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Natural cutoff in Lévy flights caused by dissipative nonlinearity

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Lévy flight models are often used to describe stochastic processes in complex systems. However, due to the occurrence of diverging position and/or velocity fluctuations Lévy flights are physically problematic if describing the dynamics of a particle of finite mass. Here we show that the velocity distribution of a random walker subject to Lévy noise can be regularized by nonlinear friction, leading to a natural cutoff in the velocity distribution and finite velocity variance.

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Lévy flights (LFs) are Markovian random processes whose probability density function (PDF) $P(X,t)$ has a characteristic function $P(k,t)=\exp(-D|k|^{\alpha}t)$ $(0<\alpha<2)$ of stretched Gaussian form, causing the asymptotic decay $P(X,t)\sim Dt/|X|^{1+\alpha}$ for $|X|\gg(Dt)^{1/\alpha}$ such that the variance diverges, $\langle X^2(t)\rangle=\infty$ [1–4]. LFs serve as good approximations to various random processes in complex systems, such as motion patterns of biological species [5–9], fluctuations in plasmas [10], optical lattices [11], or stock market dynamics [12]; see Ref. [13] for a recent review.

Although there are systems, e.g., diffusion on a polymer in chemical space mediated by jumps at places where the polymer loops back on itself [14], for which diverging fluctuations do not violate physical principles, for a particle with finite mass in position and/or velocity space the existence of a diverging variance in a strict sense must be considered a pathology. There are certain ways out of this hitch: (i) by a time cost through coupling between x and t, producing Lévy walks [15,16]; (ii) or by a cutoff in the Lévy noise to prevent the divergencies [17,18]. While (i) appears a natural choice, it gives rise to a non-Markovian process. Conversely (ii) corresponds to an *ad hoc* measure.

Here, we pursue an alternative route to processes governed by Lévy noise, based on a nonlinear friction term. Such dissipative nonlinearity occurs naturally for particles in a frictional environment at higher velocities [19]. A classical example is the Riccati equation $M dv(t)/dt = Mg - Kv(t)^2$ for the fall of a particle of mass M in a gravitational field with acceleration g[20], or autonomous oscillatory systems with a friction that is nonlinear in the velocity [19,21]. The occurrence of a nonconstant friction coefficient $\gamma(V)$ leading to a nonlinear dissipative force $-\gamma(V)V$ was highlighted in Klimontovich's theory of nonlinear Brownian motion [22]. In what follows, we show that dissipative nonlinear structures regularize a stochastic process subject to Lévy noise, leading to finite variance of velocity fluctuations and thus a well-defined kinetic energy. The velocity PDF $P(V,t)$ associated with this process turns out to preserve the properties of the Lévy process for smaller velocities, but it decays faster than a Lévy stable density (LSD) and thus possesses a physical cutoff. In what follows, we start with the asymptotic behavior for large V, and then address the remaining, central part of $P(V,T)$, that preserves the LSD property.

Dynamical equation with Lévy noise and dissipative nonlinearity. The Langevin equation for a random process in the velocity coordinate V is $M\dot{V}=-V/\mu+F(t)$, where μ denotes the mobility and $F(t)$ the white Gaussian noise. For an LF with rescaled $\gamma_0=1/(M\mu)$ and $F(t)=M\dot{L}(t)$, the Langevin equation is usually written in the form [23]

$$dV(t)+\gamma(V)V(t)dt=dL(t), \tag{1}$$

with the constant friction $\gamma_0=\gamma(0)$. $L(t)$ is the α-stable Lévy noise defined in terms of a characteristic function $p^*(\omega,t)=\mathcal{F}\{p(L,t)\}\equiv\int_{-\infty}^{\infty}p(L,t)\exp(i\omega L)dL$ of the form $p^*(\omega,t)=\exp(-D|\omega|^{\alpha}t)$ [1,24], where D of dimension $\mathrm{cm}^{\alpha}/\mathrm{sec}$ is the generalized diffusion constant. The characteristic function of the velocity PDF $P(V,t)$, $P^*(q,t)\equiv\mathcal{F}\{P(V,t)\}$ is then governed by the dynamical equation [23]

$$\frac{\partial P^*(q,t)}{\partial t}=-\gamma_0 q\frac{\partial P^*}{\partial q}-D|q|^{\alpha}P^*. \tag{2}$$

At stationarity, this characteristic function assumes the form $P^*_{\mathrm{st}}(q,t)=\exp[-D|q|^{\alpha}/(\gamma_0\alpha)]$ such that the PDF $P(V,t)$ converges towards an LSD of index α. This stationary solution possesses, however, a diverging variance [27].

To overcome the divergence of the variance $\langle V^2(t)\rangle$, we introduce into Eq. (1) the velocity dependent dissipative nonlinear form $\gamma(V)$ for the friction coefficient [19,22]. We require $\gamma(V)$ to be symmetric in V [22], assuming the virial expansion up to order $2N$

$$\gamma(V)=\gamma_0+\gamma_2V^2+\cdots+\gamma_{2N}V^{2N}, \quad \therefore \gamma_{2N}>0. \tag{3}$$

The coefficients γ_{2n} are assumed to decrease rapidly with growing n $(n\in\mathbb{N})$. To infer the asymptotic behavior, it is sufficient to take along the highest power $2N$. More generally, we will consider a power $\gamma_{\nu}|V|^{\nu}$ with $\nu\in\mathbb{R}^+$ and $\gamma_{\nu}>0$. We will show that, despite the input driving Lévy noise, by inclusion of the dissipative nonlinearity (3) the resulting process possesses a finite variance.

To this end, we pass to the kinetic equation for $P(V,t)$, the fractional Fokker-Planck equation [13,25,26,28–30]

$$\frac{\partial P(V,t)}{\partial t} = \frac{\partial}{\partial V}(V\gamma(V)P) + D\frac{\partial^\alpha P}{\partial |V|^\alpha}. \quad (4)$$

We note that, apart from the Langevin approach, Eq. (4) can be derived from a continuous time random walk with inhomogeneous jump length distribution or the generalized Chapman-Kolmogorov equation [31]. The nonlinear friction coefficient $\gamma(V)$ thereby takes on the role of a confining potential: while for $\gamma_0 = \gamma(0)$ the drift term $V\gamma_0$ is just the restoring force exerted by the harmonic Ornstein-Uhlenbeck potential, the next higher-order contribution $\gamma_2 V^3$ corresponds to a quartic potential, and so forth. The fractional operator $\partial^\alpha/\partial|V|^\alpha$ in Eq. (4) is defined through its Fourier transform, $\mathcal{F}\{\partial^\alpha/\partial|V|^\alpha f(V)\} = -|q|^\alpha f^*(q)$. For $1<\alpha<2$, it is represented explicitly by [13,30]

$$\frac{d^\alpha f(V)}{d|V|^\alpha} = -\kappa\frac{d^2}{dV^2}\int_{-\infty}^{\infty}\frac{f(V')}{|V-V'|^{\alpha-1}}dV', \quad (5)$$

with $\kappa^{-1} \equiv 2\cos(\pi\alpha/2)\Gamma(2-\alpha)$.

Asymptotic behavior. To derive the asymptotic behavior of $P(V,t)$ in the presence of a particular form of $\gamma(V)$, it is sufficient to consider the highest power, say, $\gamma(V) \sim \gamma_\nu |V|^\nu$. In particular, to infer the behavior of the stationary PDF $P_{st}(V)$ for $V\to\infty$, it is reasonable to assume that we can cut off the integral $\int_{-\infty}^{\infty} dV'$ in the fractional operator (5) at the pole $V'=V$, since the region of integration for the remaining left-side operator is much larger than the cut-off right-side region. Moreover, the remaining integral over $(-\infty, V)$ also contains the major portion of the PDF. For $V\to+\infty$, we find in the stationary state after integration by V,

$$\gamma_\nu V^{\nu+1} P_{st}(V) \simeq D\kappa\frac{d}{dV}\int_{-\infty}^{V}\frac{P_{st}(V')}{(V-V')^{\alpha-1}}dV'. \quad (6)$$

We then use the ansatz $P_{st}(V) \sim C/|V|^\mu$, $\mu>0$. With the approximation $\int_{-\infty}^{V} P_{st}(V')/(V-V')^{\alpha-1}dV' \sim V^{1-\alpha}\int_{-\infty}^{V}P_{st}(V')dV' \sim V^{1-\alpha}\int_{-\infty}^{\infty}P_{st}(V')dV' = V^{1-\alpha}$ we obtain the asymptotic form

$$P_{st}(V) \simeq \frac{C_\alpha D}{\gamma_\nu |V|^\mu}, \quad \mu = \alpha+\nu+1, \quad (7)$$

valid for $V\to\pm\infty$ due to symmetry. We conclude that for all $\nu > \nu_{cr} = 2-\alpha$ the variance $\langle V^2\rangle$ is finite, and thus a dissipative nonlinearity whose highest power ν exceeds the critical value ν_{cr} counterbalances the energy supplied by the Lévy noise $L(t)$.

Numerical solution of quadratic and quartic nonlinearity. Let us consider the case of dissipative nonlinearity up to quartic order contribution, $\gamma(V) = \gamma_0 + \gamma_2 V^2 + \gamma_4 V^4$. According to the previous result (7), the stationary PDF for the quadratic case with $\gamma_2>0$ and $\gamma_4=0$ falls off like $P_{st}(V) \sim |V|^{-\alpha-3}$, and thus $\forall \alpha \in (0,2]$ the variance $\langle V^2\rangle$ is finite. Higher-order moments, such as the fourth-order moment $\langle V^4\rangle$ are, however, still infinite. In contrast, if $\gamma_4>0$, also this fourth-order moment is finite. We investigate these claims numerically by solving the Langevin equation (1); compare Ref. [30] for details.

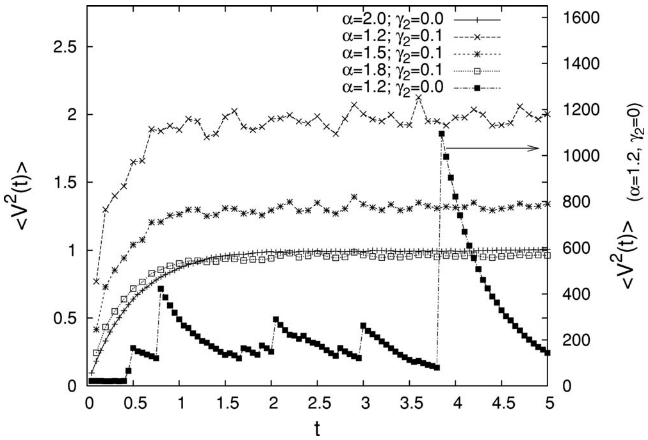

FIG. 1. Variance $\langle V^2(t)\rangle$ as function of time t, with the quartic term set to zero, $\gamma_4=0$ and $\gamma_0=1.0$ for all cases. The variance is finite for the cases $\alpha=2.0$, $\gamma_2=0.0$; $\alpha=1.2$, $\gamma_2=0.1$; $\alpha=1.5$, $\gamma_2=0.1$; and $\alpha=1.8$, $\gamma_2=0.1$. These correspond to the left ordinate. For the case $\alpha=1.2$, $\gamma_2=0.0$, the variance diverges and strong fluctuations are visible; note the large values of this curve corresponding to the right ordinate.

In Fig. 1, we show the time evolution of the variance $\langle V^2(t)\rangle$ for various combinations of Lévy index α and magnitude γ_2 of the quadratic nonlinearity ($\gamma_0=1.0$ and $\gamma_4=0.0$). For all cases with finite γ_2 ($\gamma_2=0.1$), we find convergence of the variance to a stationary value. For the two smaller α (1.2 and 1.5), we observe some fluctuations, however these are comparatively small in respect to the stationary value they oscillate around. For $\alpha=1.8$, the fluctuations are hardly visible, and in fact the stationary value is practically the same as in the Gaussian case $\alpha=2.0$. In contrast, the case with vanishing γ_2 (and $\alpha=1.2$) clearly shows large fluctuations requiring the right ordinate, whose span is roughly two orders of magnitude larger than that of the left ordinate.

Similarly, in Fig. 2, we show the fourth-order moment $\langle V^4(t)\rangle$ as function of time. It is obvious that only for a finite γ_4 ($\gamma_4=0.01$ and $\alpha=1.8$) the moment converges to a finite value that is quite close to the value for the Gaussian case ($\alpha=2.0$) for which all moments converge. Opposite to this behavior, both examples with vanishing γ_4 exhibit large fluctuations. These are naturally much more pronounced for the case with smaller Lévy index ($\alpha=1.2$, corresponding to the right ordinate).

Finally, we investigate the asymptotic behavior of the stationary PDF $P_{st}(V)$ in Fig. 3. Clearly, in all three cases the predicted power-law decay is reached, with exponents that, within the estimated error bars [33], agree well with the predicted relation for μ according to Eq. (7).

Central part of $P(V,t)$. The nonlinear damping (3) mainly affects larger velocities, while smaller velocities ($V\ll 1$) are mainly subject to the lowest-order friction $\gamma(0)$. We therefore expect that in the central region close to $V=0$, the PDF $P(V,T)$ preserves its LSD character. This is demonstrated in Fig. 4, where the initial power-law decay of the LSD eventually gives way to the steeper decay caused by the nonlinear friction term. In general, the PDF shows transitions between multiple power laws in the case when several higher-order

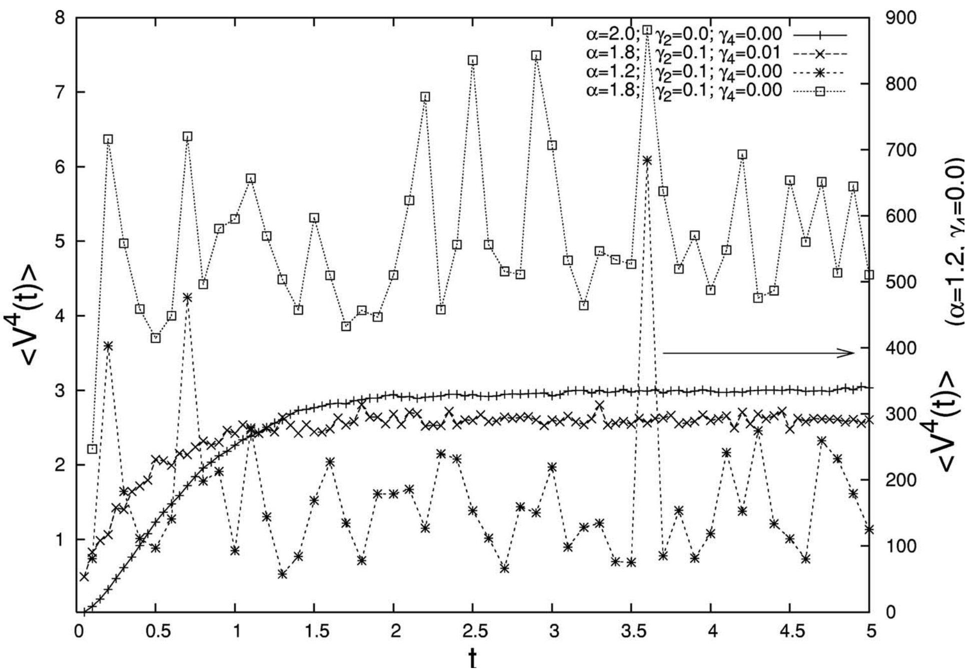

FIG. 2. Fourth-order moment $\langle V^4(t)\rangle$ as a function of t, with $\gamma_0=1$. $\langle V^4(t)\rangle$ converges to a finite value for the two cases $\alpha=2$ (Gaussian) and $\alpha=1.8$ with $\gamma_4=0.01$. The other two examples with vanishing quartic contribution ($\gamma_4=0$) show large fluctuations, i.e., diverging $\langle V^4(t)\rangle$. Note that the case $\alpha=1.2$ and $\gamma_4=0$ corresponds to the right ordinate.

friction terms are carried along. The turnover point from the unaffected LSD to steeper decay caused by nonlinear friction depends on the ratio $\gamma_0:\gamma_{2n}$, where $2n$ is the next higher-order nonvanishing friction coefficient.

Discussion and conclusions. Strictly speaking, all naturally occurring power laws in fractal or dynamic patterns are finite. Scale-free models nevertheless provide an efficient description of a broad variety of processes in complex systems [4,13,26,32]. This phenomenological fact is corroborated by the observation that the power-law properties of Lévy processes persist strongly even in the presence of cutoffs [17], and, more mathematically, by the existence of the generalized central limit theorem due to which Lévy stable laws become fundamental [2]. A categorical question is whether in the presence of Lévy noise, there exists a physical cause to remove the consequential divergencies. A possible, physically natural answer is given by a nonconstant friction coefficient $\gamma(V)$, that is known from various classical systems. Here, we present a concise derivation of the regularization of a stochastic process in velocity space that is driven by Lévy stable noise, in the presence of dissipative nonlinearities.

These dissipative nonlinearities remove the divergence of the kinetic energy from the measurable subsystem of the random walker. In the ideal mathematical language, the surrounding bath provides an infinite amount of energy through the Lévy noise, and the coupling via the nonlinear friction dissipates an infinite amount of energy into the bath, and thereby introduces a natural cutoff in the kinetic-energy distribution of the random walker subsystem. Physically, such divergencies are not expected, but correspond to the limiting procedure of large numbers in probability theory. In this work, we showed that both statements can be reconciled, and that Lévy processes are indeed physical. We believe that this

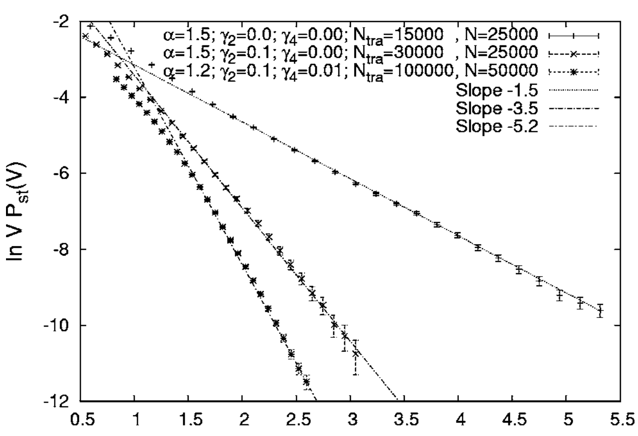

FIG. 3. Power-law asymptotics of the stationary PDF, ln-ln scale. We observe the expected scaling with exponent μ from Eq. (7). In the graph, we also indicate the number N_{tra} of trajectories of individual length N simulated to produce the average PDF.

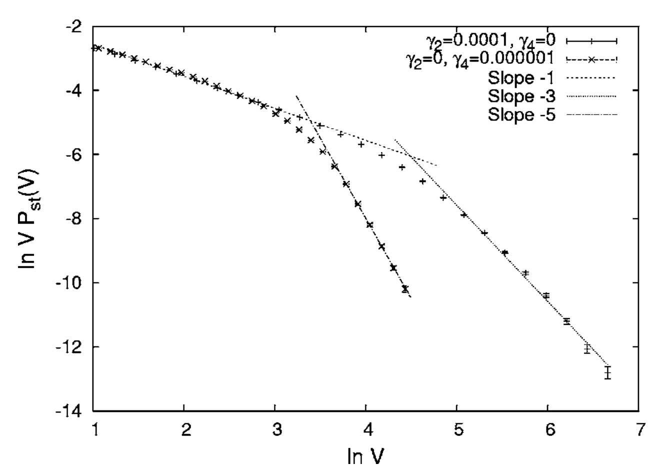

FIG. 4. Stationary PDF $P_{\text{st}}(V)$ for $\gamma_0=1.0$ and (i) $\gamma_2=0.000\,1$ and $\gamma_4=0$; (ii) $\gamma_2=0$ and $\gamma_4=0.000\,001$; with $\alpha=1.0$. The lines indicate the slopes -1, -3, and -5.

is an important progress towards the understanding of Lévy processes.

By analytical investigation of the fractional Fokker-Planck equation and its underlying Langevin equation governing the relaxation of the velocity V and its PDF $P(V,t)$, and supported by numerical simulation, we could show that the variance $\langle V^2(t)\rangle$ of the resulting stochastic process becomes finite in the presence of even the lowest-order dissipative nonlinear correction, i.e., the term $\gamma_2 V^2$. With higher-order contributions ($\gamma_4 V^4$, etc.), also higher-order moments become finite. At the same time, the resulting PDF $P(V,t)$ was shown to leave the basin of attraction of an LSD. Instead, steeper power-law asymptotics are observed whose exponent could be identified as $\mu=\alpha+\nu+1$, where ν is the highest power occurring in the expansion of $\gamma(V)$.

We note that the resulting PDF $P(V,t)$ exhibits a multimodal shape, i.e., it possesses global maxima away from the origin $V=0$ [28–30]. Especially, the stationary state is bimodal. For instance, for the case $\alpha=1$ and $\gamma(V)=V^2$, the PDF is $P_{\rm st}(V)=\pi^{-1}/(1-V^2+V^4)$ with $V_{\max}=\pm\sqrt{1/2}$. That means that the most likely velocity in the presence of the dissipative nonlinear $\gamma(V)$ does not vanish—in contrast to the usual Maxwell distribution $P_{\rm st}(V)=\sqrt{\beta m/(2\pi)}\exp(-\beta m V^2/2)$ with $\beta=1/(k_B T)$.

In conclusion, these results allow us to understand some of the fundamental physical questions concerning Lévy noise originally raised in Ref. [23]. Namely, by introducing dissipative nonlinearity elements, the diverging velocity fluctuations can be compensated, regularizing the stochastic process by producing a finite variance of the velocity through a natural cutoff of the LSD behavior.

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[33] From the scattering of the numerical data after repeated runs, see Fig. 3.