Biased continuous time random walks between parallel plates

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The generalized scheme of continuous time random walks in moving fluids [A. Compte, Phys. Rev. E **55**, 6821 (1997)] is applied to particles diffusing between parallel plates whose jumps are biased by a nonhomogeneous longitudinal velocity field. We observe that when the statistics governing diffusion is Brownian the results are those of Taylor dispersion, i.e., enhanced longitudinal diffusion due to the coupling of the transverse diffusion of the solute and the unidirectional velocity field. However, for Lévy flights with infinite mean waiting time we observe an anomalous dispersion approaching ballistic diffusion. We interpret this behavior as a consequence of the coupling between the flow and the waiting time statistics. [S1063-651X(97)01208-7]

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I. INTRODUCTION

Continuous time random walks (CTRW's) [1] are an interesting and useful generalization of Brownian random walks, since they permit the analysis of the diffusion properties of Markovian processes governed by statistics other than the standard Gaussian statistics of the central limit theorem (Lévy statistics). This feature has been exploited in a variety of applications, ranging from conduction in amorphous materials [2] to turbulent diffusion in fluids [3]. We will here apply a newly proposed scheme of CTRW's in nonhomogeneous velocity fields [4] to model the diffusion properties of Lévy flights of infinite mean waiting time evolving within a nonhomogeneous force field and restrained to stay between parallel plates. This problem has an intrinsic interest since, as we know from standard Brownian diffusion, nonhomogeneous velocity fields in a bounded fluid might have essential influences on the dispersion of a solute (Taylor dispersion [5]) or in other related topics such as the rich variety of situations encompassed by generalized Taylor dispersion [6]. The question of how such inhomogeneities might affect the diffusion of a Lévy walker is therefore relevant as a basic problem and has simultaneously considerable applied interest. Indeed, one of the first successful applications of CTRW's was to interpret the anomalous behavior of the transient current in an amorphous material (xerographic films) [2]. In that case, diffusion was biased by a homogeneous electric field; allowing it to be inhomogeneous would result in a physical instance of the system we study here. Other situations where our model finds application are separation techniques such as chromatography and electrophoresis, where long tail entanglement time distributions provide a better description than standard diffusion theory [7,8]. This system may also be of interest in diffusion in a fractal porous media subject to a pressure gradient, when the diffusing particles get stuck in the fractal matrix for a certain time before proceeding with a new jump. In this respect, analogous concepts to our distribution of waiting times have already been used in hydrodynamic dispersion in porous media by resorting to the analogy with a resistor network [9]. In this paper we will bear in mind this kind of experiments when applying the CTRW techniques to diffusion in a nonhomogeneous force field (in our previous examples the electric field or the pressure gradient, respectively). To use a general and more visual picture that encompasses all the possible experimental instances of the scheme, we shall imagine a fluid flowing between parallel plates where a tracer is released and its diffusion studied with the property that the tracer particles are not being continuously dragged by the stream but stay still between successive jumps.

To model our system we shall make use of a generalization of the CTRW scheme [4], which has been proposed to account for stochastic movements defined by a step length and waiting time probability distribution function in a velocity field. In [4] this generalized scheme was applied to linear shear flows and the results were proven to be consistent with the standard results of Brownian diffusion theory [10]. Subsequently the scheme was applied to Lévy flights of infinite mean waiting time and infinite mean square step length, respectively, to obtain their dispersion in linear shear flows and thus study their anomalous diffusive properties. At this point, the scheme was seen to be consistent with the relation of CTRW's to fractional derivatives [11,12], and with some previously proposed fractional diffusion equations in convective flows [13]. We plan to proceed here quite similarly but we now use a different flow, namely, a laminar flow constrained between two parallel plates, and focus on the longitudinal diffusion (namely, in the flow direction) of the tracer particles.

As we mentioned, our system is very reminiscent of typi-

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cal elementary instances of Taylor dispersion especially when visualized as diffusion in a flow and, as we shall see here, the asymptotic behaviors do indeed coincide with the standard results in Taylor dispersion for Brownian diffusion but differ for Lévy flights with divergent mean waiting time. The main feature of Taylor dispersion is the enhancement of the longitudinal diffusion, characterized for long times by an effective diffusivity D_{eff} proportional to the inverse of the molecular diffusivity, as was first noted by Taylor [5] in a fluid in Poiseuille streaming through a cylindrical tube and later generalized by Aris [14] to arbitrary cross section and flow profile. Many works have subsequently treated the question of the longitudinal dispersion of particles suspended in flows from a variety of points of view ([15,16], and references therein), and in some instances efforts have been made to incorporate in the system non-Brownian random walks, such as persistent random walks [16,17] with no qualitative deviation from standard behavior. It must be stressed, however, that the situation dealt with in the present paper is slightly different from the usual one in Taylor dispersion. Indeed, we consider here that the solute particles are dragged by the flow only during the jumps, but remain still between successive jumps. The case where the particles are continuously dragged by the velocity field will be included in the context of CTRW in future works. In this paper we find that for Lévy flights the asymptotic behavior for long times has essential differences to standard Taylor diffusion and approaches ballistic diffusion, whereby Taylor dispersion is only present as a higher-order phenomenon, the leading term for long times correponding to a purely convective mechanism.

The paper is structured as follows: in the next section we briefly summarize the generalized CTRW scheme of [4]. Section III contains the details of the application of the scheme to a moving fluid constrained to move laminarly between two parallel plates. The application to a particular velocity profile is made in Sec. IV and in Sec. V it is shown that for Brownian diffusion the usual mean square displacement of the tracer particles in Taylor dispersion is exactly obtained through this method. We proceed in Sec. VI to study the behavior of Lévy flights for that same velocity profile. To this aim we need to make use of some aspects of the theory of Fox's H functions, which we quickly expose in the Appendix. The conclusions of the paper are finally made in Sec. VII.

II. CTRW IN MOVING FLUIDS

This generalization of CTRW to account for diffusion in a velocity field is presented in more detail elsewhere [4] but we give here a short summary to present the scheme which we shall later apply to model our system. In [4] it was argued that in a velocity field $\mathbf{v}$ the distribution of step lengths of the random walker ϕ gets shifted proportionally to $\mathbf{v}$ with respect to the distribution of step lengths in a resting fluid ψ. Furthermore, in an inhomogeneous velocity field $\mathbf{v}(\mathbf{x})$ the probability density ϕ of a length of step $\mathbf{r}$ with waiting time t will crucially depend on the velocity of the fluid at the starting point of the jump $\mathbf{x}$ so that we have

$$\phi=\phi(\mathbf{r},t;\mathbf{x})=\psi(\mathbf{r}-\tau_a\mathbf{v}(\mathbf{x}),t), \tag{1}$$

where τ_a is a microscopic time associated with advection in the sense that $\tau_a\mathbf{v}(\mathbf{x})$ is the mean drag experienced by a random walker jumping from the point $\mathbf{x}$. This interpretation implies that, if the mean waiting time τ of the microscopic process exists, then in order that the mean velocity of the dragged particles at position $\mathbf{x}$, $\tau_a\mathbf{v}(\mathbf{x})/\tau$, coincides with the velocity field $\mathbf{v}(\mathbf{x})$ at that point we necessarily have $\tau_a=\tau$. However, if the mean waiting time is infinite (as happens in the Lévy flights that we consider here) this argument is no longer valid, since we now do not have a characteristic microscopic time scale. In this case, the relationship of τ_a to other microscopic times in the system will prove crucial for obtaining a well-defined macroscopic limit [4].

The CTRW scheme can now be reformulated taking into consideration the dependences expressed in Eq. (1): if $P(\mathbf{x},t)$ is the probability density of arriving at point $\mathbf{x}$ at time t and $\rho(x,t)$ is the probability density of being at point $\mathbf{x}$ at time t, we have

$$P(\mathbf{x},t)=\int d\mathbf{x}'\int_0^t dt'\,\phi(\mathbf{x}-\mathbf{x}',t-t';\mathbf{x}')\times P(\mathbf{x}',t')+P_i(\mathbf{x})\delta(t), \tag{2}$$

$$\rho(\mathbf{x},t)=\int_0^t dt'\,P(\mathbf{x},t-t')\Psi(t'), \tag{3}$$

where we have introduced in Eq. (3) the probability $\Psi(t')$ of remaining at least a time t' on the spot before proceeding with another jump $[\Psi(t)=\int_V d\mathbf{r}\int_t^\infty dt'\,\psi(\mathbf{r},t')]$ and we have incorporated the initial distribution function $P_i(\mathbf{x})=P(\mathbf{x},t=0)$ in Eq. (2). In Eqs. (2) and (3) we explicitly suppose that the particles stay still between successive jumps. We now combine Eqs. (2) and (3) to get

$$\rho(\mathbf{x},t)=\int d\mathbf{x}'\int_0^t dt'\,\phi(\mathbf{x}-\mathbf{x}',t-t';\mathbf{x}')\rho(\mathbf{x}',t')+P_i(\mathbf{x})\Psi(t), \tag{4}$$

or, in the Fourier-Laplace domain,

$$\rho(\mathbf{k},u)=\int d\mathbf{k}'\,\phi(\mathbf{k},u;\mathbf{k}-\mathbf{k}')\rho(\mathbf{k}',u)+P_i(\mathbf{k})\Psi(u). \tag{5}$$

Equations (1), (4), and (5) are the main equations of the generalized CTRW scheme and they are the starting point for any application to particular forms of $\mathbf{v}(\mathbf{x})$, as was done in [4] with linear shear flows.

III. CTRW BETWEEN PARALLEL PLATES

The system under study is a solute suspended in a fluid constrained between two parallel infinite plates and allowed to flow laminarly in one direction. Let us suppose that these plates are parallel to the XZ plane and that they are equidistant to the origin of coordinates and separated a distance $2a$ along the y axis. The flow direction will be chosen to be the x axis. The problem is therefore essentially two dimensional and we shall henceforth suppress all references to the variable z for the sake of clarity. To impose the nonpenetrability at the plates we could imagine a system composed by infinite repetitions of the real system along the y direction (method of images). Nevertheless, since we will eventually want to apply our scheme to Lévy flights and it has been shown in [18] that the method of images is not applicable to Lévy flights with an absorbing boundary, we will here proceed along a more intuitive path, although essentially equivalent, to justify its application when we have two reflecting barriers instead of an absorbing boundary.

Following [18] it is our aim here to express the density distribution function of the walker between the plates $\rho(x,y,t)$ in terms of the unrestricted density distribution function in an infinite medium $P(x,y,t)$, which for a walker evolving on a resting fluid is known to be given in the Fourier-Laplace domain by the formula [1,19]

$$P(\mathbf{k},u)=P_i(\mathbf{k})\frac{\Psi(u)}{1-\psi(\mathbf{k},u)},\tag{6}$$

with $P_i(\mathbf{k})$ the Fourier transform of the initial distribution $P_i(\mathbf{x})=P(\mathbf{x},t=0)$.

We now argue that the reflecting boundaries at $y=-a$ and $y=a$ might be obtained by folding the plane along the lines $y=(2n+1)a$ for all integer n onto a single stripe of width $2a$, centered at $y=0$ and of infinite length along the x axis. It is also necessary to build up a velocity field for the whole space in terms of the (arbitrary) velocity profile in the stripe $-a<y<a$, for our purposes this $\mathbf{v}(\mathbf{x})$ must be periodic of period $4a$ and symmetric with respect to the walls. We now only need to sum up at each point of the original stripe the contributions to the density distribution function of each folded stripe:

$$\rho(x,y,t)=\sum_{n=-\infty}^{\infty}[P(x,4an+y,t)+P(x,4an+2a-y,t)].$$

Applying the Fourier transform we get

$$\rho(k_x,k_y,t)=\sum_{n=-\infty}^{\infty}[e^{i4ank_y}P(k_x,k_y,t)+e^{-i(4an+2a)k_y}P(k_x,-k_y,t)].\tag{7}$$

To carry out the summations explicitly we use the identity

$$\sum_{m=-\infty}^{\infty}e^{-ikm}=e^{-ik/2}\sum_{m=-\infty}^{\infty}(-1)^m\delta\left(m+\frac{k}{2\pi}\right)$$

so that Eq. (7), after inverting the Fourier transform, turns into

$$\rho(x,y,t)=\sum_{m=-\infty}^{\infty}e^{i\pi my/2a}\frac{1}{4a}\left[P\left(x,k_y=\frac{m\pi}{2a},t\right)+(-1)^mP\left(x,k_y=-\frac{m\pi}{2a},t\right)\right],$$

where we recognize a Fourier series of period $4a$, consistent with the periodicity of the velocity profile which we use to calculate $P(\mathbf{r},t)$. We shall henceforth work with the Fourier coefficients of $\rho(k_x,y,u)$,

$$\rho_m(k_x,u)=\frac{1}{4a}\left[P\left(k_x,k_y=\frac{m\pi}{2a},u\right)+(-1)^mP\left(k_x,k_y=-\frac{m\pi}{2a},u\right)\right].\tag{8}$$

If the fluid is at rest, $v(y)=0$, Eq. (8) can be easily solved for ρ_m by using the formula (6). For more complicated flows, though, we still cannot obtain ρ_n from Eq. (8). We first need to calculate the probability density P of the random walker in the unfolded space with periodic velocity field. We take formula (5) where, following Eq. (1), ϕ is given by

$$\phi(\mathbf{k},u;\mathbf{k}')=\psi(\mathbf{k},u)\int d\mathbf{x}'\,e^{-i\mathbf{k}'\cdot\mathbf{x}'}e^{-i\tau_a\mathbf{k}\cdot\mathbf{v}(\mathbf{x}')}\tag{9}$$

in terms of the velocity field $\mathbf{v}(\mathbf{x}')$. Since $\mathbf{v}(\mathbf{x}')$ is directed along the x axis, depends only on y, and is periodic of period $4a$ we may express the exponential appearing in Eq. (9) in terms of a Fourier series as

$$e^{-i\tau_a\mathbf{k}\cdot\mathbf{v}(\mathbf{x}')}=e^{-i\tau_ak_xv(y')}=\sum_{n=-\infty}^{\infty}d_n(k_x)e^{in\pi y'/2a},$$

where

$$d_n(k_x)=\frac{1}{4a}\int_{-2a}^{2a}e^{-i\tau_ak_xv(y)}e^{-in\pi y/2a}dy.$$

This last expression can be transformed, using the fact that $v(y)$ is symmetric across the walls: $v(y+a)=v(a-y)$ and $v(y-a)=v(-a-y)$, to obtain an integral for $v(y)$ over the interval $-a<y<a$, where no further restrictions are imposed by the geometry on the functional relation $v=v(y)$,

$$d_n(k_x)=\frac{1}{4a}\int_{-a}^{a}e^{-i\tau_ak_xv(y)}[e^{-in\pi y/2a}+(-1)^ne^{in\pi y/2a}]dy.\tag{10}$$

We now introduce this result into Eqs. (9) and (5) to get

$$P(k_x,k_y,u)=\psi(k_x,k_y,u)\sum_{n=-\infty}^{\infty}d_n(k_x)P\left(k_x,k_y-\frac{n\pi}{2a},u\right)+\Psi(u)P_i(k_x,k_y),$$

which combined with Eq. (8) yields the following coupled set of linear equations for the coefficients ρ_n:

$$\rho_n(k_x,u)=\psi\left(k_x,k_y=\frac{n\pi}{2a},u\right)\sum_{m=-\infty}^{\infty}d_m(k_x)\rho_{m-n}(k_x,u)+\frac{\Psi(u)}{4a}\left[P_i\left(k_x,k_y=\frac{n\pi}{2a}\right)+(-1)^nP_i\left(k_x,k_y=-\frac{n\pi}{2a}\right)\right],\tag{11}$$

where use has been made of the relation $d_{-n}(k_x)=(-1)^nd_n(k_x)$, obvious from Eq. (10). Equation (11) is now the equation to be solved for a given velocity field, a given initial condition $P_i(\mathbf{x})$, and a given step distribution ψ. However, for the case of a symmetric velocity field $v(y)=v(-y)$ and a symmetric initial condition $P_i(x,y)=P_i(x,-y)$, Eq. (11) can be somewhat simplified by observing that the coefficients $d_n(k_x)$ in Eq. (10) vanish for uneven n. It is then easy to see that the coefficients $\rho_n(k_x,u)$ vanish for uneven n as well, and only the coefficients $\rho_n(k_x,u)$ for even n remain to be computed from Eq. (11). Since we are now in a case of higher periodicity (period $2a$) we can redefine the nonvanishing Fourier coefficients to simplify the notation as $d'_n=d_{2n}$ and $\rho'_n=\rho_{2n}$,

$$\rho'_n(k_x,u)=\psi\left(k_x,k_y=\frac{n\pi}{a},u\right)\sum_{m=-\infty}^{\infty}d'_m(k_x)\rho'_{m-n}(k_x,u)+\frac{1}{2a}\Psi(u)P_i\left(k_x,k_y=\frac{n\pi}{a}\right). \quad (12)$$

Equation (12) can be rewritten taking into account that, because of the symmetry in the velocity, the distribution function must also be symmetric and therefore $\rho'_{-n}=\rho'_n$,

$$\rho'_n=\psi_{(n)}\left[d'_0\rho'_n+\sum_{m=1}^{\infty}d'_m(\rho'_{|m-n|}+\rho'_{m+n})\right]+\frac{\Psi}{2a}P_{i(n)}, \quad (13)$$

where we have dropped the explicit dependence of the variables for they can easily be inferred from Eq. (12) and the subscript (n) indicates that k_y is to be replaced by $n\pi/a$ in the corresponding function.

The study of the solute's longitudinal dispersion will need the evaluation of the first moments of the averaged concentration $\rho_0(x,t)$. To proceed, two things are therefore needed: first, the precise form of $d_m(k_x)$ or $d'_m(k_x)$ must be computed from the velocity field $v(y)$ and introduced into Eq. (11) or Eq. (13); and secondly, a particular statistics for the jumps $\psi(\mathbf{r},t)$ must be introduced in our equations. In the next section we find the equations for the averaged density ρ_0 for a conveniently chosen velocity profile, and in the subsequent sections two statistics for the stochastic movements of the solute are analyzed: the standard Brownian case and Lévy flights with infinite mean waiting time.

IV. EQUATIONS FOR THE AVERAGED DENSITY ρ_0

The choice of the functional form of $v(y)$ is crucial to permit the analytic study of Eq. (11). We choose the symmetric velocity field

$$v(y)=v\left(1+\cos\frac{\pi y}{a}\right), \quad (14)$$

where v is the mean velocity of the flow. We look for the coefficients of the Fourier series of $\exp[-i\tau_a k_x v(y)]$ for period $2a$,

$$d'_n(k_x)=\frac{1}{2a}\int_{-a}^{a}\exp\left[-i\tau_a k_x v\left(1+\cos\frac{\pi y}{a}\right)\right]e^{-in\pi y/a}dy$$
$$=i^n e^{-i\tau_a k_x v}J_n(-\tau_a k_x v), \quad (15)$$

where $J_n(x)$ is the Bessel function of the first kind of order n and argument x. Even though the coefficients (15) might seem complicated, they have an important characteristic which will be very useful in our further developments: they are of order k_x^n as $k_x\to 0$,

$$d'_n(k_x)\simeq\frac{1}{n!}\left(\frac{-i\tau_a v}{2}\right)^n k_x^n\left[1-i\tau_a v k_x-\frac{2n+3}{n+1}\frac{\tau_a^2 v^2}{4}k_x^2+\cdots\right]. \quad (16)$$

To understand the usefulness of this fact we must focus on our main objective, which is to establish the mean square displacement $\langle\delta x^2\rangle=\langle x^2\rangle-\langle x\rangle^2$ averaged along the y direction. To this aim we will need to compute

$$\langle x\rangle=\int_{-\infty}^{\infty}dx\,x\int_{-a}^{a}dy\,\rho(x,y,t)=2ai\left.\frac{\partial\rho'_0(k_x,t)}{\partial k_x}\right|_{k_x=0}$$
$$=2ai\rho'_{0,1}(t), \quad (17)$$

$$\langle x^2\rangle=\int_{-\infty}^{\infty}dx\,x^2\int_{-a}^{a}dy\,\rho(x,y,t)=-2a\left.\frac{\partial^2\rho'_0(k_x,t)}{\partial k_x^2}\right|_{k_x=0}$$
$$=-4a\rho'_{0,2}(t), \quad (18)$$

where by $\rho'_{0,2}(t)$ and $\rho'_{0,1}(t)$ we denote the coefficients of k_x^2 and k_x in the Taylor series representation of $\rho'_0(k_x,k_z,t)$ as $k_x\to 0$, respectively. Of course at this step we are assuming the analyticity in x of $\rho(x,y,t)$ and we are therefore explicitly excluding from our analysis CTRW's with nonanalytic step length distribution (such as CTRW's with infinite mean square step length). We are thus only interested in the first terms of the Taylor series of $\rho'_0(k_x,t)$ around $k_x=0$ in order to obtain the required quantity $\langle\delta x^2\rangle$. For this reason we can proceed in a perturbative manner to solve to the lowest orders of k_x Eq. (13) for $n=0$, and the property (16) of our coefficients $d'_n(k_x)$ will be of extreme utility to truncate the infinite sum in Eq. (13). Let us first write the first orders in k_x for the quantities of interest as

$$\rho'_n(k_x,u)\simeq\rho'_{n,0}(u)+\rho'_{n,1}(u)k_x+\rho'_{n,2}(u)k_x^2+\cdots,$$

$$\psi_{(n)}(k_x,u)\simeq\psi_{(n),0}(u)+\psi_{(n),2}(u)k_x^2+\cdots,$$

$$d'_n(k_x)\simeq d'_{n,n}k_x^n+d'_{n,n+1}k_x^{n+1}+d'_{n,n+2}k_x^{n+2}+\cdots, \quad (19)$$

where we have supposed that we start from an isotropic distribution of steps ψ in the fluid at rest so that we have $\psi_{(n),1}=0$. Note that in these expansions the second subindex always indicates the order in k_x of the corresponding term in the series. We now write the relevant equations from Eq. (13) for the orders of interest: 0, 1, and 2,

$$\rho'_{n,0}=\psi_{(n),0}d'_{0,0}\rho'_{n,0}+\frac{\Psi}{2a}P_{i(n),0},$$

$$\rho'_{n,1}=\psi_{(n),0}d'_{0,0}\rho'_{n,1}+\psi_{(n),0}d'_{0,1}\rho'_{n,0}+\psi_{(n),0}d'_{1,1}(\rho'_{n+1,0}+\rho'_{n-1,0})+\frac{\Psi}{2a}P_{i(n),1},$$

$$\rho'_{n,2}=\psi_{(n),0}d'_{0,0}\rho'_{n,2}+\psi_{(n),0}d'_{0,1}\rho'_{n,1}+(\psi_{(n),0}d'_{0,2}+\psi_{(n),2}d'_{0,0})\rho'_{n,0}+\psi_{(n),0}d'_{1,1}(\rho'_{n+1,1}+\rho'_{n-1,1})+\psi_{(n),0}d'_{1,2}(\rho'_{n+1,0}+\rho'_{n-1,0})$$
$$+\psi_{(n),0}d'_{2,2}(\rho'_{n+2,0}+\rho'_{n-2,0})+\frac{\Psi}{2a}P_{i(n),2},$$

where we have not written the dependencies of each function but they can be inferred from Eqs. (12) and (19). Now, to compute $\rho'_{0,1}$ and $\rho'_{0,2}$, we shall need

$$\rho'_{n,0}=\frac{1}{1-\psi_{(n),0}}\frac{\Psi}{2a}P_{i(n),0},\quad n=-2,-1,\ldots,2$$

$$\rho'_{n,1}=\frac{\psi_{(n),0}d'_{0,1}\rho'_{n,0}+\psi_{(n),0}d'_{1,1}(\rho'_{n+1,0}+\rho'_{n-1,0})+(\Psi/2a)P_{i(n),1}}{1-\psi_{(n),0}},\quad n=-1,0,1$$

$$\rho'_{0,2}=\frac{\psi_{(0),0}d'_{0,1}\rho'_{0,1}+(\psi_{(0),0}d'_{0,2}+\psi_{(0),2})\rho'_{0,0}+2\psi_{(0),0}d'_{1,1}\rho'_{1,1}+2\psi_{(0),0}d'_{1,2}\rho'_{1,0}+2\psi_{(0),0}d'_{2,2}\rho'_{2,0}+(\Psi/2a)P_{i(n),2}}{1-\psi_{(0),0}},\tag{20}$$

where we have set $d'_{0,0}=1$, as is obvious from Eq. (16), and we have used that $\rho'_n=\rho'_{-n}$ for symmetric velocity fields. We now only need to choose a symmetric initial distribution $P_i(x,y)$, define our spatially isotropic and analytic $\psi(x,y,t)$ for the particular kind of CTRW which we want to study, and solve the coupled system of linear algebraic equations (20) to obtain $\rho'_{0,1}$ and $\rho'_{0,2}$. In the next two sections we do this for Brownian diffusion and for Lévy flights, respectively. For each case we shall also consider two different initial distributions $P_i(x,y)$.

V. BROWNIAN DIFFUSION

We choose a Gaussian distribution of step lengths for our random walker:

$$\psi(k_x,k_y,u)=\varphi(u)\exp[-\sigma^2(k_x^2+k_y^2)],\tag{21}$$

$\varphi(u)$ being the Laplace transform of the distribution of waiting times, and we solve Eq. (20) for two different initial distributions: one homogeneously distributed on the line $x=0$ for $-a<y<a$ and one initially concentrated at the origin. For the first of these initial conditions we have $P_i(\mathbf{x})=(1/2a)\theta(a-|y|)\delta(x)$, $\theta(x)$ being the Heaviside function. Upon Fourier transformation it is found that $P_{i(n)}(k_x)=\delta_{n,0}$ and the system (20) is readily solved to get

$$\langle x\rangle=2ai\rho'_{0,1}(u)=\frac{\tau_a v}{u}\frac{\varphi}{1-\varphi},\tag{22}$$

$$\langle x^2\rangle=-4a\rho'_{0,2}(u)=2\frac{\sigma^2}{u}\frac{\varphi}{1-\varphi}+\frac{\tau_a^2v^2}{u}\frac{\varphi}{1-\varphi}\times\left[\frac{3}{2}+2\frac{\varphi}{1-\varphi}+\frac{\varphi}{e^{\pi^2\sigma^2/a^2}-\varphi}\right].\tag{23}$$

For Brownian diffusion, we now introduce an exponential decreasing distribution of waiting times $\varphi(t)=\tau^{-1}\exp(t/\tau)$, which upon a Laplace transform turns into $\varphi(u)=(1+u\tau)^{-1}$. Notice that τ is now the mean waiting time between steps, so that one has $\tau_a=\tau$ as was argued in Sec. II. By taking the limit $\tau\to0$ and $\sigma\to0$ and keeping $D=\sigma^2/\tau$ constant we obtain the macroscopic results $\langle X\rangle$ and $\langle X^2\rangle$ (henceforth we write capital X to indicate that the macroscopic limit has already been taken on the corresponding quantities with lower case x), which retain only the essential properties of the random walk and thus discard all spurious behaviors possibly dependent on the model chosen. We finally invert the Laplace transform to get

$$\langle X\rangle(t)=vt,$$

$$\langle X^2\rangle(t)=v^2t^2+\left(2D+\frac{v^2a^2}{\pi^2D}\right)t-\frac{v^2a^4}{\pi^4D^2}(1-e^{-\pi^2Dt/a^2}).$$

The mean square displacement is now computed from these results as

$$\langle\Delta X^2\rangle\equiv\langle X^2\rangle-\langle X\rangle^2=\left(2D+\frac{v^2a^2}{\pi^2D}\right)t-\frac{v^2a^4}{\pi^4D^2}(1-e^{-\pi^2Dt/a^2}),\tag{24}$$

which is exactly the same functional relation, even for the numerical constants and the transient terms, that is obtained by different stochastic methods in Taylor dispersion for the velocity profile (14) [20]. Asymptotically we now find the typical behaviors of the mean square displacement for small and large times, which show the standard Taylor diffusion results:

$$\langle \Delta X^2 \rangle = \begin{cases} 2Dt + \frac{1}{2}v^2t^2 + O(t^3), & t \ll a^2/D \\ \left(2D + \frac{v^2a^2}{\pi^2 D}\right)t + O(1), & t \gg a^2/D. \end{cases}$$

For the other initial distribution $P_i(\mathbf{x}) = \delta(\mathbf{x})$, we proceed analogously from Eq. (20) with now $P_{i(n)}(k_x) = 1$. The calculations to obtain $\rho'_{0,1}$ and $\rho'_{0,2}$ are somewhat more intricate than in the previous case and yield the following expressions for the quantities in which we are interested:

$$\langle x \rangle = \tau_a v \frac{\varphi}{u}\left[\frac{e^{\pi^2\sigma^2/a^2}}{e^{\pi^2\sigma^2/a^2} - \varphi} + \frac{1}{1-\varphi}\right], \tag{25}$$

$$\begin{aligned} \langle x^2 \rangle = {} & 2\frac{\sigma^2}{u}\frac{\varphi}{1-\varphi} \\ & + \frac{\tau_a^2 v^2}{2u}\frac{5e^{4\pi^2\sigma^2/a^2} - \varphi^2 e^{2\pi^2\sigma^2/a^2} - 4\varphi}{(e^{\pi^2\sigma^2/a^2} - \varphi)^2(e^{4\pi^2\sigma^2/a^2} - \varphi)} e^{2\pi^2\sigma^2/a^2}\varphi \\ & + \frac{\tau_a^2 v^2}{2u}\left[\frac{(3+4\varphi)e^{\pi^2\sigma^2/a^2} - \varphi}{e^{\pi^2\sigma^2/a^2} - \varphi}\frac{\varphi}{1-\varphi} + 4\frac{\varphi^2}{(1-\varphi)^2}\right]. \end{aligned} \tag{26}$$

We now introduce the distribution of waiting times $\varphi(u) = (1 + u\tau)^{-1}$ and take the macroscopic limit as we did before. Reverting again to the time representation through an inverse Laplace transform we get

$$\langle X \rangle(t) = vt + \frac{va^2}{\pi^2 D}\left[1 - \exp\left(-\frac{\pi^2}{a^2}Dt\right)\right],$$

$$\begin{aligned} \langle X^2 \rangle(t) = {} & v^2t^2 + \left[2D + \frac{3v^2a^2}{\pi^2 D}\left(1 - \frac{2}{3}e^{-\pi^2 Dt/a^2}\right)\right]t \\ & - \frac{3}{4}\frac{v^2a^4}{\pi^4 D^2}\left[1 - \frac{8}{9}e^{-\pi^2 Dt/a^2} - \frac{1}{9}e^{-4\pi^2 Dt/a^2}\right], \end{aligned}$$

from where the mean square displacement $\langle \Delta X^2 \rangle \equiv \langle X^2 \rangle - \langle X \rangle^2$ of a Brownian random walker initially concentrated at $\mathbf{x} = \mathbf{0}$ in Taylor dispersion is readily derived,

$$\begin{aligned} \langle \Delta X^2 \rangle = {} & \left(2D + \frac{v^2a^2}{\pi^2 D}\right)t - \frac{v^2a^4}{\pi^4 D^2}\left(\frac{7}{4} - \frac{8}{3}e^{-\pi^2 Dt/a^2}\right. \\ & \left. + e^{-2\pi^2 Dt/a^2} - \frac{1}{12}e^{-4\pi^2 Dt/a^2}\right). \end{aligned} \tag{27}$$

Hence, the behavior for long and short times is found to be

$$\langle \Delta X^2 \rangle = \begin{cases} 2Dt + O(t^3), & t \ll a^2/D \\ \left(2D + \frac{v^2a^2}{\pi^2 D}\right)t + O(1), & t \gg a^2/D. \end{cases}$$

We see that the asymptotics prove to be independent of the initial conditions (as was expected) and these initial conditions are only relevant in the derivation of the transient terms for $\langle \Delta X^2 \rangle$ in Eqs. (24) and (27). It is now worth noting that for Brownian diffusion our model, designed for particles not being continuously dragged by the fluid stream, yields exactly the same results as the traditional computations for liquid systems in Taylor dispersion. This point becomes clear if we focus on the waiting period between successive jumps, this being the only difference between the two models. If we have a finite mean waiting time between successive steps τ and we take the macroscopic limit as $\tau \to 0$ we are effectively making the waiting period between steps vanish on average, whereby no differences should arise from models only differing in the behavior of the particles during this interval of time.

VI. LÉVY FLIGHTS

We now turn to the question of determining the longitudinal dispersion for Lévy flights with infinite mean waiting time evolving in a fluid constrained between parallel plates at $y = -a$ and $y = a$ and subject to the velocity field (14). This point is easy after the developments of the preceding section if we take a Lévy flight of infinite mean waiting time and a Gaussian distribution of step lengths as in Eq. (21), where now

$$\varphi(u) = \frac{1}{1 + (u\tau)^{\gamma}}, 0 < \gamma < 1. \tag{28}$$

This choice allows us to make use of all the previous calculations up to Eqs. (22) and (23) or Eqs. (25) and (26), depending on the initial conditions. We introduce there our $\varphi(u)$ and take the macroscopic limit $\tau \to 0$, $\tau_a \to 0$, $\sigma \to 0$ keeping $D = \sigma^2/\tau^{\gamma}$ and $A = \tau_a/\tau^{\gamma}$ constant. A must be kept as a macroscopic parameter associated with convection with dimensions of time to the power $1-\gamma$. As it was argued in [4] the introduction of the constant A seems to be necessary to obtain well-defined equations free of all microscopic parameters τ, τ_a, and σ; it is nevertheless still unclear what physical foundations might support it. We first apply this procedure to the case of a homogeneous initial distribution on the segment $-a < y < a$ of the y axis, that is to say we introduce Eq. (28) in Eqs. (22) and (23) and then take the macroscopic limit to get

$$\langle X \rangle = Avu^{-\gamma-1}, \tag{29}$$

$$\langle X^2 \rangle = 2Du^{-\gamma-1} + A^2v^2u^{-2\gamma-1}\left(2 + \frac{1}{1 + (D\pi^2/a^2)u^{-\gamma}}\right). \tag{30}$$

We now need to apply the inverse Laplace transform to these expressions. To this aim the only difficulty is to invert the expression $u^{-n\gamma-1}(1 + Ku^{-\gamma})^{-1}$, which appears in Eq. (30) for $n = 2$. We express it as a geometrical series

$$\frac{u^{-n\gamma-1}}{1+Ku^{-\gamma}}=u^{-n\gamma-1}\sum_{m=0}^{\infty}(-Ku^{-\gamma})^m$$

and now invert it term by term to obtain

$$\mathcal{L}^{-1}\left(\frac{u^{-n\gamma-1}}{1+Ku^{-\gamma}}\right)=t^{n\gamma}\sum_{m=0}^{\infty}\frac{(-Kt^{\gamma})^m}{\Gamma(m\gamma+n\gamma+1)}$$
$$=t^{n\gamma}E_{\gamma,1+n\gamma}(-Kt^{\gamma}), \tag{31}$$

where in the last equality we have identified the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ [21]. Applying now this result to Eqs. (29) and (30) we obtain

$$\langle X\rangle=\frac{Av}{\Gamma(\gamma+1)}t^{\gamma},$$

$$\langle X^2\rangle=\frac{2D}{\Gamma(\gamma+1)}t^{\gamma}+\frac{2A^2v^2}{\Gamma(2\gamma+1)}t^{2\gamma}+A^2v^2t^{2\gamma}E_{\gamma,1+2\gamma}\left(-\frac{D\pi^2}{a^2}t^{\gamma}\right)$$

in terms of the Mittag-Leffler function. To compute the mean square displacement we now apply the definition $\langle\Delta X^2\rangle\equiv\langle X^2\rangle-\langle X\rangle^2$ and get

$$\langle\Delta X^2\rangle=\frac{2D}{\Gamma(\gamma+1)}t^{\gamma}+A^2v^2t^{2\gamma}\left[\frac{2}{\Gamma(2\gamma+1)}-\frac{1}{\Gamma(\gamma+1)^2}+E_{\gamma,1+2\gamma}\left(-\frac{D\pi^2}{a^2}t^{\gamma}\right)\right]. \tag{32}$$

We now study the other initial condition, namely, a pulse initially concentrated at the origin. We introduce our distribution of waiting times (28) into Eqs. (25) and (26) and take the macroscopic limit. We obtain, in the Laplace domain,

$$\langle X\rangle=Avu^{-\gamma-1}\left[1+\frac{1}{1+(D\pi^2/a^2)u^{-\gamma}}\right], \tag{33}$$

$$\langle X^2\rangle=2u^{-2\gamma-1}\left[Du^{\gamma}+A^2v^2+\frac{2}{3}\frac{A^2v^2}{1+4(D\pi^2/a^2)u^{-\gamma}}+\frac{4}{3}\frac{A^2v^2}{1+(D\pi^2/a^2)u^{-\gamma}}+\frac{A^2v^2}{[1+(D\pi^2/a^2)u^{-\gamma}]^2}\right]. \tag{34}$$

By using now the result (31) it is easy to invert Eq. (33) but to invert Eq. (34) we need another special function, namely, Maitland's generalized hypergeometric function ${}_1\Psi_1$ [22]. To come up with this result and to obtain a unified expression, we make use of the properties of Fox's H function [22], a brief summary of which is given in the Appendix. Introducing the Fox functions (A1) and (A2), we arrive at

$$\langle X\rangle(t)=\frac{Avt^{\gamma}}{\Gamma(\gamma+1)}\left\{1+\Gamma(\gamma+1)H_{1,2}^{1,1}\left[\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(0,1)\\(0,1),(-\gamma,\gamma)\end{matrix}\right]\right\},$$

$$\langle X^2\rangle(t)=\frac{2D}{\Gamma(\gamma+1)}t^{\gamma}+2A^2v^2t^{2\gamma}\left\{\frac{1}{\Gamma(2\gamma+1)}+\frac{2}{3}H_{1,2}^{1,1}\left[4\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(0,1)\\(0,1),(-2\gamma,\gamma)\end{matrix}\right]+\frac{4}{3}H_{1,2}^{1,1}\left[\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(0,1)\\(0,1),(-2\gamma,\gamma)\end{matrix}\right]+H_{1,2}^{1,1}\left[\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(-1,1)\\(0,1),(-2\gamma,\gamma)\end{matrix}\right]\right\}.$$

We can now compute the mean square displacement for the particles initially concentrated at the origin and get

$$\langle\Delta X^2\rangle=\frac{2D}{\Gamma(\gamma+1)}t^{\gamma}+A^2v^2t^{2\gamma}\left\{\frac{2}{\Gamma(2\gamma+1)}-\frac{1}{\Gamma(\gamma+1)^2}+\frac{4}{3}H_{1,2}^{1,1}\left[4\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(0,1)\\(0,1),(-2\gamma,\gamma)\end{matrix}\right]+\frac{8}{3}H_{1,2}^{1,1}\left[\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(0,1)\\(0,1),(-2\gamma,\gamma)\end{matrix}\right]+2H_{1,2}^{1,1}\left[\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(-1,1)\\(0,1),(-2\gamma,\gamma)\end{matrix}\right]-\frac{2}{\Gamma(\gamma+1)}H_{1,2}^{1,1}\left[\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(0,1)\\(0,1),(-\gamma,\gamma)\end{matrix}\right]-H_{1,2}^{1,1}\left[\frac{D\pi^2}{a^2}t^{\gamma}\middle|\begin{matrix}(0,1)\\(0,1),(-\gamma,\gamma)\end{matrix}\right]^2\right\}. \tag{35}$$

To compute the asymptotic behaviors of the results (32) and (35) we need to expand conveniently these H functions as we show in the Appendix, and we obtain

$$\langle\Delta X^2\rangle=\begin{cases}\dfrac{2D}{\Gamma(\gamma+1)}t^{\gamma}+A^2v^2\left(\dfrac{a_i}{\Gamma(2\gamma+1)}-\dfrac{b_i}{\Gamma(\gamma+1)^2}\right)t^{2\gamma}+O(t^{3\gamma}), & t\ll(a^2D^{-1}\pi^{-2})^{1/\gamma},\\ A^2v^2\left(\dfrac{2}{\Gamma(2\gamma+1)}-\dfrac{1}{\Gamma(\gamma+1)^2}\right)t^{2\gamma}+\dfrac{1}{\Gamma(\gamma+1)}\left(2D+\dfrac{A^2v^2a^2}{D\pi^2}\right)t^{\gamma}+O(1), & t\gg(a^2D^{-1}\pi^{-2})^{1/\gamma},\end{cases} \tag{36}$$

where the coefficients a_i and b_i depend on the initial conditions and are here $a_1=3$ and $b_1=1$ for a homogeneously distributed (in the axis $x=0$) initial condition and $a_2=8$ and $b_2=4$ for an initial pulse at $\mathbf{x}=\mathbf{0}$. We observe in Eq. (36) that, for long times, Lévy flights $(\gamma\neq 1)$ might present superdiffusion (if $1/2<\gamma<1$). Therefore we encounter a paradoxical situation, where a stochastic mechanism which introduces a delaying term (Lévy flights with infinite mean waiting time) turns out to bring about superdiffusion when constrained to move between parallel plates and subject to the nonhomogeneous velocity field (14). A similar paradox appeared in [4] when this Lévy flight was evolving in a purely sheared two-dimensional flow. It can nevertheless be argued that an analogous paradoxical situation arises in standard Taylor diffusion, where the smaller the molecular diffusion coefficient D the larger the effective diffusion rate along direction x. In the latter case, this is not after all paradoxical if one follows a particle near the maximum of the velocity profile (here $y=0$) and one realizes that the closer the particle remains to this maximum the farther it travels along the direction x and, at the same time, the particles near the minimum of velocity remain more stagnant the closer they remain in their diffusive motion [16]. It is therefore logical that a small molecular diffusion rate leads to a large longitudinal Taylor diffusion rate. However, the mechanism that accounts for the superdiffusion in Eq. (36) for $t\gg(a^2D^{-1}\pi^{-2})^{1/\gamma}$ is now of a different nature and stems from the fact that our diffusing particles stay still between successive jumps and are not being continuously dragged by the stream. We encounter arbitrarily long waiting time between the biased steps of the walker, which means that some particles remain forever stagnated after a finite number of steps whereas others keep jumping in the flow. This leads to an enhancement in the dispersion, which does not have anything to do with the transverse diffusive motion and this is clear in Eq. (36), where the leading term for long times does not depend on D but only on the velocity v and the parameter A. The next-order term, though, does indeed contain the coupling of convection to the diffusive transverse motion, which is characteristic of Taylor dispersion. The mechanism of Taylor dispersion is therefore present but does not represent the dominant order for long times. This interpretation also explains why the dispersion for long times in Eq. (36) is accomplished by means of a new term in $t^{2\gamma}$ and not through an increase of the effective diffusion coefficient in front of t^{γ}. For if an appreciable number of particles remains fixed after a small number of steps and the rest keep moving with $\langle X\rangle\propto t^{\gamma}$, the dispersion must at least increase as $\langle\Delta X^2\rangle\propto t^{2\gamma}$, in a sort of ''ballistic'' way. The fact that the average displacement of the particles does not increase linearly with time, as should appear to be logical after a sufficiently long time, now has an easy explanation since this appreciable portion of ''frozen'' particles slows down the advancement of the center of mass of the cloud of particles. A particular physical picture of this behavior is provided quite clearly by macromolecular separation in gel electrophoresis, where the motion of the macromolecules can be modeled as a succession of periods of advancement interrupted by periods of immobility due to the entanglement with the gel matrix [8].

We can corroborate this interpretation of the long-time result (36) through some easy calculations involving *exclusively* the convection of the tracer particles: let us neglect the transverse diffusion of the particles so that the particles advance on average the same quantity $\tau_a v$ whenever they jump. We then have the following expressions for the mean and the mean square displacement, respectively,

$$\langle x\rangle(t)=\tau_a v\,\overline{N(t)},\tag{37}$$

$$\langle x^2\rangle(t)=\tau_a^2 v^2\,\overline{N^2}(t),\tag{38}$$

where $N(t)$ stands for the number of steps that a particle has taken up to time t, and the bar over it indicates the average over all the tracer particles diffusing in the system. We denote by $p_n(t)$ the probability that a tracer particle has performed n jumps before time t,

$$p_n(t)=\int_0^t dt'\int_0^{t-t'}dt''\cdots\int_0^{t-t'-\cdots-t^{(n-1)}}dt^{(n)}\varphi(t')\times\varphi(t'')\cdots\varphi(t^{(n)})\Psi(t-t'-t''-\cdots-t^{(n)}),$$

or, in the Laplace domain

$$p_n(u)=\Psi(u)[\varphi(u)]^n.$$

The quantities $\overline{N(t)}$ and $\overline{N^2}(t)$ are now easily seen to be

$$\overline{N(t)}=\sum_{n=0}^{\infty}np_n(t)\ \Rightarrow\ \overline{N(u)}=\frac{1}{u}\frac{\varphi(u)}{1-\varphi(u)},$$

$$\overline{N^2}(t)=\sum_{n=0}^{\infty}n^2p_n(t)\ \Rightarrow\ \overline{N^2}(u)=\frac{\varphi(u)}{u}\frac{1+\varphi(u)}{[1-\varphi(u)]^2},\tag{39}$$

whence it is now straightforward to calculate $\langle x\rangle$ and $\langle x^2\rangle$ for our Lévy flights using Eq. (28) in Eqs. (39) and introducing the results in Eqs. (37) and (38). The results obtained are, after the corresponding macroscopic limit,

$$\langle X\rangle=\frac{Av}{\Gamma(\gamma+1)}t^{\gamma},$$

$$\langle X^2\rangle=\frac{2}{\Gamma(2\gamma+1)}A^2v^2t^{2\gamma},$$

$$\langle\Delta X^2\rangle=A^2v^2\left(\frac{2}{\Gamma(2\gamma+1)}-\frac{1}{\Gamma(\gamma+1)^2}\right)t^{2\gamma},$$

which are in absolute accordance with the leading terms for long times in Eq. (36) and the previous expressions for $\langle X\rangle$ and $\langle X^2\rangle$. This therefore makes it clear that for Lévy flights with divergent mean waiting time the main dispersion mechanism is convection decoupled from diffusion and we therefore do not properly have Taylor dispersion for sufficiently long times.

It must be noted here that such a convectively originated dispersion has already been observed in gel electrophoresis [8], and it has been interpreted through the modeling of the kinetics of the transition between the adsorbed and mobile phases in terms of a waiting time distribution function with a finite first moment and divergent variance [7]. It is straightforward to prove that, if the waiting time distribution in [7] is taken as here, the results agree perfectly with ours as long as we identify τ_a^{-1} as the first-order rate constant of the transition from the mobile phase to the adsorbed phase. Conversely, our model yields the asymptotic results of [7] as well, as is readily seen by substituting their waiting time distribution in Eq. (39). Both from the random walk formalism in this paper and from a kinetic approach in [7] it is therefore concluded that, when a long tail waiting time distribution is present, the main dispersion mechanism is neither diffusion nor Taylor dispersion but pure convection. Thus the long-time analysis of such a system is most easily performed through the statistical considerations leading to Eqs. (37) and (38).

VII. CONCLUSIONS

In this paper we have further developed the generalized CTRW scheme [4], which permits the analysis of the diffusive properties of CTRW's evolving in convective flows. We have applied it here to diffusion constrained between parallel plates and subject to a nonhomogeneous force field. First, we have shown that for Brownian diffusion the scheme reproduces exactly the mean square displacement obtained through more standard methods for Taylor dispersion [20] and then we have used the scheme on Lévy flights. We have thus obtained for the first time the mean square displacement for a Lévy flight with infinite mean waiting time subjected to a nonhomogeneous longitudinal flow and constrained to move between two parallel plates. As is customary in anomalous diffusion problems [12,23], the results involve Fox's H functions and this permits the derivation of both the long-time and the short-time limits, $t \gg (a^2 D^{-1} \pi^{-2})^{1/\gamma}$ and $t \ll (a^2 D^{-1} \pi^{-2})^{1/\gamma}$, respectively. This long-time behavior is especially remarkable because it presents an essential deviation from Taylor dispersion results, where the long-time mean square displacement only differs from diffusion in a resting fluid through an enhancement of the diffusion coefficient. In this generalized case, though, the leading term for long times is only velocity dependent and increases as $t^{2\gamma}$ instead of t^{γ}, the temporal characteristic dependence of Lévy dispersion in a resting fluid. We therefore encounter a ''ballistic'' kind of diffusion asymptotically independent of the diffusion constant D as $t \to \infty$. This strange behavior, as we have argued in the preceding section, must be explained by different arguments than those interpreting the enhancement of the dispersion coefficient in standard Taylor diffusion [16]. Now the arbitrarily long waiting time between biased steps of the random walker facilitates the advancement of part of the particles jumping in the x direction and the separation from the ''frozen'' particles, which enter arbitrarily long waiting periods. Therefore the presence of a well-defined microscopic time scale, the mean waiting time between steps, abruptly marks the transition from a process diffusing superdiffusively to a standard diffusive regime. A similar situation was encountered in [12]. It is important to note at this point that this behavior, anomalous in the frame of Taylor dispersion, is only due to the fact that in our system the particles remain still before proceeding with a new jump.

As a conclusion, in any experimental setup in accordance with the scheme considered here (electronic transport in amorphous materials, diffusion in fractal porous media, interactive dispersion in gel electrophoresis [8], etc.) the property of microscopic scale invariance in time of the underlying diffusion process should dramatically manifest in the longitudinal dispersion of the tracer particles for sufficiently long times.

In this paper we have not applied the scheme to Lévy flights with infinite mean square step length, because we incur some mathematical difficulties when defining the dispersion of the random walker, much as it happened in [4], but with the additional difficulty of not having now a symmetric problem in the x direction. Because of the intricate mathematics involved, we will include the analysis of this situation in a future publication.

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APPENDIX

Fox's H function is defined via a contour integral of Mellin-Barnes type [22]. Originally applied in statistics, it arises in the physical sciences naturally in the solution of linear differential equations of fractional order [12,23,24]. To avoid unnecessary confusion, we omit the explicit discussion of the mathematical definition and properties of Fox's H function. The interested reader may find the essentials in [12,22–24]. At this point it suffices to note that the Fox function is a very general class so that Laplace and Fourier transforms of an H function only change its parameters. Similarly, the fractional derivative of an H function is just another H function. In addition, an H function is computable by use of its series expansion.

In the mathematical manipulations following Eq. (33), the corresponding expression in the Laplace domain can be identified with a simple H function, which can easily be Laplace inverted due to the well-known theorems for H functions [24]. Consulting the tables in [22], one can identify the obtained results with either the generalized Mittag-Leffler function $E_{\alpha,\beta}(x)$ or Maitland's generalized hypergeometric (or Wright's) function ${}_p\Psi_q(x)$. These identities have the following form:

$$E_{\alpha,\beta}(-x) = H^{1,1}_{1,2}\left[x \middle| \begin{matrix} (0,1) \\ (0,1),(1-\beta,\alpha) \end{matrix} \right], \tag{A1}$$

$$
{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots, & (a_p,A_p)\\ (b_1,B_1),(b_2,B_2),\ldots, & (b_q,B_q)\end{matrix};-x\right]
=H^{1,p}_{p,q+1}\left[x\left|\begin{matrix}(1-a_1,A_1),(1-a_2,A_2),\ \ldots,(1-a_p,A_p)\\ (0,1),(1-b_1,B_1),\ \ldots,(1-b_q,B_q)\end{matrix}\right.\right]. \tag{A2}
$$

For the calculation of the asymptotic behavior, we can now employ the standard properties of H functions and its series expansions.

For a small argument $x \ll 1$, both Fox functions can be expanded in a series as follows:

$$
E_{\gamma,1+n\gamma}(-x)=H^{1,1}_{1,2}\left[x\left|\begin{matrix}(0,1)\\ (0,1),(-n\gamma,\gamma)\end{matrix}\right.\right]
=\frac{1}{\gamma x^n}H^{1,1}_{1,2}\left[x\left|\begin{matrix}(n,1/\gamma)\\ (n,1/\gamma),(0,1)\end{matrix}\right.\right]
=\sum_{m=0}^{\infty}\frac{(-1)^m x^m}{\Gamma(1+(n+m)\gamma)},
$$

$$
{}_1\Psi_1\left[\begin{matrix}(2,1)\\ (1+n\gamma,\gamma)\end{matrix};-x\right]=H^{1,1}_{1,2}\left[x\left|\begin{matrix}(-1,1)\\ (0,1),(-n\gamma,\gamma)\end{matrix}\right.\right]
=\sum_{m=0}^{\infty}\frac{(-1)^m(m+1)x^m}{\Gamma(1+(n+m)\gamma)}. \tag{A3}
$$

On the other hand, for $x \gg 1$ large, we find the following asymptotes:

$$
H^{1,1}_{1,2}\left[x\left|\begin{matrix}(0,1)\\ (0,1),(-n\gamma,\gamma)\end{matrix}\right.\right]\sim\frac{1}{\Gamma(1+(n-1)\gamma)}x^{-1}, \tag{A4}
$$

$$
H^{1,1}_{1,2}\left[x\left|\begin{matrix}(-1,1)\\ (0,1),(-n\gamma,\gamma)\end{matrix}\right.\right]\sim\frac{1}{\Gamma(1+(n-2)\gamma)}x^{-2}, \tag{A5}
$$

whereby higher orders may be neglected for our computations.

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