ELSEVIER

Physica A 211 (1994) 13–24

PHYSICA A

Fractional model equation for anomalous diffusion

Ralf Metzler, Walter G. Glöckle, Theo F. Nonnenmacher

Department of Mathematical Physics, University of Ulm, Albert-Einstein-Allee 11, 89069 Ulm, Germany

Received 24 March 1994

Abstract

In recent years the phenomenon of anomalous diffusion has attracted more and more attention. One of the main impulses was initiated by de Gennes' idea of the "ant in the labyrinth". Several authors presented asymptotic probability density functions for the location of a random walker on a fractal object. As this density function and the time dependence of its second moment are now well established, a modified diffusion equation providing the correct result is formulated. The parameters of this fractional partial differential equation are uniquely determined by the fractal Hausdorff dimension of the underlying object and the anomalous diffusion exponent. The presented equation reduces exactly to the ordinary isotropic diffusion equation by appropriate choice of the parameters. A closed form solution is given in terms of Fox's H-function. In the asymptotic case a "halved" diffusion equation can be established. Furthermore, the differences to equations considered previously are discussed.

1. Introduction

In regular Euclidean spaces with dimension d, d a positive integer, the mean-square displacement of a random walker is given by $\langle r^2(t)\rangle \sim t$. Nevertheless, anomalous diffusion is theoretically predicted and experimentally observed [1–5]. A process is referred to as anomalous diffusion if

$$\langle r^2(t)\rangle \sim t^{2/d_w}, \tag{1}$$

where d_w is the anomalous diffusion exponent. Both cases enhanced and reduced diffusion speed are possible, though, for a random walk on a fractal object only the second case occurs, i.e. $d_w > 2$. The reason is given by the geometric obstacles existing on all length scales that slow down the random walker.

The quantity d_w can be computed for exact fractals and for percolation clusters near criticality. In their review, Havlin and Ben-Avraham [4] also present the limiting

SSDI 0378-4371(94)000184-7

behaviour for the random walk's asymptotic probability density. The argumentation bases on scaling arguments as well as computer simulations for corresponding random walks. The result is a "stretched exponential" of the form [4]

$$P(r,t) \sim At^{-d_f/d_w}\,\mathrm{e}^{-c(r/R)^u}, \tag{2}$$

valid in the asymptotic range $r/R \gg 1$ and $t \to \infty$. R and u are defined by

$$R = \sqrt{\langle r^2(t)\rangle}, \tag{3}$$

$$u = \frac{d_w}{d_w - 1}, \tag{4}$$

so that the probability density (2) has a very unique shape. Nevertheless, this behaviour can be regarded as a general property of random walks on fractal objects of any kind. In the following the asymptotic limitation where (2) is valid is always referred to when we speak of the asymptotic behaviour of probability densities. In (2) two free parameters occur: the anomalous diffusion exponent d_w and the fractal dimension d_f of the underlying object. The fraction $2d_f/d_w$ is just the fracton dimension (spectral dimension) d_s of the fractal [6].

In this paper we demonstrate that anomalous diffusion with the asymptotic behaviour (2) is provided by the phenomenological diffusion equation

$$\frac{\partial^{2/d_w}}{\partial t^{2/d_w}}P(r,t) = \frac{1}{r^{d_s-1}}\frac{\partial}{\partial r}\left(r^{d_s-1}\frac{\partial}{\partial r}P(r,t)\right), \tag{5}$$

which is a fractional partial differential equation. In comparison with the ordinary isotropic diffusion equation in d dimensions,

$$\frac{\partial}{\partial t}P(r,t) = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}P(r,t)\right), \tag{6}$$

the Laplacian operator is generalized by introducing a non-integer dimension d_s and the temporal differential operator is replaced by a fractional time derivation of order $2/d_w$, which is defined via the convolution integral [12]

$$\frac{\partial^{2/d_w}}{\partial t^{2/d_w}}P(r,t) = \frac{1}{\Gamma(1-2/d_w)}\frac{\partial}{\partial t}\int_0^t d\tau\,\frac{P(r,\tau)}{(t-\tau)^{2/d_w}} \tag{7}$$

for $0 \le 2/d_w < 1$. Starting out from a very general modified diffusion equation it will be shown in Section 3 that the fractional diffusion equation (5) is compatible with the conditions (1) and (2) and that the parameters in (5) are uniquely determined. Furthermore, the solution of Eq. (5) can be expressed in closed form by an H-function [13]. In order to discuss the connection between (5) and the equation proposed by Giona and Roman [10], a "halved" diffusion equation is constructed on the basis of (5) in Section 4. In the final section an interpretation of our Eq. (5) and the occurring parameters is given.

2. Résumé of earlier proposed equations

Before we start "deriving" Eq. (5) we, at first, briefly review some modified diffusion equations. Taking care of the anomalous time-behaviour (1) only, Schneider and Wyss [7] presented a fractional diffusion equation of the form

$$P(r,t) = f_0(r) + \frac{1}{\Gamma(2/d_w)} \int_0^t d\tau \ (t-\tau)^{2/d_w - 1} \frac{\partial^2}{\partial r^2} P(r,\tau), \tag{8}$$

where the convolution integral represents an integration of order $2/d_w$, and $f_0(r)$ is the initial distribution. The formulation as a fractional integral equation has the advantage of directly including a given initial value [8]. The solution can be given analytically in terms of Fox's H-function and the computation of its second moment delivers (1). Nevertheless, this equation can be denied at once for our purposes by taking into consideration that two free parameters are needed in the distribution function's asymptotic. In Eq. (8) only one, $2/d_w$, occurs.

Another attempt is given by O'Shaugnessy and Procaccia [9]. Based on scaling and renormalization arguments they derive the equation

$$\frac{\partial P(r,t)}{\partial t} = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left(K(r) r^{D-1} \frac{\partial P(r,t)}{\partial r} \right), \tag{9}$$

where an r-dependent diffusion coefficient

$$K(r) = K r^{-\Theta} \tag{10}$$

is assumed. K is a constant and $\Theta = D + \alpha - 2$. The Laplacian operator is modified by introducing a general dimension D in the exponent of r. D can be identified with the Hausdorff dimension of the underlying fractal structure. Additionally, a second parameter, Θ, is amended. This is to be seen as a direct consequence of the Einstein relation connecting the diffusion constant $K(r)$ and the underlying object's electrical conductivity. α is the scaling power of r for the electrical resistance [9]. The exact solution to (9) is [9]

$$P(r,t) = \frac{2+\Theta}{D\,\Gamma(D/(2+\Theta))} \left(\frac{1}{K(2+\Theta)^2 t} \right)^{D/(2+\Theta)} \times \exp\left(- \frac{r^{2+\Theta}}{K(2+\Theta)^2 t} \right). \tag{11}$$

Comparing the second moment of (11) with (1), $2 + \Theta$ is readily identified with d_w, the anomalous diffusion exponent. But this solution (11) lacks the unique scaling in the exponential function as required by (2). Thus, this equation also does not provide the requested asymptotic behaviour (2).

Giona and Roman [10] attacked both the time derivative and the Laplacian. Taking the semi-differential diffusion equation of Oldham and Spanier [11,12] as starting point, they construct the modified equation

$$\frac{\partial^{1/d_w} P(r,t)}{\partial t^{1/d_w}} = -G\frac{1}{r^\kappa}\frac{\partial}{\partial r}\left(r^\kappa P(r,t)\right), \tag{12}$$

which can be considered as a "halved" diffusion equation. In Section 4 we will go into more detail concerning this "halved" equation. In (12), a fractional time derivative of order $1/d_w$ is introduced and the abbreviation κ is given by $\kappa = d_f/d_w - 1/2$. In [10] the asymptotic behaviour of $P(r,t)$ is calculated via the method of steepest descent leading to the correct asymptotic form (2). In terms of an H-function the solution of Eq. (12) can be given exactly as

$$P(r,t) = Br^{-d_f} H^{1,0}_{1,1}\left[\frac{r}{Gt^{1/d_w}}\,\middle|\,\begin{matrix}(1\,,\,1/d_w)\\(d_f-\kappa\,,\,1)\end{matrix}\right], \tag{13}$$

where B is a constant. The asymptotic behaviour (2) can therefore be calculated from (13), as well. But by an appropriate choice of the parameters $d_w \to 2$ and $d_f \to d$ neither (12) nor (13) do reduce to the d-dimensional ordinary diffusion equation (6) and Gaussian solution, respectively. Only for the special values $d = 1$ and $d = 3$ the reduction is working. The reason for this inconsistency, especially for $d = 2$, is due to the formulation of a "halved" equation which is only exact for the special values $d = 1$ and $d = 3$ (see Section 4). To overcome this limitation a "full" diffusion equation has to be formulated.

3. General case

In the following a generalized diffusion equation is introduced and it is shown that only a particular set of parameters can reproduce (1) and (2). Considering the diffusion-type Eqs. (9) and (12), a generalized (and still exactly solvable) fractional partial differential equation embracing Eqs. (9) and (12) can be written out as follows:

$$\frac{\partial^\gamma}{\partial t^\gamma}P(r,t) = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{-\Theta} r^{D-1}\frac{\partial}{\partial r}P(r,t)\right). \tag{14}$$

Here, yet three parameters, γ, D and Θ are present. They are to be specified in the further procedure. The reason for the consideration of the additional power Θ of r is to investigate whether the argumentation of O'Shaugnessy and Procaccia [9] can be saved somehow. It should not be denied from the very beginning. But before starting off towards the solution it is worth remarking that a term of the form

$$\alpha/r^2 \cdot P(r,t), \tag{15}$$

α an arbitrary constant, may be added to (14) without changing the asymptotical shape of the solution. Such an additionary term cancels out in the asymptotic expansion. In fact, this peculiarity enables one to obtain the correct form (2) from Eq. (12).

Let us first summarize the following procedure as a kind of recipe providing a helpful survey over the steps to come: After the *(i) Solution of the fractional partial differential equation in the Laplace domain* including the *(ii) Fulfilling of the asymptotic decrease* *for large r* the *(iii) Normalization condition* determines the shape of the solution in Laplace-space unequivocally. Subsequent *(iv) Inversion of the Laplace transformation* reveals the general solution of Eq. (5) in the time domain and the free parameters are successively determined by the *(v) Calculation of the second moment and comparison with (1)* and the *(vi) Adjustment of the solution's asymptotic expansion to Eq. (2).*

(i) Application of the Laplace transformation

$$\tilde{P}(r;s) = \int_0^\infty dt\ \mathrm{e}^{-st}\, P(r,t) \tag{16}$$

to Eq. (14) results in the ordinary differential equation

$$s^\gamma \tilde{P}(r;s) = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{-\Theta} r^{D-1}\frac{\partial}{\partial r}\tilde{P}(r;s)\right) \tag{17}$$

in the Laplace space. The solution can be given in terms of Bessel functions of the first and second kind depending on a complex variable or in terms of modified Bessel functions of a real argument.

(ii) By physical reasons, the result is not allowed to tend to infinity for r going to infinity. Thus the result of (17) is

$$\tilde{P}(r;s) = A\left(\frac{r^{(2+\Theta)/2}}{s^{\gamma/2}}\right)^{1-D/(2+\Theta)} K_\nu\left(\frac{2}{2+\Theta} r^{(2+\Theta)/2} s^{\gamma/2}\right), \tag{18}$$

where $\nu = 1 - D/(2+\Theta)$ is the order of the modified Bessel function K_ν [14]. A is a constant of integration possibly depending on s.

(iii) The arbitrariness of $A = A(s)$ can be removed by demanding $P(r,t)$ to be normalized for all times, i.e. $\langle P_N(r,t)\rangle = 1$ or

$$\left\langle \tilde{P}_N(r;s)\right\rangle = \frac{1}{s}. \tag{19}$$

The index N denotes the normalized distribution function. The volume element in the procedure of normalization shall follow a scaling law

$$dV \propto r^{\tilde{d}-1} dr, \tag{20}$$

with $\tilde{d}$ left unspecified for the moment. Of course, it is expected to be d_f, the Hausdorff dimension of the underlying fractal geometry. But it will be shown in course of our argumentation that it is actually completely determined. Thus the normalization condition can be written in the form

$$\left\langle \tilde{P}_N(r;s)\right\rangle = Q(s)\int_0^\infty dr\, r^{\tilde{d}-1}\tilde{P}(r;s) \;=\; \frac{1}{s}, \tag{21}$$

where $Q(s)$ guarantees the correct s-dependence. Finally, the normalized result in s-space

$$\tilde{P}_N(r;s) = \tilde{A}\left(r^{(2+\Theta)/2}\right)^{2/\gamma-2\tilde{d}/(2+\Theta)} \times H^{2,0}_{0,2}\left[\left(\frac{2}{2+\Theta}\right)^{2/\gamma} r^{(2+\Theta)/\gamma}s \,\middle|\, \left(\frac{\alpha+\nu}{2},\frac{1}{\gamma}\right),\left(\frac{\alpha-\nu}{2},\frac{1}{\gamma}\right)\right] \tag{22}$$

is obtained. Here, the modified Bessel function is already expressed by its corresponding H-function [13]. $\tilde{A}$ is a constant and the abbreviations α and ν are standing for

$$\alpha = \frac{2\tilde{d}}{2+\Theta} - \frac{2}{\gamma} + \nu \tag{23}$$

and

$$\nu = 1 - \frac{D}{2+\Theta}, \tag{24}$$

respectively.

(iv) The next step in the general procedure is the back-transformation to time-space. This step is quite easy for the class of H-functions because the integral transformation reduces to a formal manipulation of the parameters of the H-function [15]. The result is

$$P_N(r,t) = A^* \frac{\left(r^{(2+\Theta)/2}\right)^{2/\gamma-2\tilde{d}/(2+\Theta)}}{t} \times H^{2,0}_{1,2}\left[\left(\frac{2}{2+\Theta}\right)^{2/\gamma} \frac{r^{(2+\Theta)/\gamma}}{t} \,\middle|\, \begin{matrix}(0,1)\\ \left(\frac{\alpha+\nu}{2},\frac{1}{\gamma}\right),\left(\frac{\alpha-\nu}{2},\frac{1}{\gamma}\right)\end{matrix}\right] \tag{25}$$

with A^* being a constant. For an H-function

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right], \tag{26}$$

with $n = 0$, the asymptotic behaviour for $|z| \gg 1$ becomes exponential [13]

$$H^{m,0}_{p,q}[z] \sim O\left(\exp\left(-\mu z^{1/\mu}\beta^{1/\mu}\right) z^{(\rho+1/2)/\mu}\right), \tag{27}$$

provided that $\lambda > 0$, $|\arg z| < \pi\lambda/2$, and $\mu > 0$. O denotes the well known Landau symbol and the Greek letters symbolize

$$\beta = \prod_{j=1}^{p} (A_j)^{A_j} \prod_{j=1}^{q} (B_j)^{-B_j}, \tag{28}$$

$$\rho = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p}{2} - \frac{q}{2}, \tag{29}$$

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \tag{30}$$

$$\lambda = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j - \sum_{j=1}^{p} A_j. \tag{31}$$

For the special case (25) this means $\rho = \alpha - 1$ and $\mu = 2/\gamma$. Since ν cancels out according to (29), and with it every term of the form (15), it does not affect the asymptotic behaviour. In the asymptotic domain, (25) is now revealed to be of the form

$$P_N(r,t) \sim r^{\frac{\tilde{d}-D/\gamma}{2/\gamma-1}} t^{\frac{D-2\tilde{d}}{(2/\gamma-1)(2+\Theta)}} \exp\left[-c\left(r^{(2+\Theta)/2}/t^{\gamma/2}\right)^{1/(1-\gamma/2)}\right], \tag{32}$$

which is of the required type.

(v) The time-dependence of the distribution's second moment may be calculated from

$$\left\langle r^2(s)\right\rangle = \int_0^\infty dr\, r^{\tilde{d}-1} r^2 \tilde{P}_N(r;s) \propto \left(\frac{1}{s}\right)^{\frac{2\gamma}{2+\Theta}+1}, \tag{33}$$

leading to

$$\left\langle r^2(t)\right\rangle \propto t^{\frac{2\gamma}{2+\Theta}} \tag{34}$$

in the time domain. We require the power of t in (34) to be equal to $2/d_w$. A relation

$$\frac{2\gamma}{2+\Theta} = \frac{2}{d_w} \tag{35}$$

emanates connecting γ and Θ. Three different cases are worth mentioning. For arbitrary values of γ, Θ must be equal to $d_w\gamma - 2$. The special cases with $\gamma = 1$ requiring $\Theta = d_w - 2$, and $\gamma = 2/d_w$ demanding $\Theta = 0$ regain the models discussed by O'Shaugnessy and Procaccia [9], and Giona and Roman [10], respectively.

(vi) Using (35), the result of the introduced diffusion equation (14) in the asymptotic domain has the form

$$P_N(r,t) \sim r^{\frac{\tilde{d}-D/\gamma}{2/\gamma-1}} t^{\frac{D-2\tilde{d}}{(2-\gamma)d_w}} \exp\left[-c\left(r^{d_w\gamma/2}/t^{\gamma/2}\right)^{1/(1-\gamma/2)}\right], \tag{36}$$

where c is a constant. Comparison of the exponential functions in (36) and (2) leads to

$$\frac{d_w}{2/\gamma - 1} = \frac{d_w}{d_w - 1}. \tag{37}$$

Thus, the only possible value for γ is $\gamma = 2/d_w$. The powers of r and t before the exponential function demand

$$\frac{\tilde{d} - d_w D/2}{d_w - 1} = 0 \tag{38}$$

and

$$\frac{D - 2\tilde{d}}{2(d_w - 1)} = -\frac{d_f}{d_w}, \tag{39}$$

respectively.

With (38) and (39) we arrive at the generalized diffusion equation (5) with $d_s = 2d_f/d_w$. Hence, all the parameters occurring in (14) are uniquely determined and in addition the volume element (20), necessary in the calculation of the moments scales with $\tilde{d} = d_f$, as expected. According to (25), the solution of (5) is given by

$$P_N(r,t) = A^* t^{-d_f/d_w} H^{2,0}_{1,2}\left[\frac{r^{d_w}}{t} \,\middle|\, \begin{matrix} \left(1 - \frac{d_f}{d_w}, 1\right) \\ \left(1 - \frac{d_f}{d_w}, \frac{d_w}{2}\right), \left(0, \frac{d_w}{2}\right) \end{matrix} \right], \tag{40}$$

with the constant A^* defined by

$$A^* = \frac{2^{-d_w-3} d_w}{\Gamma\left(\frac{1}{2} + d_f - \frac{1}{2}d_s\right) \Gamma\left(\frac{1}{2}d_f\right)}. \tag{41}$$

The H-function occurring in (40) can be expressed via a convergent power series expansion [13] with the result

$$\begin{aligned} P_N(r,t) = {} & A^* \frac{2}{d_w} t^{-d_f/d_w} \\ & \times \left[\sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{d_f}{d_w} - 1 - j\right)}{\Gamma\left(1 - \frac{d_f}{d_w} - \frac{2}{d_w}\left(1 - \frac{d_f}{d_w} + j\right)\right)} \frac{(-1)^j}{j!} \left(\frac{r^{d_w}}{t}\right)^{\frac{2}{d_w}(1 - d_f/d_w + j)} \right. \\ & \left. + \sum_{j=0}^{\infty} \frac{\Gamma\left(1 - \frac{d_f}{d_w} - j\right)}{\Gamma\left(1 - \frac{d_f}{d_w} - \frac{2}{d_w}j\right)} \frac{(-1)^j}{j!} \left(\frac{r^{d_w}}{t}\right)^{\frac{2}{d_w}j} \right]. \end{aligned} \tag{42}$$

For $r \to 0$ the probability density becomes $P_N(r = 0, t) \propto t^{-d_f/d_w}$ which is the temporal decreasing of the returning probability to the origin [4]. For $t \to 0$, $r^{d_f - 1} P_N(r,t)$ approaches the δ-function, $\delta(r)$, so that $P_N(r,t)$ actually describes the location probability of a particle starting at $r = 0$. In the limit of a small argument, i.e. $r^{d_w}/t \ll 1$, an algebraic behaviour is revealed. The density $P_N(r,t)$ is monotonically decreasing in r. The spreading of the probability in course of time is slower than in the normally diffusive case. In Fig. 1 $P_N(r,t)/A^*$ is plotted versus r for the times $t = 0.5, 1, 3, 10$. The chosen values of the fractal dimension, $d_f = 1.58$ and the anomalous diffusion exponent, $d_w = 2.32$, correspond to the two-dimensional Sierpiński gasket [4].

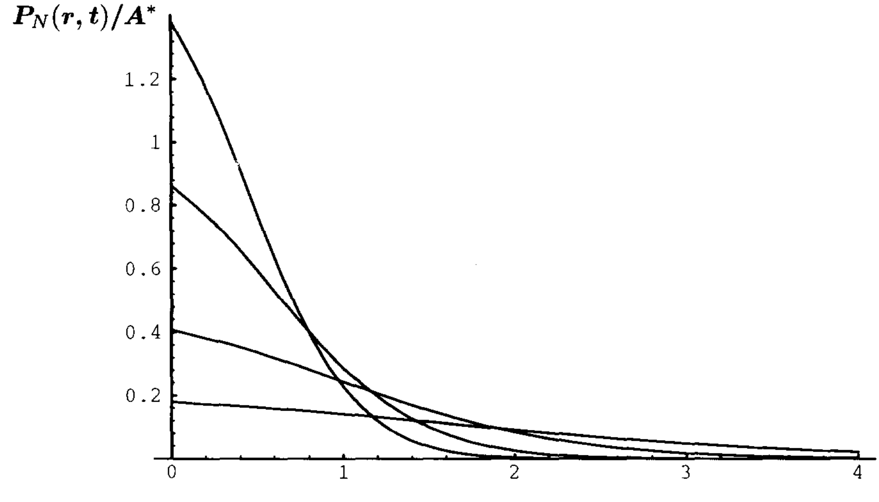

Fig. 1. Probability density (40) for $d_f = 1.58$ and $d_w = 2.32$ at the times $t = 0.5, 1, 3, 10$ (the curve with the largest value at $r = 0$ corresponds to the smallest time).

4. The "halved" diffusion equation

What is actually meant by this term? To see this, and how Eq. (12) fits into the whole scheme, let us briefly remind the procedure of Oldham and Spanier [11,12] for the one-dimensional case before starting out for the general case of re-writing Eq. (5) as a "halved" one.

The standard diffusion equation in one dimension is

$$\frac{\partial}{\partial t}P(r,t) = \frac{\partial^2}{\partial r^2}P(r,t) \tag{43}$$

and the solution in the Laplace domain is $\tilde{P}(r;s) = \alpha \exp(\sqrt{s}r) + \beta \exp(-\sqrt{s}r)$. By requiring a finite total probability, α must be zero. So the result in s-space is $\tilde{P}(r;s) = \beta \exp(-\sqrt{s}r)$, where β may still be dependent on s. The first derivative of this exponential function is

$$\frac{\partial}{\partial r}\tilde{P}(r;s) = -\sqrt{s}\,\tilde{P}(r;s). \tag{44}$$

Back-transformation results – by use of the definition of a fractional derivative – in

$$\frac{\partial^{1/2}}{\partial t^{1/2}}P(r,t) = -\frac{\partial}{\partial r}P(r,t), \tag{45}$$

which is the semi-differential or "halved" one-dimensional diffusion equation.

Following this scheme, we now start out from the generalized diffusion equation (5) and rewrite it as a "halved" equation. In s-space the solution of (5) may be written as

$$\tilde{P}(r;s) = \tilde{\beta} r^{1-d_s/2} K_{d_s/2-1}\left(r s^{1/d_w}\right) \tag{46}$$

where $\tilde{\beta}$ is an s-dependent coefficient.

A general feature of Bessel functions is the following property [14]:

$$\frac{d}{dz} z^{-\nu} K_\nu(z) = -z^{-\nu} K_{\nu+1}(z), \tag{47}$$

so that we can deduce for $\tilde{P}(r;s)$ in the Laplace domain

$$\begin{aligned}\frac{1}{s^{1/d_w}} \frac{\partial}{\partial r} \tilde{P}(r;s) &= -\tilde{\beta} r^{1-d_s/2} K_{d_s/2}(r s^{1/d_w}) \\ &= -\tilde{P}(r;s) \frac{K_{d_s/2}(r s^{1/d_w})}{K_{d_s/2-1}(r s^{1/d_w})}.\end{aligned} \tag{48}$$

A recursion formula for successive Bessel functions K_ν may be written as [14]

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = 1 + \frac{1/2-\nu}{z} \left\{ 1 - \frac{1/2+\nu}{2z} \left[1 + o\left(\frac{1}{z}\right) \right] \right\} + \frac{2\nu}{z}. \tag{49}$$

The infinite summation breaks off if and only if $\nu = \pm 1/2$. This means $d_s/2 = 1/2,\ 3/2$ or $d_f/d_w = 1/2,\ 3/2$. Especially for the case of normal diffusion, i.e. $d_w = 2$, the transition to a "halved" diffusion equation is only possible for one- and three-dimensional geometries. Nevertheless, in the limit $z \gg 1$,

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} \sim 1 + \frac{1/2+\nu}{z}, \tag{50}$$

so that in this limit the transition to such a "halved" equation for arbitrary d_s is allowed. The result is

$$\frac{\partial}{\partial r} \tilde{P}(r;s) = -\left(s^{1/d_w} + \frac{d_s/2 - 1/2}{r} \right) \tilde{P}(r;s). \tag{51}$$

Rewritten in time-space we arrive at

$$\frac{\partial^{1/d_w}}{\partial t^{1/d_w}} P(r,t) = -r^{1/2-d_s/2} \frac{\partial}{\partial r} r^{d_s/2-1/2} P(r,t), \tag{52}$$

which is valid in the asymptotic case, only. Thus, we have constructed Eq. (12) properly and have shown its limitation to the asymptotic domain.

5. Conclusions

We have formulated a generalized diffusion equation composed of an extended Laplacian operator and a fractal time derivative. By comparison to the known results, second moment (1) and asymptotic probability density function (2), the introduced parameters could be reduced and the remaining two could be assigned definite values, d_f and d_w, the fractal Hausdorff dimension and the anomalous diffusion exponent, respectively. The presented equation reduces exactly to the ordinary diffusion equation in d-dimensional Euclidean space for appropriate choice of the free parameters. The equation can be solved exactly in terms of an H-function. This solution is valid for all t and for all r. A corresponding "halved" diffusion equation in space-time is revealed as an alternative formulation which is valid in the asymptotic domain. A generalization in the sense of O'Shaugnessy and Procaccia, i.e. by amending $r^{-\Theta}$ inside the Laplacian operator, cannot provide the demanded result.

In our diffusion equation (5) three types of generalizations occur: the scaling of the volume element in the normalization integral, the fractal time derivative and the altered Laplacian operator. Those facets shall now be accounted for.

For calculating the moments of $P(r,t)$ the integration has to be performed by use of a volume element scaling with the fractal Hausdorff dimension of the underlying geometrical object. By definition, the integration for the calculation of the moments, especially the normalization, must average a given function on the hypervolume on which it is defined. Thus it seems to be quite clear that dV must be of the revealed form, i.e. V must scale with the fractal dimension d_f since the integral is to be calculated on the static fractal structure.

For the fractional time derivative the following reasoning applies. Given the asymptotic distribution function one can calculate its moments. The moment-operator $\hat{A}^n$ shall have the property $\hat{A}^n P(r,t) = \langle r^n(t)\rangle$. A straightforward integration results in

$$\hat{A}^n P(r,t) = a_n t^{n/d_w}, \tag{53}$$

where a_n is a constant. The proposed Eq. (14) with $\Theta = 0$ can be restated in a more general way using an arbitrary time-operator $\hat{T}$ that shall be dependent on time only,

$$\hat{T} P(r,t) = r^{1-D} \frac{\partial}{\partial r} r^{D-1} \partial_r P(r,t). \tag{54}$$

Now $\hat{A}^n$ is applied on both sides of Eq. (54). The rhs can be restated by use of an integration by parts as

$$\text{rhs} = \int_0^\infty dr\, B r^{d_f+n-3} P(r,t), \tag{55}$$

valid for all $n \geq 2$. B is a constant. Thus the rhs of (53) can be re-written as rhs $= B a_{n-2} t^{n/d_w - 2/d_w}$. The exponentially decreasing shape of the probability functions' asymptotic form causes $[\hat{T}, \hat{A}^n] = 0$ so that the lhs of (54) after application of $\hat{A}^n$ becomes lhs $= \hat{T} a_n t^{n/d_w}$. Hence the following equation emanates for $\hat{T}$

$$\hat{T} t^{n/d_w} = \tilde{B}_n t^{n/d_w - 2/d_w}, \tag{56}$$

which is fulfilled by a fractional time derivation operator of the order $2/d_w$. For every $n \geq 2$, the n-dependent coefficients cancel out in the same way, so that $\hat{T}$ is uniquely determined for all moments with $n \geq 2$.

The effect of the occurring differential operator is the introduction of a memory effect. Contrary to the Euclidean case, former values of $P(r,t)$ influence the present value of

$P(r,t)$. The memory effect in the time domain reflects the correlation in space caused by the fractal geometry.

A question that is still open, is why the fracton dimension happens to occur in the Laplacian operator. The proposed diffusion equation can be considered in terms of an effective volume $V_{eff} = r^{d_s}$ entering into the Laplacian operator. Thus the diffusing particles "see" their environment not as a fractal of its Hausdorff-dimension d_f but as a fractal of the fracton dimension d_s, where $d_s < d_f$.

Acknowledgement

A poster version of this paper was presented at the Gordon Research Conference on Fractals in San Miniato, Toscany, in early May this year.

This paper was supported by the Deutsche Forschungsgemeinschaft (project SFB 239 C8).

References

[1] H.G.E. Hentschel and I. Procaccia, Phys. Rev. A 29 (1984) 1461.
[2] W. Young, A. Pumir and Y. Pomeau, Phys. Fluids A 1 (1989) 462.
[3] O. Cardoso and P. Tabeling, Europhys. Lett. 7 (1988) 225.
[4] S. Havlin and D. Ben-Avraham, Adv. in Phys. 36 (1987) 695.
[5] H.W. Weber and R. Kimmich, Macromolecules 26 (1993) 2597.
[6] R. Rammal and G. Toulouse, J. Phys. Lett. 44 (1983) L-13.
[7] W.R. Schneider and W. Wyss, J. Math. Phys. 30 (1989) 134.
[8] W.G. Glöckle and T.F. Nonnenmacher, Macromolecules 24 (1991) 6426.
[9] B. O'Shaugnessy and I. Procaccia, Phys. Rev. Lett. 54 (1985) 455.
[10] M. Giona and H.E. Roman, Physica A 185 (1992) 87.
[11] K.B. Oldham and J. Spanier, J. Math. Anal. Appl. 39, 655.
[12] K.B. Oldham and J. Spanier, The Fractional Calculus (Academic, New York, 1974).
[13] A.M. Mathai and R.K. Saxena, The H-function with Applications in Statistics and Other Disciplines (Wiley Eastern, New Delhi, 1978).
[14] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions (Dover, New York, 1970).
[15] W.G. Glöckle and T.F. Nonnenmacher, J. Stat. Phys. 71 (1993) 741.