp((2l+1)\pi)/2$, leading to

$$
\begin{aligned}
\mathrm{Pr}_{\mathrm{wall}}(x=L|x_1,x_2) &\approx \frac{8}{\pi}\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)x_2\pi}{L}\right)}{2l+1}\sinh\left(\frac{(2l+1)x_1\pi}{L}\right)\times\exp\left(-(2l+1)\pi\right)\\
&=\frac{2}{\mathrm{i}\pi}\sum_{l=0}^{\infty}\left(\frac{\exp\left(\frac{\pi}{L}(\mathrm{i}x_2+x_1-L)\right)^{2l+1}}{2l+1}+\frac{\exp\left(\frac{\pi}{L}(-\mathrm{i}x_2-x_1-L)\right)^{2l+1}}{2l+1}\right.\\
&\quad\left.-\frac{\exp\left(\frac{\pi}{L}(\mathrm{i}x_2-x_1-L)\right)^{2l+1}}{2l+1}-\frac{\exp\left(\frac{\pi}{L}(-\mathrm{i}x_2+x_1-L)\right)^{2l+1}}{2l+1}\right)\\
&=\frac{1}{\mathrm{i}\pi}\left(\ln\left(\frac{1+\exp\left(\frac{\pi}{L}(\mathrm{i}x_2+x_1-L)\right)}{1-\exp\left(\frac{\pi}{L}(\mathrm{i}x_2+x_1-L)\right)}\frac{1-\exp\left(\frac{\pi}{L}(-\mathrm{i}x_2+x_1-L)\right)}{1+\exp\left(\frac{\pi}{L}(-\mathrm{i}x_2+x_1-L)\right)}\right)\right.\\
&\quad\left.-\ln\left(\frac{1+\exp\left(\frac{\pi}{L}(\mathrm{i}x_2-x_1-L)\right)}{1-\exp\left(\frac{\pi}{L}(\mathrm{i}x_2-x_1-L)\right)}\frac{1-\exp\left(\frac{\pi}{L}(-\mathrm{i}x_2-x_1-L)\right)}{1+\exp\left(\frac{\pi}{L}(-\mathrm{i}x_2-x_1-L)\right)}\right)\right)\\
&=\frac{1}{\mathrm{i}\pi}\left(\ln\left(\frac{1+\mathrm{i}\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi(L-x_1)}{L}\right)}}{1-\mathrm{i}\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi(L-x_1)}{L}\right)}}\right)-\ln\left(\frac{1+\mathrm{i}\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi(L+x_1)}{L}\right)}}{1-\mathrm{i}\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi(L+x_1)}{L}\right)}}\right)\right)\\
&=\frac{2}{\pi}\left(\arctan\left(\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi(L-x_1)}{L}\right)}\right)-\arctan\left(\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi(L+x_1)}{L}\right)}\right)\right).
\end{aligned}\tag{8}
$$

The maximal deviation of the approximate result (8) from the exact expression is around 0.2%. Consistently, equation (8) vanishes at $x_1 = 0$, $x_2 = 0$, or $x_2 = L$. At $x_1 = L$, expression (8) is approximately 1.

To link this result with the meeting probability of two random walkers in a one-dimensional domain with absorbing boundaries, we consider a random walker on the two-dimensional $L\times L$ square with diffusion coefficient D, starting from (x_1, x_2). Since the process terminates when $x = y$, the line $x = y$ is considered absorbing. We use the images method to obtain the propagator $P_T(x, y, t|x_1, x_2)$ with absorbing boundary conditions on the lines $x = y$, $x = L$ and $y = 0$ in terms of the propagator P determined above with absorbing conditions only on the square boundary:

$$P_T(x,y,t|x_1,x_2) = P(x,y,t|x_1,x_2) - P(x,y,t|x_2,x_1). \tag{9}$$

This expression is a direct generalization of the image method for one walker [1]. The probability $P_M(x_1, x_2)$ to reach the line $x = y$ before the walls $y = 0$ and $x = L$ starting from (x_1, x_2) at $t = 0$ is 1 minus the probability to reach the border $x = L$ or $y = 0$. Using equations (9) and (8) to compute these probabilities, $P_M(x_1, x_2)$ is thus given by the sum

$$
\begin{aligned}
P_M(x_1,x_2) &= 1-(\mathrm{Pr}_{\mathrm{wall}}(x=L|x_1,x_2)-\mathrm{Pr}_{\mathrm{wall}}(x=L|x_2,x_1)\\
&\quad+\mathrm{Pr}_{\mathrm{wall}}(y=0|x_1,x_2)-\mathrm{Pr}_{\mathrm{wall}}(y=0|x_2,x_1))\\
&=1-(\mathrm{Pr}_{\mathrm{wall}}(x=L|x_1,x_2)-\mathrm{Pr}_{\mathrm{wall}}(x=0|x_1,x_2)\\
&\quad+\mathrm{Pr}_{\mathrm{wall}}(y=0|x_1,x_2)-\mathrm{Pr}_{\mathrm{wall}}(y=L|x_1,x_2))\\
&=2\left(\mathrm{Pr}_{\mathrm{wall}}(x=0|x_1,x_2)+\mathrm{Pr}_{\mathrm{wall}}(y=L|x_1,x_2)\right)\\
&=\frac{8}{\pi}\left(\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)x_2\pi}{L}\right)}{2l+1}\frac{\sinh\left(\frac{(2l+1)(L-x_1)\pi}{L}\right)}{\sinh\left((2l+1)\pi\right)}\right.\\
&\quad\left.+\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2l+1)x_1\pi}{L}\right)}{2l+1}\frac{\sinh\left(\frac{(2l+1)x_2\pi}{L}\right)}{\sinh\left((2l+1)\pi\right)}\right).
\end{aligned}\tag{10}
$$

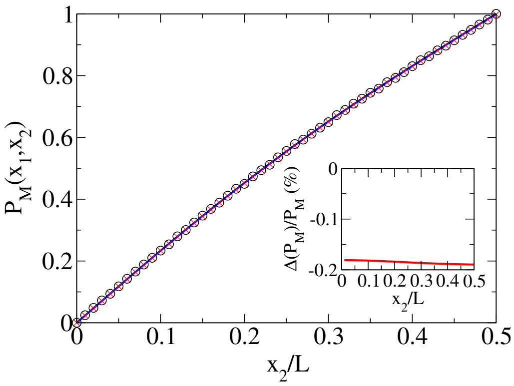

Figure 1. Encounter probability of two random walkers on the interval [0, 1], initially placed at (x_1, x_2), where $x_1 = 0.5$ and x_2 varies between 0 and 0.5. The simulation is in discrete time, on a segment of size $L = 100$, where the random walker makes Gaussian steps of variance $\sigma = 1$. We compare simulation results (black circles) with the approximate result given by equation (11) (red crosses) and the exact result from equation (10) (blue line). The inset shows the relative difference between the approximate and exact results. The error is of the same order of magnitude for all x_2 (below 0.2%).

Note that to pass from the first to the second equality we used the symmetry of the square geometry. Equation (10) is an exact equivalent to the expression in [17]. Our result has the advantage of being much easier to compute numerically, for the occurrence of simple trigonometric functions. Moreover, as before we can approximate this expression by the arctan function, yielding

$$P_M(x_1, x_2) \approx \frac{4}{\pi}\left(\arctan\left(\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi x_1}{L}\right)}\right) - \arctan\left(\frac{\sin\left(\frac{\pi x_2}{L}\right)}{\sinh\left(\frac{\pi(2L-x_1)}{L}\right)}\right)\right.$$
$$\left. + \arctan\left(\frac{\sin\left(\frac{\pi x_1}{L}\right)}{\sinh\left(\frac{\pi(L-x_2)}{L}\right)}\right) - \arctan\left(\frac{\sin\left(\frac{\pi x_1}{L}\right)}{\sinh\left(\frac{\pi(L+x_2)}{L}\right)}\right)\right). \tag{11}$$

This is the first main result of this study.

Figures 1 and 2 show excellent agreement between the exact and the approximate formula of equations (10) and (11), as well as with numerical simulations of the encounter process. The approximate formula provided by equation (11) obviously provides an excellent approximation. Its numerical evaluation is significantly quicker than the exact result involving the Weierstrass elliptic function with complex argument. Moreover, the accuracy is more than sufficient for most purposes: the relative error is always smaller than 0.2%. The approximate result is far easier to handle analytically than the exact expression of equation (2) proposed in [17], and the dependence on the geometrical parameters is more explicit.

3. First encounter time

We now consider the analogous problem with reflective boundaries at the endpoints of the one-dimensional interval: the orthogonal speed of the continuous random walker is inverted when hitting a boundary. We determine the mean first encounter time, namely the average of the first time when the two random walkers encounter each other. To compute this time, we again transform the problem of two random walkers in the one-dimensional domain into

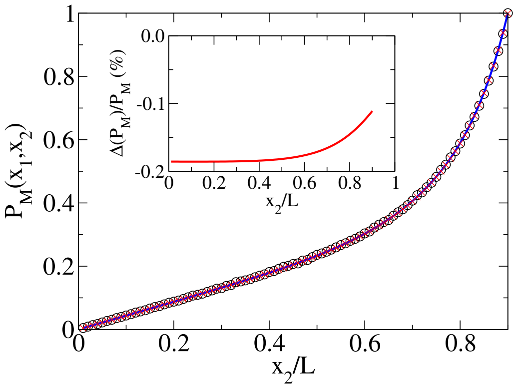

Figure 2. Encounter probability of two random walkers initially placed at (x_1, x_2), where $x_1 = 0.9$ and x_2 varies between 0 and 0.9. The simulation is in discrete time, on a segment of size $L = 100$, where the random walker makes Gaussian steps of variance $\sigma = 1$. We compare the simulation results (black circles) with the approximate result given by equation (11) (red crosses) and the exact result, equation (10) (blue line). The inset shows the relative difference between the approximate and the exact results (red line). The error is of the same order of magnitude for all x_2 (below 0.2%).

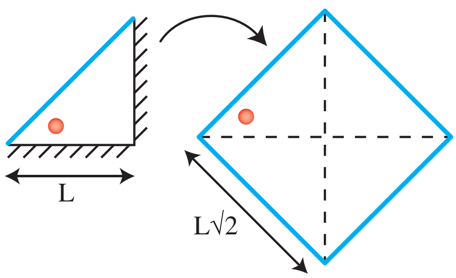

Figure 3. Symmetry used to simplify the first encounter problem: the first-passage time to the diagonal on a triangle is equivalent to a first exit time on a $\sqrt{2}L \times \sqrt{2}L$ square.

a two-dimensional single-walker problem. This two-dimensional walker moves on one half of the square domain $L \times L$, separated by the diagonal. In this half-square, the diagonal is absorbing while the two equilateral edges are reflecting. As shown in figure 3, we transform by symmetry this problem to the first exit time on a $\sqrt{2}L \times \sqrt{2}L$ square, where, if we take $x_1 > x_2$, the initial coordinates are $(x_0, y_0) = ((x_1 - x_2)/\sqrt{2}, (x_1 + x_2)/\sqrt{2})$.

We start with the propagator on the $\sqrt{2}L \times \sqrt{2}L$ square:

$$P(x, y, t|x_0, y_0) = \frac{2}{L^2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sin\left(\frac{kx\pi}{\sqrt{2}L}\right) \sin\left(\frac{kx_0\pi}{\sqrt{2}L}\right) \sin\left(\frac{ly\pi}{\sqrt{2}L}\right) \times \sin\left(\frac{ly_0\pi}{\sqrt{2}L}\right) \exp\left(-\frac{(k^2 + l^2)D\pi^2 t}{2L^2}\right). \tag{12}$$

The survival probability, namely the probability $\mathscr{S}(t|x_0, y_0)$ that at time t, a continuous random walker starting from (x_0, y_0) at $t = 0$ has not hit any boundary, becomes

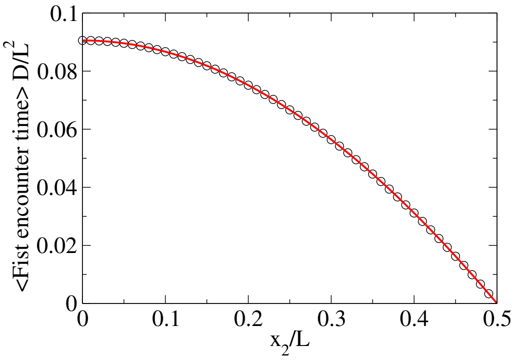

Figure 4. Mean first encounter time $\langle T\rangle$ between two random walkers initially placed at (x_1, x_2) in a one-dimensional domain, where $x_1 = 0.5$, and x_2 varies between 0 and 0.5. The simulation is in discrete time, on a segment of size $L = 100$, where the random walker makes Gaussian steps of variance $\sigma = 1$. We compare the simulation results (black circles) with the exact result, equation (14) (red line).

$$\mathcal{S}(t|x_0,y_0) = \int_0^{\sqrt{2}L}\int_0^{\sqrt{2}L} P(x,y,t|x_0,y_0)dx\,\mathrm{d}y = \frac{16}{\pi^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{2k+1}\frac{\sin\left(\frac{(2l+1)y_0\pi}{\sqrt{2}L}\right)}{2l+1}$$
$$\times\exp\left(-\frac{((2k+1)^2+(2l+1)^2)D\pi^2 t}{2L^2}\right), \tag{13}$$

from which we deduce the mean first encounter time $\langle T\rangle(x_0, y_0)$ as function of the initial positions x_0 and y_0:

$$\begin{aligned}\langle T\rangle(x_0,y_0) &= \int_0^{\infty}\mathcal{S}(t|x_0,y_0)\,\mathrm{d}t\\
&= \frac{32L^2}{D\pi^4}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{2k+1}\frac{\sin\left(\frac{(2l+1)y_0\pi}{\sqrt{2}L}\right)}{2l+1}\frac{1}{(2k+1)^2+(2l+1)^2}\\
&= \frac{32L^2}{D\pi^4}\sum_{k=0}^{\infty}\frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{(2k+1)^3}\sum_{l=0}^{\infty}\left(\frac{\sin\left(\frac{(2l+1)y_0\pi}{\sqrt{2}L}\right)}{2l+1}-\frac{(2l+1)\sin\left(\frac{(2l+1)y_0\pi}{\sqrt{2}L}\right)}{(2k+1)^2+(2l+1)^2}\right)\\
&= \frac{8L^2}{D\pi^3}\sum_{k=0}^{\infty}\frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{(2k+1)^3}\left(1-\frac{\sinh\left(\frac{(2k+1)y_0\pi}{\sqrt{2}L}\right)+\sinh\left(\frac{(2k+1)(\sqrt{2}L-y_0)\pi}{\sqrt{2}L}\right)}{\sinh((2k+1)\pi)}\right)\\
&= \frac{x_0}{2D}(\sqrt{2}L-x_0)-\frac{8L^2}{D\pi^3}\sum_{k=0}^{\infty}\frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{(2k+1)^3}\\
&\quad\times\frac{\sinh\left(\frac{(2k+1)y_0\pi}{\sqrt{2}L}\right)+\sinh\left(\frac{(2k+1)(\sqrt{2}L-y_0)\pi}{\sqrt{2}L}\right)}{\sinh((2k+1)\pi)}.\end{aligned} \tag{14}$$

Here, we made use of the expansion of the sinh function. The exact result (14) shows perfect agreement with numerical simulations, as demonstrated in figure 4.

In equation (14) the first term is the mean first exit time of a one-dimensional random walker confined to a domain of size $\sqrt{2}L$, with diffusion coefficient D. The second term is the correction in a square domain. We approximate this second term in some limits. For instance, when the two particles are initially near a corner of the one-dimensional domain, i.e.

$x_2 + x_1 \ll L$, such that $y_0/L \ll 1$ and $x_0/L \ll 1$, we have

$$\sum_{k=0}^{\infty} \frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{(2k+1)^3} \frac{\sinh\left(\frac{(2k+1)y_0\pi}{\sqrt{2}L}\right) + \sinh\left(\frac{(2k+1)(\sqrt{2}L-y_0)\pi}{\sqrt{2}L}\right)}{\sinh((2k+1)\pi)}$$
$$\approx \sum_{k=0}^{\infty} \frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{(2k+1)^3} \exp\left(-\frac{(2k+1)y_0\pi}{\sqrt{2}L}\right). \tag{15}$$

This expression can be simplified in the limit $x_0/L \ll 1$, using

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3} \exp(-ky) = x\mathrm{Li}_2(\exp(-y)) + \mathcal{O}(x^3), \tag{16}$$

where Li_2 is the dilogarithm defined as

$$\mathrm{Li}_2(z) = \sum_{k=0}^{\infty} \frac{z^k}{k^2}. \tag{17}$$

The series expansion of $\mathrm{Li}_2(z)$ around 1^- is

$$\mathrm{Li}_2(z) = \frac{\pi^2}{6} + (1 - \ln(1-z))(z-1) + \mathcal{O}((z-1)^2). \tag{18}$$

We thus obtain

$$\sum_{k=0}^{\infty} \frac{\sin\left(\frac{(2k+1)x_0\pi}{\sqrt{2}L}\right)}{(2k+1)^3} \frac{\sinh\left(\frac{(2k+1)y_0\pi}{\sqrt{2}L}\right) + \sinh\left(\frac{(2k+1)(\sqrt{2}L-y_0)\pi}{\sqrt{2}L}\right)}{\sinh((2k+1)\pi)}$$
$$\approx \frac{x_0\pi}{2\sqrt{2}L}\left(\frac{\pi^2}{4} + \left(\ln\left(\frac{y_0\pi}{\sqrt{2}L}\right) - 1 - \ln(2)\right)\frac{y_0\pi}{\sqrt{2}L}\right). \tag{19}$$

Thus, when both particles are initially close to an endpoint of the interval, we find

$$\langle T\rangle(x_0, y_0) \approx \frac{2x_0y_0}{D\pi}\left(1 + \ln(2) - \ln\left(\frac{y_0\pi}{\sqrt{2}L}\right)\right) - \frac{x_0^2}{2D}. \tag{20}$$

A similar approach leads to the same result when $\sqrt{2}L - y_0 \ll L$ and $x_0/L \ll 1$, if we replace y_0 by $\sqrt{2}L - y_0$. In the original variables we rewrite the previous expression as follows:

$$\langle T\rangle(x_1, x_2) \approx \frac{x_1^2 - x_2^2}{D\pi}\left(1 + \ln(2) - \ln\left(\frac{(x_1+x_2)\pi}{2L}\right)\right) - \frac{(x_1-x_2)^2}{4D}. \tag{21}$$

Figure 5 demonstrates excellent agreement with the simulations over a quite large range.

Finally, we calculate the associated first-passage density $p_T(x_0, y_0, t)$. It directly follows from the survival probability $\mathscr{S}(t|x_0, y_0)$ in equation (13), through

$$p_T(x_0, y_0, t) = -\frac{\partial\mathscr{S}(t|x_0, y_0)}{\partial t}$$
$$= \frac{8D}{L^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{2k+1}{2l+1} + \frac{2l+1}{2k+1}\right)\sin\left(\frac{(2k+1)x_0\pi}{L\sqrt{2}}\right)\sin\left(\frac{(2l+1)y_0\pi}{L\sqrt{2}}\right)$$
$$\times \exp\left(-\frac{((2k+1)^2 + (2l+1)^2)D\pi^2 t}{2L^2}\right). \tag{22}$$

Figure 6 shows that only the first few terms of this infinite sum are sufficient to reproduce very accurately the simulations results. We observe that the leading term is given by the exponential

$$p_T(x_0, y_0, t) \sim \frac{16D}{L^2}\sin\left(\frac{x_0\pi}{\sqrt{2}L}\right)\sin\left(\frac{y_0\pi}{\sqrt{2}L}\right)\exp\left(-\frac{D\pi^2 t}{L^2}\right). \tag{23}$$

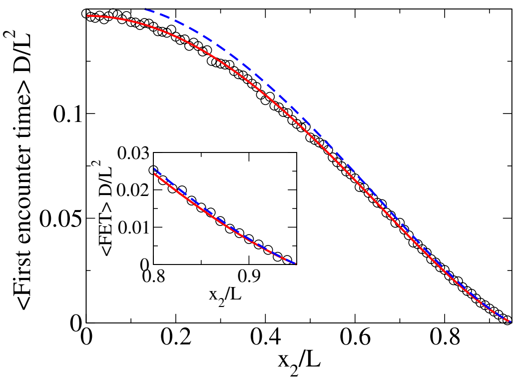

Figure 5. Mean first encounter time $\langle T \rangle$ between two random walkers initially placed at (x_1, x_2) on a one-dimensional domain, where $x_1 = 0.95$ and x_2 varies between 0 and 0.95. The simulation is in discrete time, on a segment of size $L = 100$, where the random walker makes Gaussian steps of variance $\sigma = 1$. We compare the simulation results (black circles) with the exact result given by equation (14) (red line) and the approximation in equation (21) (blue dashed line). The inset shows a zoom into the area around 0.95, where the approximation is valid ($\sqrt{2}L - y_0 \ll L$ and $x_0/L \ll 1$).

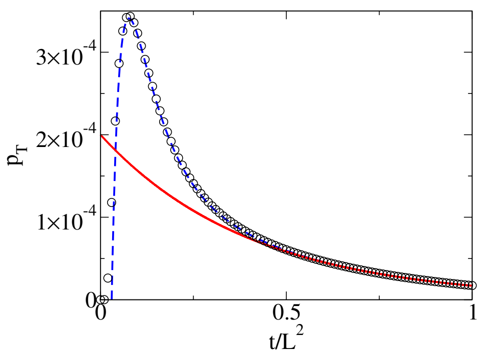

Figure 6. First encounter time distribution p_T of two random walkers on a finite one-dimensional domain, initially positioned at (x_1, x_2), where $x_1 = 0$ and $x_2 = 0.5$, as a function of time t. The simulation is in discrete time, on a segment of size $L = 100$, where the random walker makes Gaussian steps of variance $\sigma = 1$. We compare the simulations results (black circles) with the leading exponential of the exact result given by equation (22) (red line) and the first ten terms of the same expression (dashed blue line).

4. Conclusions

We obtained a simple formula for the encounter probability of two random walkers on a finite one-dimensional domain with absorbing boundary conditions at the interval endpoints. The resulting infinite sum can be approximated by analytical functions to high accuracy. The obtained result facilitates significantly the analytical and numerical handling of this problem, compared to the rigorous mathematical result. For the first encounter time of two random walkers in the presence of reflective boundary conditions, analogous results are presented and approximations are obtained. While the calculations are performed in continuous time, the simulations are performed in discrete time. The excellent agreement between these corroborate the equivalence of both approaches for the case of Brownian motion. We expect that this contribution will be useful in a broad range of applications of first passage theories.

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