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Stochastic dynamics with multiplicative dichotomic noise: Heterogeneous telegrapher's equation, anomalous crossovers and resetting

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A R T I C L E I N F O

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A B S T R A C T

We analyze diffusion processes with finite propagation speed in a non-homogeneous medium in terms of the heterogeneous telegrapher's equation. In the diffusion limit of infinite-velocity propagation we recover the results for the heterogeneous diffusion process. The heterogeneous telegrapher's process exhibits a rich variety of diffusion regimes including hyperdiffusion, ballistic motion, superdiffusion, normal diffusion and subdiffusion, and different crossover dynamics characteristic for complex systems in which anomalous diffusion is observed. The anomalous diffusion exponent in the short time limit is twice the exponent in the long time limit, in accordance to the crossover dynamics from ballistic diffusion to normal diffusion in the standard telegrapher's process. We also analyze the finite-velocity heterogeneous diffusion process in presence of stochastic Poissonian resetting. We show that the system reaches a non-equilibrium stationary state. The transition to this non-equilibrium steady state is analyzed in terms of the large deviation function.

1. Introduction

It is well known that the Green's function of the classical diffusion equation, the Gaussian distribution, has non-zero values for any x at $t > 0$, which means that some of the particles move with an arbitrarily chosen large velocity. To avoid this unphysical property, a finite-velocity diffusion process governed by the so-called telegrapher's (or Cattaneo) equation was introduced, and a corresponding persistent random walk model was proposed. Historically, the telegrapher's equation has been derived by Heaviside for a voltage u along a lossy transmission line in electrodynamic theory [1],

$$\tau \frac{\partial^2}{\partial t^2} u(x,t) + \frac{\partial}{\partial t} u(x,t) = D \frac{\partial^2}{\partial x^2} u(x,t). \tag{1}$$

Here τ is a time parameter, and D is the diffusion coefficient, which relates to a finite propagation velocity $v = \sqrt{D/\tau}$. Contrary to the diffusion equation which is parabolic, the telegrapher's equation is a hyperbolic partial differential equation. A simple interpretation of this process is that the probability flux is delayed over time by the interval τ with respect to the probability gradient, $J(x,t+\tau) = -D\frac{\partial}{\partial x}u(x,t)$. Assuming $\tau \ll t$, then

$$J(x,t) + \tau \frac{\partial}{\partial t} J(x,t) = -D \frac{\partial}{\partial x} u(x,t). \tag{2}$$

This equation was proposed by Cattaneo in 1948 [2] (see also [3,4]) to extend the standard constitutive relation. Combining this equation with continuity equation

$$\frac{\partial}{\partial t} u(x,t) = -\frac{\partial}{\partial x} J(x,t), \tag{3}$$

one obtains the telegrapher's equation (1) that is often alternatively referred to as Cattaneo equation. The persistent random walk was suggested first by Fürth [5] and Taylor [6], who considered it as a suitable model for transport in turbulent diffusion, while Goldstein gave solutions of various forms of the telegrapher's equation [7] (see also [8]). The telegrapher's equation can be considered as a particular

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case of a spatio-temporally coupled Lévy walk model with exponential waiting time probability density [9–11]. Extended Poisson–Kac theory provides a unifying framework for stochastic processes with finite propagation velocity and was developed recently [12]. The telegrapher's equation was also used to study finite-velocity diffusion on a comb [13] and in random media [14], as well as the telegraph processes with random velocities [15]. Fractional generalizations of the telegrapher's equation were considered in [16–21], while non-Markovian discrete-time versions of the telegraph process were studied in [22]. For more details on the Cattaneo equation, as well as derivation and application of the telegrapher's equation, we refer to the literature, see, e.g., [23–26].

In the telegrapher's equation (1) it is assumed that the diffusion coefficient D and the time interval τ are constants. In present paper we consider the case of space-dependent diffusion coefficient. In pure diffusion models, a space-dependent diffusivity is introduced to describe heterogeneous diffusion process (HDP), i.e., relative diffusion of passive tracers in the atmosphere [27,28], transport processes in heterogeneous media [29–40], and on random fractals [41], including comb structures [42,43]. The mean first passage time and related search problems [44,45], ergodicity breaking [35–37] and infinite ergodic theory for HDPs [46], as well as Lévy flights in inhomogeneous media [47] have been investigated, as well.

In Section 2 we introduce the telegrapher's equation for a finite-velocity HDP. We derive a general solution of the problem in Section 3. Exact results for the probability density function (PDF) are obtained, and the asymptotic behaviors are analyzed. In Section 4 we present general result for the MSD, for which we observe different crossovers between diffusion regimes in the system. Several special cases are recovered, as well. We then introduce exponential resetting in the heterogeneous telegrapher's equation in Section 5 and report exact results for the PDF and MSD. It is shown that in the long time limit the system approaches a non-equilibrium stationary state (NESS). The transition to the NESS is analyzed in terms of the large deviation function. In Section 6 we summarize our findings.

2. From the Langevin equation with dichotomic noise to the telegrapher's equation

The master equation for a persistent random walk leading to a Langevin equation with dichotomic noise was considered in [48,49]. The Langevin equation takes the form

$$\dot{x}(t) = \upsilon\,\zeta(t), \tag{4}$$

where υ is a positive constant with physical dimension of a speed, and $\zeta(t)$ is a stationary dichotomic Markov process that jumps between two states ± 1 with the mean rate ν, i.e., the inverse mean sojourn time for each state. The corresponding equation for the PDF $P(x,t)$ of such a process is the telegrapher's equation (1), where $\tau = \frac{1}{2\nu}$ and $D = \upsilon^2\tau$. For an elegant and simple derivation of Eq. (1) starting from Eq. (4) we refer the reader to Ref. [49]. In the limit $\nu \to \infty$, $\upsilon \to \infty$ such that D is a finite constant the dichotomic noise reduces to a Gaussian white noise, and the diffusion equation is obtained.

There exist different generalizations of the standard telegrapher's equation for inhomogeneous cases [23,50–55]. In this paper, we consider the form originating from the general nonlinear Langevin equation with multiplicative dichotomic noise

$$\dot{x}(t) = \upsilon(x)\zeta(t), \tag{5}$$

where $\upsilon(x) > 0$ is a position-dependent speed, and $\zeta(t)$ is the same dichotomic process as in Eq. (4). The result is

$$\frac{\partial^2}{\partial t^2}P(x,t) + \frac{1}{\tau}\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left\{\upsilon(x)\frac{\partial}{\partial x}\left[\upsilon(x)P(x,t)\right]\right\}, \tag{6}$$

where $\upsilon(x) = \sqrt{D(x)/\tau}$. A detailed derivation of the heterogeneous telegrapher's equation from the Langevin equation (5) is given in Ref [55], see Theorem 3.1.. Other forms of the heterogeneous telegrapher's equation can be derived for a voltage and current in a inhomogeneous lossy transmission line, from the generalized Fick's law with position-dependent diffusivity, as well as from the persistent random walk in inhomogeneous medium. For details please see Appendix A. The motivation to consider heterogeneous models comes from the application of the telegrapher's equation in the description of turbulent diffusion [56–58], as well as in cosmic-ray transport [59]. Moreover, the heterogeneous telegrapher's equation may also be important in the description of turbulent relative dispersion of particle pairs [60,61] and represents a generalization of the Richardson model [27], since it takes into consideration the long-time correlation of the Lagrangian relative velocity of a particle pair, which exists in turbulent flows [62,63].

3. Solution of the heterogeneous telegrapher's equation

3.1. Solution for $x_0 \neq 0$

In what follows we consider equation of form (6) for power-law position dependent speed $\upsilon(x) = \upsilon_\alpha |x|^{\frac{\alpha}{2}}$, where $\upsilon_\alpha > 0$ is with physical dimension $[\upsilon_\alpha] = \mathrm{m}^{1-\alpha/2}s^{-1}$. Thus, we write the heterogeneous telegrapher's equation (6) in the form

$$\tau\frac{\partial^2}{\partial t^2}P(x,t) + \frac{\partial}{\partial t}P(x,t) = \mathcal{D}_\alpha\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}P(x,t)\right]\right\}, \tag{7}$$

where $\mathcal{D}_\alpha = \upsilon_\alpha^2\tau$ is a diffusion coefficient with physical dimension $[\mathcal{D}_\alpha] = \mathrm{m}^{2-\alpha}s^{-1}$. For $\tau \to 0$ and $\upsilon_\alpha \to \infty$ such that $\mathcal{D}_\alpha = const$, Eq. (7) becomes the heterogeneous (infinite-velocity) diffusion equation [35],[1]

$$\frac{\partial}{\partial t}p(x,t) = \mathcal{D}_\alpha\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}p(x,t)\right]\right\}, \tag{8}$$

which is derived from the Langevin equation in the Stratonovich interpretation [35]

$$\dot{x}(t) = \sqrt{2\,\mathcal{D}_\alpha|x|^\alpha}\,\eta(t), \tag{9}$$

with position dependent diffusion coefficient, where $\eta(t)$ is a white Gaussian noise of zero mean. Here we use $\alpha < 2$ to ensure the growth condition for existence and uniqueness of the solution of a Markovian stochastic differential equation, see Ref. [35]. The case with $\alpha = 2$ requires a separate consideration and is related to the problem of geometric Brownian motion in the Stratonovich interpretation [64].

To solve Eq. (7), we consider the initial conditions[2]

$$P(x,t=0) = \delta(x-x_0), \quad \frac{\partial}{\partial t}P(x,t=0) = 0, \tag{10}$$

and the boundary conditions are set to zero at infinity, i.e.,

$$P(\pm\infty,t) = 0, \quad \frac{\partial}{\partial x}P(\pm\infty,t) = 0.$$

The Laplace transform[3] of Eq. (7) yields

$$s(1+\tau s)\hat{P}(x,s) - (1+\tau s)\delta(x-x_0) = \mathcal{D}_\alpha\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}\hat{P}(x,s)\right]\right\}, \tag{11}$$

which can be rewritten in the form

$$s\hat{P}(x,s) - \delta(x-x_0) = \mathcal{D}_\alpha\frac{1}{1+\tau s}\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}\hat{P}(x,s)\right]\right\}. \tag{12}$$

We note that by inverse Laplace transform, we obtain an equivalent formulation for Eq. (7),

$$\frac{\partial}{\partial t}P(x,t) = \mathcal{D}_\alpha\int_0^t K(t-t',\tau)\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}P(x,t')\right]\right\}dt', \tag{13}$$

[1] Here we note that we use $p(x,t)$ for the PDF of the HDP, while $P(x,t)$ is the PDF of the heterogeneous telegrapher's process.

[2] See the discussion of initial conditions in Refs. [23,65,66].

[3] The Laplace transform is defined by $\hat{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$, while the inverse Laplace transform by $f(t) = \mathcal{L}^{-1}\left[\hat{f}(s)\right] = \frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty}\hat{f}(s)e^{st}ds$.

where

$$K(t,\tau)=\frac{1}{\tau}e^{-t/\tau} \quad \rightarrow \quad \hat{K}(s,\tau)=\frac{1}{1+s\tau}.$$

Therefore, the heterogeneous telegrapher's equation (7) can be considered as a heterogeneous diffusion equation with an non-local memory kernel. Such an equation with exponential memory kernel was analyzed for $\alpha=0$ in [67,68].

For $\tau=0$, we obtain the HDP equation in Laplace space,

$$s\hat{p}(x,s)-\delta(x-x_0)=\mathcal{D}_\alpha\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}\hat{p}(x,s)\right]\right\}. \tag{14}$$

Using $s\to s(1+\tau s)$ in Eq. (14) we obtain

$$s(1+\tau s)\hat{p}(x,s(1+\tau s))-\delta(x-x_0)=\mathcal{D}_\alpha\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}\hat{p}(x,s(1+\tau s))\right]\right\}. \tag{15}$$

Now, one can see that by substituting

$$\hat{P}(x,s)=(1+s\tau)\,\hat{p}(x,s(1+s\tau)), \tag{16}$$

from Eq. (15) we arrive at Eq. (12). Thus, we can directly obtain the solution $P(x,t)$ from the solution $p(x,t)$. Relation (16) is not affected by the inhomogeneity in space and was derived in [67] for the standard telegrapher's equation, Eq. (13) with $\alpha=0$, by using the subordination approach. The solution of the diffusion equation for the HDP (8) is given by (see equation (18) in [69])

$$p(x,t)=\frac{|x|^{1/p-1}}{\sqrt{4\pi\mathcal{D}_p t}}\times\exp\left(-\frac{p^2\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|^2}{4\mathcal{D}_p t}\right), \tag{17}$$

where $\mathcal{D}_\alpha\to\mathcal{D}_p$ and $p=\frac{2}{2-\alpha}$, and which in Laplace space reads

$$\begin{aligned}\hat{p}(x,s)=&\frac{|x|^{1/p-1}}{2\sqrt{\mathcal{D}_p}}s^{-1/2}\\&\times\exp\left(-\frac{p}{\sqrt{\mathcal{D}_p}}\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|s^{1/2}\right).\end{aligned} \tag{18}$$

By integration of (18) it is shown that $\int_{-\infty}^{\infty}\hat{p}(x,s)\,dx=\frac{1}{s}$ (see Appendix A in Ref. [69]), which means that $p(x,t)$ is normalized. Therefore, from Eq. (16), we have

$$\int_{-\infty}^{\infty}\hat{P}(x,s)\,dx=(1+s\tau)\int_{-\infty}^{\infty}\hat{p}(x,s(1+s\tau))\,dx=(1+s\tau)\frac{1}{s(1+s\tau)}=\frac{1}{s}, \tag{19}$$

which means that the PDF $P(x,t)$ is normalized, as well. We also note that in the limit $\alpha\to2$ and $x,x_0>0$, the PDF (18) turns into the log-normal distribution for geometric Brownian motion [64]. Thus, for the PDF $P(x,t)$, we obtain in Laplace space

$$\begin{aligned}\hat{P}(x,s)=&\frac{|x|^{1/p-1}}{2v_p}(\tau^{-1}+s)(s+\tau^{-1})^{-1/2}s^{-1/2}\\&\times\exp\left(-\frac{p}{v_p}\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|(s+\tau^{-1})^{1/2}s^{1/2}\right),\end{aligned} \tag{20}$$

where $v_p=\sqrt{\mathcal{D}_p/\tau}$. This result can be rewritten in the form

$$\hat{P}(x,s)=\frac{|x|^{1/p-1}}{2v_p}(\tau^{-1}+s)\hat{L}(x,s), \tag{21}$$

with

$$\begin{aligned}\hat{L}(x,s)=&(s+\tau^{-1})^{-1/2}s^{-1/2}\\&\times\exp\left(-\frac{p}{v_p}\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|(s+\tau^{-1})^{1/2}s^{1/2}\right).\end{aligned} \tag{22}$$

The PDF then becomes

$$P(x,t)=\frac{|x|^{1/p-1}}{2v_p}\left[\tau^{-1}L(x,t)+\frac{\partial}{\partial t}L(x,t)\right], \tag{23}$$

where

$$\begin{aligned}L(x,t)=&\theta\left(v_pt-p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)e^{-\frac{t}{2\tau}}\\&\times I_0\left(\frac{\sqrt{v_p^2t^2-p^2\left[\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right]^2}}{2v_p\tau}\right).\end{aligned} \tag{24}$$

Finally, we obtain the form

$$\begin{aligned}P(x,t)=&\frac{|x|^{1/p-1}}{2}e^{-\frac{t}{2\tau}}\delta\left(v_pt-p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)\\&+\frac{|x|^{1/p-1}}{4v_p\tau}\theta\left(v_pt-p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)\\&\times e^{-\frac{t}{2\tau}}\left[I_0(\xi)+\frac{t}{2\tau}\frac{I_1(\xi)}{\xi}\right],\end{aligned} \tag{25}$$

where

$$\xi=\frac{\sqrt{t^2-\frac{p^2}{v_p^2}\left[\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right]^2}}{2\tau}. \tag{26}$$

Here we note that $I_\nu(z)$ is the modified Bessel function of the first kind with asymptotic behavior $I_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}$. Thus, in the long time limit, from (25) we arrive at the PDF (17) for the heterogeneous diffusion equation, while in short time limit we have

$$P(x,t)=\frac{|x|^{1/p-1}}{2}\delta\left(v_pt-p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right). \tag{27}$$

For $\alpha=0$, i.e., $p=1$, we obtain the solution for the standard telegrapher's equation [7] (see also equation (16) in Ref. [70], equation (2.5.3) in Ref. [66] or equation (6) in [71])

$$\begin{aligned}P(x,t)=&\frac{e^{-\frac{t}{2\tau}}\delta\left(vt-|x-x_0|\right)}{2}\\&+\frac{e^{-\frac{t}{2\tau}}}{4v\tau}\theta\left(vt-|x-x_0|\right)\\&\times\left[I_0\left(\frac{\sqrt{v^2t^2-|x-x_0|^2}}{2v\tau}\right)+vt\frac{I_1\left(\frac{\sqrt{v^2t^2-|x-x_0|^2}}{2v\tau}\right)}{\sqrt{v^2t^2-|x-x_0|^2}}\right].\end{aligned} \tag{28}$$

For $x_0=0$, the PDF (25) reduces to

$$\begin{aligned}P(x,t)=&\frac{|x|^{1/p-1}}{2}e^{-\frac{t}{2\tau}}\delta\left(v_pt-p|x|^{1/p}\right)+\frac{|x|^{1/p-1}}{4v_p\tau}e^{-\frac{t}{2\tau}}\theta\left(v_pt-p|x|^{1/p}\right)\\&\times\left[I_0\left(\frac{\sqrt{v_p^2t^2-p^2|x|^{2/p}}}{2v_p\tau}\right)+v_pt\frac{I_1\left(\frac{\sqrt{v_p^2t^2-p^2|x|^{2/p}}}{2v_p\tau}\right)}{\sqrt{v_p^2t^2-p^2|x|^{2/p}}}\right].\end{aligned} \tag{29}$$

A graphical representation of the PDF (25) for different parameter values is shown in Fig. 1. We observe the distinct finite-velocity propagation where the PDF drops to zero beyond the front $v_pt=p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|$. We compare the PDFs for the telegrapher's equation and the diffusion equation in Fig. 2. Due to the finite-velocity propagation, we see the drop to zero of the PDF for the telegrapher's process (red solid lines), while instantaneous propagation

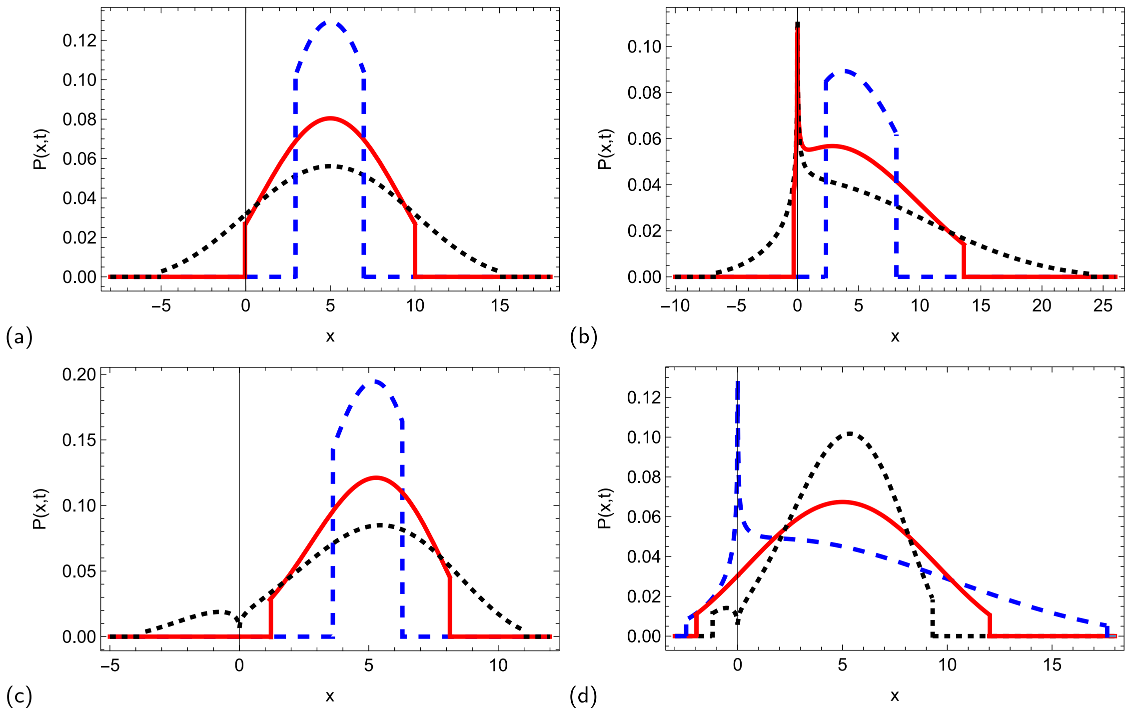

Fig. 1. PDF (25) for $x_0 = 5$, $\tau = 1$, $v_\alpha = 1$. (a) $\alpha = 0$, $t = 2$ (blue dashed line), $t = 5$ (red solid line), $t = 10$ (black dotted line). (b) $\alpha = 0.5$, $t = 2$ (blue dashed line), $t = 5$ (red solid line), $t = 10$ (black dotted line). (c) $\alpha = -0.5$, $t = 2$ (blue dashed line), $t = 5$ (red solid line), $t = 10$ (black dotted line). (d) Comparison between PDFs for $t = 7$, and $\alpha = 0.5$ (blue dashed line), $\alpha = 0$ (red solid line), $\alpha = -0.5$ (black dotted line). The delta functions at the endpoints, which ensure normalization of the PDFs are not shown in the figure.

is observed for the HDP (blue dashed lines). We also note that the PDF is unimodal for $\alpha > 0$ and bimodal for $\alpha < 0$. Here we note that the contribution of the delta functions which ensure normalization of the PDF at the endpoints x, that satisfy $v_p t = p\left|\mathrm{sgn}(x)|x|^{1/p} - \mathrm{sgn}(x_0)|x_0|^{1/p}\right|$, are not shown in the figures. We have already proven in Eq. (19) that the PDF satisfies the normalization condition $\int_{-\infty}^{\infty} P(x,t)\,dx = 1$, however, for the readers' convenience, in Appendix B we provide additional proof of the normalization directly by integration of the PDF (23).

3.2. Alternative solution for $x_0 = 0$

For the specific initial condition at the origin, $P(x,0) = \delta(x)$, we find the solution of Eq. (7) in Laplace space yielding

$$s(1+\tau s)\hat{P}(x,s) - (1+\tau s)\delta(x) = \mathcal{D}_\alpha \frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}\hat{P}(x,s)\right]\right\}. \quad (30)$$

From here, by differentiation with respect to x, we find

$$\begin{aligned} &s(1+\tau s)\hat{P}(x,s) - (1+\tau s)\delta(x) \\ &= \mathcal{D}_\alpha \frac{\partial}{\partial x}\left\{(2\theta(x)-1)\frac{\alpha}{2}|x|^{\alpha-1}\hat{P}(x,s) + |x|^\alpha \frac{\partial}{\partial x}\hat{P}(x,s)\right\}, \end{aligned} \quad (31)$$

where $\theta(x)$ is the Heaviside step function. By further differentiation with respect to x, we obtain

$$\begin{aligned} &s(1+\tau s)\hat{P}(x,s) - (1+\tau s)\delta(x) \\ &= \mathcal{D}_\alpha\left\{\frac{\alpha(\alpha-1)}{2}|x|^{\alpha-2}\hat{P}(x,s) + \alpha|x|^{\alpha-1}\delta(x)\hat{P}(x,s)\right. \\ &\left. + \frac{3\alpha}{2}\left[2\theta(x)-1\right]|x|^{\alpha-1}\frac{\partial}{\partial x}\hat{P}(x,s) + |x|^\alpha\frac{\partial^2}{\partial x^2}\hat{P}(x,s)\right\}. \end{aligned} \quad (32)$$

We see that Eq. (32) is invariant with respect to inversion $x \to -x$, and we use $y = |x|$. Eq. (32), then becomes

$$\begin{aligned} &s(1+\tau s)\hat{P}(y,s) - (1+\tau s)\delta(x) \\ &= \mathcal{D}_\alpha\frac{(\alpha-1)\alpha}{2}y^{\alpha-2}\hat{P}(y,s) + \mathcal{D}_\alpha\alpha y^{\alpha-1}\hat{P}(y,s)\delta(x) \\ &+ \mathcal{D}_\alpha\frac{3\alpha}{2}y^{\alpha-1}\frac{\partial}{\partial y}\hat{P}(y,s) + 2\mathcal{D}_\alpha y^\alpha\frac{\partial}{\partial y}\hat{P}(y,s)\delta(x) + \mathcal{D}_\alpha y^\alpha\frac{\partial^2}{\partial y^2}\hat{P}(y,s), \end{aligned} \quad (33)$$

where $\hat{P}(|x|,s) = C(s)\hat{f}(|x|,s) = C(s)\hat{f}(y,s)$, and $C(s)$ is a function of s. From Eq. (33), we obtain a system of two equations

$$\frac{\partial^2}{\partial y^2}\hat{f}(y,s) + \frac{3\alpha/2}{y}\frac{\partial}{\partial y}\hat{f}(y,s) + \left[-\frac{s(1+s\tau)}{\mathcal{D}_\alpha}y^{-\alpha} + \frac{(\alpha-1)\alpha}{2}\frac{1}{y^2}\right]\hat{f}(y,s) = 0, \quad (34)$$

$$-(1+s\tau) = C(s)\,\mathcal{D}_\alpha\left[\alpha y^{\alpha-1}\hat{f}(y,s) + 2y^\alpha\frac{\partial}{\partial y}\hat{f}(y,s)\right]\Big|_{y=0}. \quad (35)$$

Eq. (34) is the Lommel-type equation

$$z''(y) + \frac{1-2\beta'}{y}z'(y) + \left[\left(a\alpha' y^{\alpha'-1}\right)^2 + \frac{\beta'^2 - \nu^2\alpha'^2}{y^2}\right]z(y) = 0, \quad (36)$$

which solution is given by

$$z(y) = y^{\beta'}Z_\nu\left(ay^{\alpha'}\right),$$

where $Z_\nu(y) = C_1 J_\nu(y) + C_2 N_\nu(y)$ and $J_\nu(y)$ and $N_\nu(y)$ are the Bessel functions of first and second kind, respectively. Therefore, we have

$$\begin{aligned} \hat{f}(y,s) &= y^{\frac{2-3\alpha}{4}}Z_{\frac{1}{2}}\left(\imath\frac{2}{2-\alpha}\sqrt{\frac{s(1+s\tau)}{\mathcal{D}_\alpha}}y^{\frac{2-\alpha}{2}}\right) \\ &= y^{\frac{2-3\alpha}{4}}K_{\frac{1}{2}}\left(\frac{2}{2-\alpha}\sqrt{\frac{s(1+s\tau)}{\mathcal{D}_\alpha}}y^{\frac{2-\alpha}{2}}\right), \end{aligned} \quad (37)$$

where $Z_\nu(\imath z) = C_1 I_\nu(z) + C_2 K_\nu(z)$, and $I_\nu(z)$ and $K_\nu(y)$ are the modified Bessel functions [72], where $K_\nu(z)$ satisfies the zero boundary conditions at infinity. The PDF then reads

$$\hat{P}(x,s) = C(s)\,|x|^{\frac{2-3\alpha}{4}}K_{\frac{1}{2}}\left(\frac{2}{2-\alpha}\sqrt{\frac{s(1+s\tau)}{\mathcal{D}_\alpha}}|x|^{\frac{2-\alpha}{2}}\right). \quad (38)$$

From Eq. (35), by using the series representation of $K_\nu(y)$

$$\begin{aligned} K_\nu(z) \sim &\frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}\left[1 + \frac{z^2}{4(1-\nu)} + \cdots\right] \\ &+ \frac{\Gamma(-\nu)}{2}\left(\frac{z}{2}\right)^{\nu}\left[1 + \frac{z^2}{4(\nu+1)} + \cdots\right], \end{aligned} \quad (39)$$

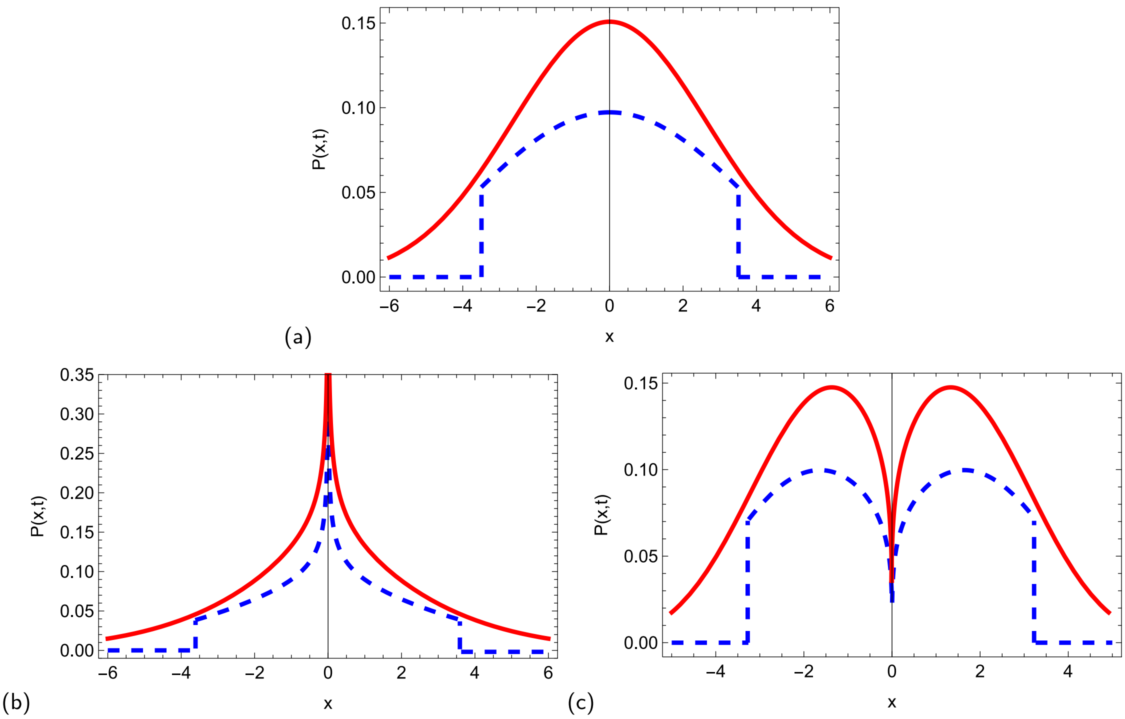

Fig. 2. PDF (29) for $\tau = 1$, $\upsilon_\alpha = 1$ (blue dashed line) versus PDF (17) (red solid line) for $\mathcal{D}_\alpha = 1$, at $t = 3.5$. (a) $\alpha = 0$ – standard telegrapher's process versus standard diffusion; (b) $\alpha = 0.5$ (long-time superdiffusion); (c) $\alpha = -0.5$ (long-time subdiffusion). The contribution of the delta function at the endpoints in the solution of the telegrapher's equation (blue dashed lines), which ensure normalization of the PDF, are not shown in the figure.

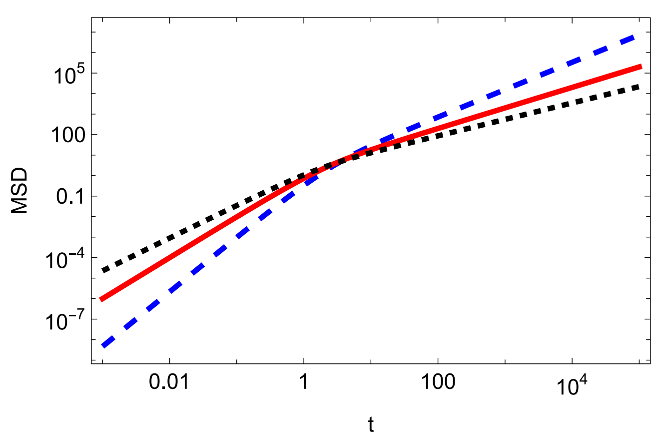

Fig. 3. MSD (43) for $\tau = 1$, $\mathcal{D}_\alpha = 1$, and $\alpha = 0.5$ (blue dashed line), $\alpha = 0$ (red solid line), $\alpha = -0.5$ (black dotted line).

for $z \to 0$ and $\nu \notin Z$, for $C(s)$ we obtain

$$C(s) = \mathcal{D}_\alpha^{-3/4} \frac{s^{-1/4}(1+s\tau)^{3/4}}{\sqrt{(2-\alpha)\pi}}. \tag{40}$$

The PDF in Laplace space then reads

$$\begin{aligned}\hat{P}(x,s) &= \mathcal{D}_\alpha^{-3/4} \frac{s^{-1/4}(1+s\tau)^{3/4}}{\sqrt{(2-\alpha)\pi}} |x|^{\frac{2-3\alpha}{4}} K_{\frac{1}{2}}\left(\frac{2}{2-\alpha}\sqrt{\frac{s(1+s\tau)}{\mathcal{D}_\alpha}}|x|^{\frac{2-\alpha}{2}}\right) \\ &= \frac{|x|^{-\alpha/2}}{2\upsilon_\alpha}\left(s+\tau^{-1}\right)^{1/2} s^{-1/2} \\ &\quad\times \exp\left(-\frac{2}{2-\alpha}\frac{s^{1/2}\left(s+\tau^{-1}\right)^{1/2}}{\upsilon_\alpha}|x|^{(2-\alpha)/2}\right),\end{aligned} \tag{41}$$

where we use $K_{\frac{1}{2}}(x) = \sqrt{\pi/(2x)}\, e^{-x}$. This is the same result as (20) for $x_0 = 0$, as it should be.

4. Calculation of the mean squared displacement

From the PDF (16), we find the MSD $\left\langle \hat{x}^2(s)\right\rangle_\tau = \int_{-\infty}^{\infty} x^2\, \hat{P}(x,s)\, dx$,

$$\left\langle \hat{x}^2(s)\right\rangle_\tau = (1+s\,\tau)\int_{-\infty}^{\infty} x^2\, \hat{p}(x, s(1+s\,\tau))\, dx = (1+s\,\tau)\left\langle \hat{x}^2(s(1+s\tau))\right\rangle_0, \tag{42}$$

where $\left\langle x^2(u)\right\rangle_0 = \frac{\Gamma(1+2p)}{p^{2p}}\frac{(\mathcal{D}_p u)^p}{\Gamma(1+p)}\,{}_1F_1\left(-p, \frac{1}{2}, -p^2\frac{|x_0|^{2/p}}{4\mathcal{D}_p u}\right)$ is the MSD for $\tau = 0$, see equation (20) in Ref. [69]. Here ${}_1F_1(a,b,z)$ is the confluent hypergeometric function of the first kind.

For $x_0 = 0$, the MSD is $\left\langle x^2(u)\right\rangle_0 = \frac{\Gamma(1+2p)}{p^{2p}}\frac{(\mathcal{D}_p u)^p}{\Gamma(1+p)}$, i.e., $\left\langle \hat{x}^2(s)\right\rangle_0 = \frac{\Gamma(1+2p)}{p^{2p}}\mathcal{D}_p u^p s^{-p-1}$, and thus, from Eq. (42) we find

$$\begin{aligned}\left\langle x^2(t)\right\rangle_\tau &= \frac{\Gamma(1+2p)}{p^{2p}}\left(\frac{\mathcal{D}_p}{\tau}\right)^p \mathcal{L}^{-1}\left[\frac{s^{-p-1}}{(s+\tau^{-1})^p}\right] \\ &= \frac{\Gamma(1+2p)\left(\mathcal{D}_p\tau\right)^p}{p^{2p}}\left(\frac{t}{\tau}\right)^{2p} E^p_{1,2p+1}\left(-\frac{t}{\tau}\right),\end{aligned} \tag{43}$$

where

$$E^\delta_{\rho,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\rho n+\beta)}\frac{z^n}{n!} \tag{44}$$

is the three-parameter Mittag–Leffler function [73], and $(\delta)_n = \Gamma(\delta+n)/\Gamma(\delta)$ is the Pochhammer symbol. To perform the inverse Laplace transform in Eq. (43) we use the formula, see Eq. (5.1.33) in Ref. [74],

$$\mathcal{L}^{-1}\left[\frac{s^{\rho\delta-\beta}}{(s^\rho+\lambda)^\delta}\right] = t^{\beta-1}E^\delta_{\rho,\beta}(-\lambda t^\rho), \tag{45}$$

with $|\lambda/s^\rho| < 1$ (note that in Eq. (43) we have $\rho \to 1$, $\delta \to p$, and $\rho\delta - \beta \to -p-1$, which means $\beta \to 2p+1$). From the definition of the Mittag–Leffler function and the known formula [75]

$$E^\delta_{\rho,\beta}(-z) = \frac{z^{-\delta}}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)}{\Gamma(\beta-\rho(\delta+n))}\frac{(-z)^{-n}}{n!}, \tag{46}$$

Table 1
Characteristic crossover regimes in finite-velocity HDPs.

	MSD – short time behavior	MSD – long time behavior
	$\langle x^2(t)\rangle \sim t^{\mu_1},\ \mu_1 = 4/(2-\alpha)$	$\langle x^2(t)\rangle \sim t^{\mu_2},\ \mu_2 = \mu_1/2$
$1<\alpha<2$	$\mu_1 > 4$ – hyperdiffusion	$\mu_2 > 2$ – hyperdiffusion
$\alpha = 1$	$\mu_1 = 4$ – hyperdiffusion	$\mu_2 = 2$ – ballistic motion
$0<\alpha<1$	$2<\mu_1<4$ – hyperdiffusion	$1<\mu_2<2$ – superdiffusion
$\alpha = 0$	$\mu_1 = 2$ – ballistic motion	$\mu_2 = 1$ – normal diffusion
$-2<\alpha<0$	$1<\mu_1<2$ – superdiffusion	$1/2<\mu_2<1$ – subdiffusion
$\alpha = -2$	$\mu_1 = 1$ – normal diffusion	$\mu_2 = 1/2$ – subdiffusion
$\alpha < -2$	$0<\mu_1<1$ – subdiffusion	$0<\mu_2<1/2$ – subdiffusion

with $z > 1$, and $0 < \rho < 2$, we find the asymptotic behavior of the MSD in the short and long time limits,

$$\left\langle x^2(t)\right\rangle_\tau \sim \frac{\Gamma(1+2p)}{p^{2p}}\left(\mathcal{D}_p\tau\right)^p \begin{cases} \frac{(t/\tau)^{2p}}{\Gamma(1+2p)}, & t/\tau \ll 1, \\ \frac{(t/\tau)^{p}}{\Gamma(1+p)}, & t/\tau \gg 1. \end{cases} \quad (47)$$

From these results we take the following conclusions: **(i)** for $0 < \alpha < 2$ we have a crossover from hyperdiffusion $\left\langle x^2(t)\right\rangle \simeq t^{\mu_1}$, $\mu_1 = \frac{4}{2-\alpha}$, $\mu_1 > 2$ to $\left\langle x^2(t)\right\rangle \simeq t^{\mu_2}$, $\mu_2 = \frac{2}{2-\alpha}$, which means: (a) either superdiffusion for $0 < \alpha < 1$, (b) ballistic motion for $\alpha = 1$, (c) or hyperdiffusion for $1 < \alpha < 2$; **(ii)** for $\alpha = 0$ we observe a crossover from ballistic motion $\left\langle x^2(t)\right\rangle \simeq t^2$, to normal diffusion $\left\langle x^2(t)\right\rangle \simeq t$; **(iii)** for $-2 < \alpha < 0$ we have a crossover from superdiffusion $\left\langle x^2(t)\right\rangle \simeq t^{\mu_1}$, $\mu_1 = \frac{4}{2-\alpha}$, $1 < \mu_1 < 2$ to subdiffusion with $\left\langle x^2(t)\right\rangle \simeq t^{\mu_2}$, $\mu_2 = \frac{2}{2-\alpha}$, $1/2 < \mu_2 < 1$; **(iv)** for $\alpha = -2$ we have a crossover from normal diffusion $\left\langle x^2(t)\right\rangle \simeq t^{\mu_1}$, $\mu_1 = \frac{4}{2-\alpha} = 1$ to subdiffusion with $\left\langle x^2(t)\right\rangle \simeq t^{\mu_2}$, $\mu_2 = \frac{2}{2-\alpha} = \frac{1}{2}$; (iv) for $\alpha < -2$ we obtain a crossover from subdiffusion $\left\langle x^2(t)\right\rangle \simeq t^{\mu_1}$, $\mu_1 = \frac{4}{2-\alpha}$, $0 < \mu_1 < 1$ to subdiffusion with $\left\langle x^2(t)\right\rangle \simeq t^{\mu_2}$, $\mu_2 = \frac{2}{2-\alpha}$, $0 < \mu_2 < 1/2$. Therefore, various diffusive crossovers are observed, rendering the considered model a suitable basis for the description of anomalous dynamics in complex systems. The obtained results are summarized in Table 1. In Fig. 3, we show the graphical representation of the MSD (43) where we observe the characteristic crossover dynamics from $\langle x^2(t)\rangle \sim t^{2p}$ to $\langle x^2(t)\rangle \sim t^p$.

5. Finite-velocity HDP with stochastic resetting

5.1. Probability density function and non-equilibrium stationary state

We now turn to the analysis of the effect of stochastic resetting on the finite-velocity HDP. We consider a Poissonian stochastic resetting mechanism [76,77] with instantaneous resetting events, whereas we leave the case of non-instantaneous resetting for future research. Thus from the simple renewal equation approach we deduce [78–81]

$$P_r(x,t) = e^{-rt}P(x,t) + \int_0^t r\,e^{-rt'}P(x,t')\,dt', \quad (48)$$

where $P(x,t)$ is the PDF (25) in absence of resetting, which means that each resetting event to the initial position x_0 renews the process at a rate r. In this equation, the first term on the right-hand side corresponds to the fraction that there is no resetting event up to time t, while the second term describes multiple resetting events up to time t. The standard telegrapher's equation, which is a special case of our model for $p = 1$ ($\alpha = 0$), in presence of stochastic resetting was analyzed in [71,82]. Numerous examples of space-dependent diffusion in soft matter systems and recent experimental advances [83] motivated new studies on inhomogeneous diffusion processes with resetting. Thus, the particular cases of heterogeneous diffusion processes with $\alpha = 1$ and stochastic Poissonian resetting were considered [84,85]. In this context, our study is a natural generalization of these very recent advances. Moreover, in view of the importance of resetting phenomena in the context of search problems, we can speculate that the heterogeneous telegrapher's equation with resetting represents a toy model of random search in a turbulent environment. We also note that run-and-tumble particle motion under stochastic resetting [82,86–88] and more generalized models of Lévy walks under resetting [89,90] are of current interest.

By Laplace transform of Eq. (48), it follows that

$$\hat{P}_r(x,s) = \frac{s+r}{s}\hat{P}(x,s+r). \quad (49)$$

Using this relation, and in combination with Eq. (11), we arrive at the relation

$$\begin{aligned}\tau\left[s^2\hat{P}_r(x,s) - s\delta(x-x_0)\right] &+ (2r\tau+1)\left[s\hat{P}_r(x,s) - \delta(x-x_0)\right] \\ &= \mathcal{D}_\alpha\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}\hat{P}_r(x,s)\right]\right\} \\ &- r(r\tau+1)\left[\hat{P}_r(x,s) - \frac{1}{s}\delta(x-x_0)\right]. \end{aligned} \quad (50)$$

By inverse Laplace transform we derive the corresponding telegrapher's equation with position-dependent diffusion coefficient in the presence of stochastic resetting,

$$\begin{aligned}\tau\frac{\partial^2}{\partial t^2}P_r(x,t) + (2r\tau+1)\frac{\partial}{\partial t}P_r(x,t) &= \mathcal{D}_\alpha\frac{\partial}{\partial x}\left\{|x|^{\frac{\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{\alpha}{2}}P_r(x,t)\right]\right\} \\ &- r(r\tau+1)\left[P_r(x,t) - \delta(x-x_0)\right].\end{aligned} \quad (51)$$

Next, we will show that in the long time limit the system reaches a non-equilibrium stationary state (NESS). From Eqs. (20) and (49), we obtain

$$\begin{aligned}\hat{P}_r(x,s) =& \frac{|x|^{1/p-1}}{2v_p}\frac{(s+r+\tau^{-1})^{1/2}(s+r)^{1/2}}{s} \\ &\times\exp\left(-\frac{p}{v_p}\left|\mathrm{sgn}(x)|x|^{1/p} - \mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right. \\ &\left.\times(s+r+\tau^{-1})^{1/2}(s+r)^{1/2}\right).\end{aligned} \quad (52)$$

From here, from the final value theorem $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\hat{f}(s)$ [91], we find the NESS

$$\begin{aligned}P_r^{st}(x) &= \lim_{t\to\infty}P_r(x,t) = \lim_{s\to 0}s\hat{P}_r(x,s) \\ &= \frac{|x|^{1/p-1}}{2v_p}\sqrt{r(r+\tau^{-1})} \\ &\times\exp\left(-\sqrt{r(r+\tau^{-1})}\frac{p}{v_p}\left|\mathrm{sgn}(x)|x|^{1/p} - \mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right).\end{aligned} \quad (53)$$

For $\tau \to 0$ (note that $v_p = \sqrt{\mathcal{D}_p/\tau} \to \infty$), we recover the result for HDPs with stochastic resetting, see equation (26) in Ref. [69],

$$p_r^{st}(x) = \frac{|x|^{1/p-1}}{2\sqrt{\mathcal{D}_p/r}}\times\exp\left(-\frac{p}{\sqrt{\mathcal{D}_p/r}}\left|\mathrm{sgn}(x)|x|^{1/p} - \mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right). \quad (54)$$

A graphical representation of the NESS is given in Fig. 4. For $\alpha = 0$ ($p = 1$) this is in fact a Laplace distribution [71,82]

$$P_r^{st}(x) = \frac{\sqrt{r(r+\tau^{-1})}}{2v}\times\exp\left(-\frac{\sqrt{r(r+\tau^{-1})}}{v}|x-x_0|\right). \quad (55)$$

In Fig. 4(a) we observe that the PDF has a cusp at $x_0 > 0$ since the resetting mechanism introduces a source of probability at x_0. For $\alpha > 0$ we observe another cusp at $x = 0$ since for small x the intensity of the multiplicative noise in the Langevin equation becomes very small such that the particle spends more time around the origin before it is reset to the initial position x_0. For $\alpha < 0$ the PDF shows an anti-cusp at $x = 0$ since for small x the intensity of the multiplicative noise becomes very large, and the particle does not spend much time near the origin. For $x_0 = 0$, see Fig. 4(b), the PDFs show a cusp only at $x = 0$.

5.2. Transition to the non-equilibrium stationary state

In order to find the relaxation dynamics to the NESS, we consider the renewal Eq. (48). We see that in the long time limit the dominant term is the integral term, which will be estimated by the Laplace approximation for large t. For the integral, we obtain (see Appendix C)

$$\int_0^t r\,e^{-rt'}P(x,t')\,dt' \sim r\frac{|x|^{1/p-1}}{4v_p\sqrt{\tau\pi}}\sqrt{t}\int_0^1\left[1+\frac{\tau_0}{\sqrt{\tau_0^2-\frac{w^2}{v_p^2}}}\right]\times\frac{e^{-t\Phi(\tau_0,w)}}{\left(\tau_0^2-\frac{w^2}{v_p^2}\right)^{1/4}}d\tau_0, \tag{56}$$

where

$$\Phi(\tau_0,w)=\left(r+\frac{1}{2\tau}\right)\tau_0-\frac{1}{2\tau}\sqrt{\tau_0^2-\frac{w^2}{v_p^2}}, \tag{57}$$

and

$$w=\frac{p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|}{t}.$$

Note that the integral in (56) is always convergent in spite of the singularity arising from the denominator in the integrand. From the Laplace approximation [92]

$$\mathcal{I}(t)\approx e^{-t\,f(z_0)}g(z_0)\sqrt{\frac{2\pi}{t|f''(z_0)|}}, \tag{58}$$

of the integral

$$\mathcal{I}(t)=\int_0^1 e^{-tf(z)}g(z)\,dz \tag{59}$$

for large t, which requires the evaluation of the minimum of the function $f(z)$, i.e., $f'(z_0)=0$, if $0<z_0<1$ (if the extremum point z_0 is outside the integration limits, $z_0>1$, then the approximation result is calculated at $z_0=1$), we find the integral (56). The extremum point can be calculated from $\frac{\partial}{\partial\tau_0}\Phi(\tau_0,w)\Big|_{\tau_0=\tau_0^*}=0$, which gives

$$\tau_0^*=\frac{2r\tau+1}{\sqrt{(2r\tau+1)^2-1}}\frac{w}{v_p}. \tag{60}$$

From here, we find that the PDF behaves as

$$P_r(x,t)\sim e^{-t\,I(w)}, \tag{61}$$

where the large deviation function (LDF) reads

$$I(w)=\begin{cases}\sqrt{r(r+\tau^{-1})}\frac{w}{v_p}, & \left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right| \le \frac{2\sqrt{r\tau(r\tau+1)}}{p(2r\tau+1)}v_pt,\\ \left(r+\frac{1}{2\tau}\right)-\frac{1}{2\tau}\sqrt{1-\frac{w^2}{v_p^2}}, & \left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right| \ge \frac{2\sqrt{r\tau(r\tau+1)}}{p(2r\tau+1)}v_pt.\end{cases} \tag{62}$$

From here we conclude that the length scale grows like $\xi(t)\sim\left(v_pt\right)^p$.

For $x_0=0$, we have ($w=p\frac{|x|^{1/p}}{t}$)

$$I(|x|/\xi(t))=\begin{cases}p\sqrt{r(r+\tau^{-1})}\frac{(|x|/t^p)^{1/p}}{v_p}, & |x|\le\left(\frac{2\sqrt{r\tau(r\tau+1)}}{p(2r\tau+1)}\right)^p v_p^pt^p,\\ \left(r+\frac{1}{2\tau}\right)-\frac{1}{2\tau}\sqrt{1-p^2\frac{(|x|/t^p)^{2/p}}{v_p^2}}, & |x|\ge\left(\frac{2\sqrt{r\tau(r\tau+1)}}{p(2r\tau+1)}\right)^p v_p^pt^p,\end{cases} \tag{63}$$

and the length scale is $\xi(t)\sim\left(v_pt\right)^p$. The trajectories corresponding to the first line of Eq. (63) are relaxed to the NESS (note that the LDF corresponds to the one of the PDF (53), as it should), while those satisfying the second line of Eq. (63) are not relaxed and are still in transient regime. The boundary between the NESS region and the transient region moves with a non-constant velocity $v(t)\sim v_p^pt^{p-1}$ (see Fig. 5). For $\tau\to 0$ the LDF reduces to the one for the HDP [69]. For the standard telegrapher's equation ($p=1$, i.e., $\alpha=0$), the LDF becomes

$$I(|x|/t)=\begin{cases}\sqrt{r(r+\tau^{-1})}\frac{|x|/t}{v}, & |x|\le\frac{2\sqrt{r\tau(r\tau+1)}}{2r\tau+1}vt,\\ \left(r+\frac{1}{2\tau}\right)-\frac{1}{2\tau}\sqrt{1-\frac{x^2/t^2}{v^2}}, & |x|\ge\frac{2\sqrt{r\tau(r\tau+1)}}{2r\tau+1}vt.\end{cases} \tag{64}$$

For $\tau\to 0$ (HDP), we arrive at the known LDF [93]

$$I(|x|/t)=\begin{cases}\sqrt{r/D}\,\frac{|x|}{t}, & |x|\le\sqrt{4Dr}\,t,\\ r+\frac{1}{4D}\left(\frac{|x|}{t}\right)^2, & |x|\ge\sqrt{4Dr}\,t.\end{cases} \tag{65}$$

In Fig. 5, we show the boundaries between the regions in which the particles have already relaxed to the NESS and the region in which the particles are in a transient regime. It is evident that the length scale depends on the parameter α.

5.3. Mean squared displacement

The MSD can be calculated from Eq. (49), yielding

$$\langle x^2(s)\rangle_r=\frac{s+r}{s}\langle\hat{x}^2(x,s+r)\rangle_\tau, \tag{66}$$

where $\langle\hat{x}^2(x,s)\rangle_\tau$ is given by Eq. (42). From here we conclude that in the short time limit ($s\to\infty$) the MSD in presence of resetting behaves analogously to the MSD in absence of resetting $\langle x^2(t)\rangle_r=\langle\hat{x}^2(x,t)\rangle_\tau$, while in the long time limit ($s\to 0$) it saturates to $\langle x^2(t)\rangle_r=r\langle\hat{x}^2(x,r)\rangle_\tau$.

For $x_0=0$, the MSD becomes

$$\left\langle x^2(t)\right\rangle_r=\frac{\Gamma(1+2p)\left(\mathcal{D}_\alpha/\tau\right)^p}{p^{2p}}\mathcal{L}^{-1}\left[\frac{s^{-1}(s+r)^{-p}}{(s+r+\tau^{-1})^p}\right]=\frac{\Gamma(1+2p)\left(\mathcal{D}_\alpha\tau\right)^p}{p^{2p}\tau}\int_0^t e^{-rt'}\left(\frac{t'}{\tau}\right)^{2p-1}E_{1,2p}^p\left(-\frac{t'}{\tau}\right)dt'. \tag{67}$$

In the short time limit it behaves as the MSD in absence of resetting, Eq. (43),

$$\left\langle x^2(t)\right\rangle_r\sim\frac{\Gamma(1+2p)\left(\mathcal{D}_\alpha\tau\right)^p}{p^{2p}\tau}\int_0^t\left(\frac{t'}{\tau}\right)^{2p-1}E_{1,2p}^p\left(-\frac{t'}{\tau}\right)dt'=\frac{\Gamma(1+2p)\left(\mathcal{D}_\alpha\tau\right)^p}{p^{2p}}\left(\frac{t}{\tau}\right)^{2p}E_{1,2p+1}^p\left(-\frac{t}{\tau}\right), \tag{68}$$

while in the long time limit the MSD saturates to $\langle x^2(t)\rangle_r\sim\frac{1}{r^p(r+\tau^{-1})^p}$, due to the resetting mechanism. A graphical representation of the MSD (67) is shown in Fig. 6, where its saturation due to the stochastic resetting is clearly observed.

6. Summary

We reported exact results for the heterogeneous telegrapher's equation. A rich range of different diffusion regimes were observed, such as a crossover from hyperdiffusion to either superdiffusion, ballistic motion, or hyperdiffusion with different exponent, from ballistic motion to normal diffusion, from superdiffusion to subdiffusion, from normal diffusion to subdiffusion, or from subdiffusion with larger exponent to subdiffusion with lower exponent. Therefore, the considered model is suitable to describe anomalous diffusion in complex systems exhibiting characteristic crossover dynamics, including finite-velocity diffusion in random media. We also analyzed the finite-velocity HDP with stochastic resetting and we showed that the system reaches a NESS. The transition to the NESS was analyzed in terms of the large deviation function. We also found the boundaries between the region in which the system relaxed to the NESS and the transient region as a function of α. Exact results for the MSD under resetting were obtained, as well. The anomalous diffusive regime saturates in the long time limit due to the resetting mechanism.

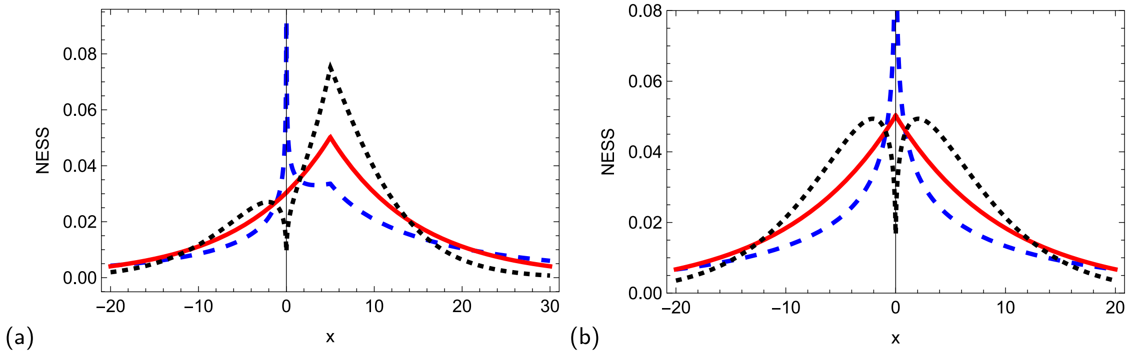

Fig. 4. NESS (53) for $\tau = 1$, $v_\alpha = 1$, $r = 0.01$ and $\alpha = 0.5$ (blue dashed line), $\alpha = 0$ (red solid line), $\alpha = -0.5$ (black dotted line). (a) $x_0 = 5$, (b) $x_0 = 0$.

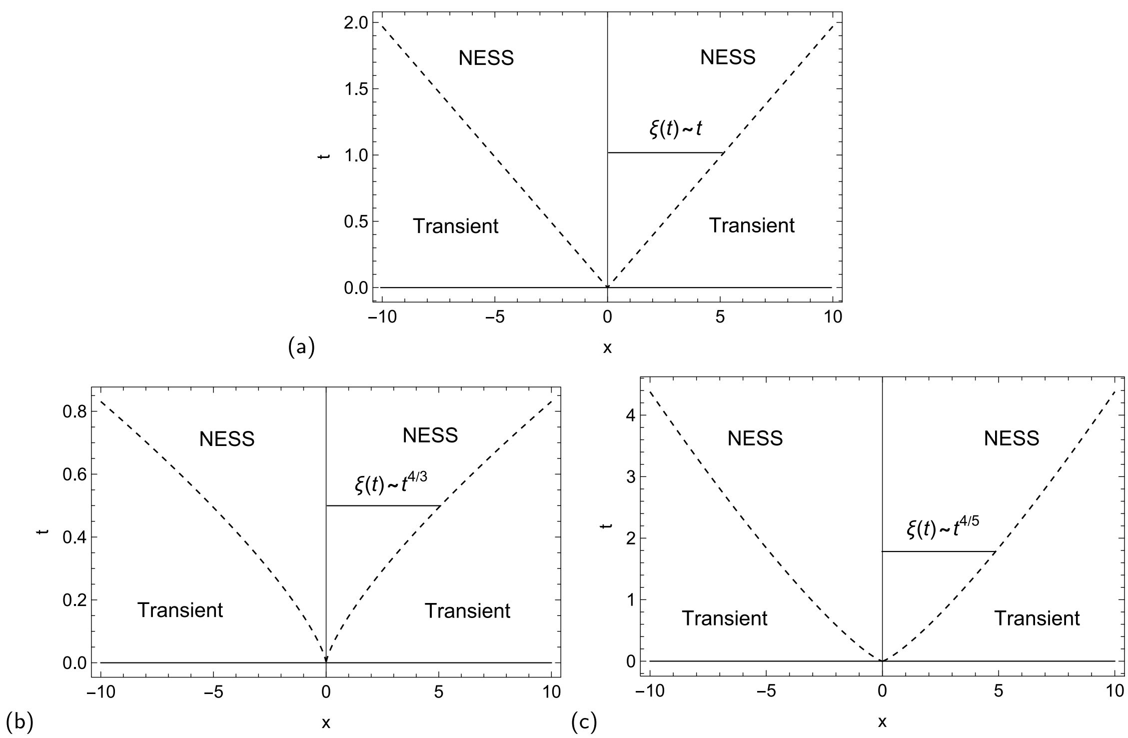

Fig. 5. Boundary between the region where the NESS is achieved and the transient region for $\tau = 1$, $v_\alpha = 1$, $r = 0.01$ and (a) $\alpha = 0$ – standard telegrapher's process, (b) $\alpha = 0.5$, (c) $\alpha = -0.5$.

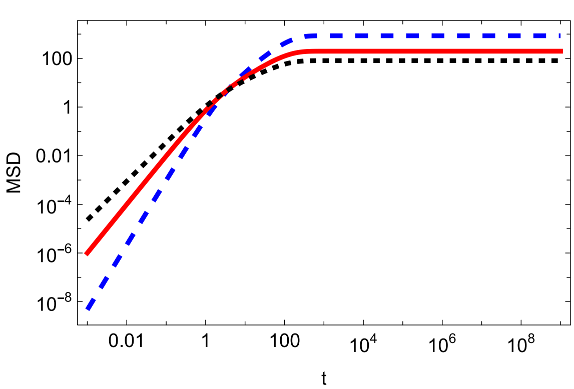

Fig. 6. MSD (67) for $\tau = 1$, $\mathcal{D}_\alpha = 1$, $r = 0.01$, and $\alpha = 0.5$ (blue dashed line), $\alpha = 0$ (red solid line), $\alpha = -0.5$ (black dotted line).

Future research could be related to the investigation of ergodic properties of finite-velocity HDPs in absence and presence of resetting [35–37,94–96], including also corresponding higher-dimensional formulations [38,39]. Infinite- and finite-velocity HDPs in presence of time-dependent resetting [97], non-instantaneous [81,98] and space–time coupled returns [99], HDPs in presence of resetting in an interval [100,101] and bounded in complex potential [102], as well as discrete space–time resetting models [103] for HDPs, are other topics worth investigating.

CRediT authorship contribution statement

Trifce Sandev: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization. **Ljupco Kocarev:** Conceptualization, Writing – original draft, Writing – review & editing, Supervision. **Ralf Metzler:** Conceptualization, Methodology, Investigation, Writing – original draft, Writing – review & editing, Supervision. **Aleksei**

Chechkin: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Trifce Sandev reports financial support was provided by Alexander von Humboldt Foundation. Trifce Sandev, Ljupco Kocarev and Ralf Metzler report financial support was provided by Deutsche Forschungsgemeinschaft. Aleksei Chechkin reports financial support was provided by the Polish National Agency for Academic Exchange (NAWA).

Data availability

No data was used for the research described in the article.

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Appendix A. Different forms of heterogeneous telegrapher's equation

Here we note that one can derive different form of the heterogeneous telegrapher's equations for a voltage and current in a lossy transmission inhomogeneous line [104], which have the form

$$\tau\frac{\partial^2}{\partial t^2}I(x,t)+\frac{\partial}{\partial t}I(x,t)=\frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}I(x,t)\right], \quad (69)$$

$$\tau\frac{\partial^2}{\partial t^2}V(x,t)+\frac{\partial}{\partial t}V(x,t)=D(x)\frac{\partial^2}{\partial x^2}V(x,t), \quad (70)$$

where $\tau = L/R =$ const, $R =$ const is the resistance, $L =$ const is the inductance, $D(x) = [RC(x)]^{-1}$, and $C(x)$ is the capacitance.

Telegrapher's equation of form (69) can also be derived from the continuity equation

$$\frac{\partial}{\partial t}n(x,t)+\frac{\partial}{\partial x}J(x,t)=0, \quad (71)$$

where $n(x,t)$ is the concentration and the flow of particles $J(x,t)$ obeys the generalized Fick's law with memory,

$$J(x,t)=\frac{1}{\tau}\int_0^t e^{(t-t')/\tau}D(x)\frac{\partial}{\partial x}n(x,t')\,dt', \quad (72)$$

where τ is a time parameter. From Eqs. (71) and (72), one arrives at the heterogeneous telegrapher's equation

$$\tau\frac{\partial^2}{\partial t^2}n(x,t)+\frac{\partial}{\partial t}n(x,t)=\frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}n(x,t)\right]. \quad (73)$$

Another form of the heterogeneous telegrapher's equation can be derived from the persistent random walk in inhomogeneous medium. It takes the form [23,50,51]

$$\frac{\partial^2}{\partial t^2}P(x,t)+\frac{1}{\tau}\frac{\partial}{\partial t}P(x,t)=v(x)\frac{\partial}{\partial x}\left[v(x)\frac{\partial}{\partial x}P(x,t)\right], \quad (74)$$

where $v(x)$ is the position-dependent velocity.

Appendix B. Normalization of the PDF

Here we provide detailed proof of the normalization condition $\int_{-\infty}^{\infty}P(x,t)\,dx=1$. From Eq. (23), we have

$$\int_{-\infty}^{\infty}P(x,t)\,dx=\mathcal{J}+\tau\frac{\partial}{\partial t}\mathcal{J}, \quad (75)$$

with

$$\mathcal{J}=\frac{e^{-\frac{t}{2\tau}}}{2v_p\tau}\int_{-\infty}^{\infty}|x|^{1/p-1}\theta\left(v_pt-p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)\times I_0\left(\frac{\sqrt{v_p^2t^2-p^2\left[\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right]^2}}{2v_p\tau}\right)dx = \mathcal{J}_1+\mathcal{J}_2, \quad (76)$$

where

$$\mathcal{J}_1=\frac{e^{-\frac{t}{2\tau}}}{2v_p\tau}\int_0^{\infty}|x|^{1/p-1}\theta\left(v_pt-p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)\times I_0\left(\frac{\sqrt{v_p^2t^2-p^2\left[\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right]^2}}{2v_p\tau}\right)dx \quad (77)$$

and

$$\mathcal{J}_2=\frac{e^{-\frac{t}{2\tau}}}{2v_p\tau}\int_{-\infty}^{0}|x|^{1/p-1}\theta\left(v_pt-p\left|\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)\times I_0\left(\frac{\sqrt{v_p^2t^2-p^2\left[\mathrm{sgn}(x)|x|^{1/p}-\mathrm{sgn}(x_0)|x_0|^{1/p}\right]^2}}{2v_p\tau}\right)dx. \quad (78)$$

For the first integral, by introducing $p\,x^{1/p}=y$, i.e., $x^{1/p-1}dx=dy$ and then $z=y-p\,\mathrm{sgn}(x_0)|x_0|^{1/p}$, i.e., $dy=dz$, we find

$$\mathcal{J}_1=\frac{e^{-\frac{t}{2\tau}}}{2v_p\tau}\int_0^{\infty}\theta\left(v_pt-\left|y-p\,\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)\times I_0\left(\frac{\sqrt{v_p^2t^2-\left[y-p\,\mathrm{sgn}(x_0)|x_0|^{1/p}\right]^2}}{2v_p\tau}\right)dy = \frac{e^{-\frac{t}{2\tau}}}{2v_p\tau}\int_0^{\infty}\theta\left(v_pt-|z|\right)I_0\left(\frac{\sqrt{v_p^2t^2-z^2}}{2v_p\tau}\right)dz. \quad (79)$$

Then we introduce new variable $\frac{\sqrt{v_p^2t^2-z^2}}{2v_p t}=r$ to obtain

$$\mathcal{J}_1=\frac{e^{-\frac{t}{2\tau}}}{2v_p\,\tau}\int_0^{\frac{t}{2\tau}}I_0(r)\frac{r\,dr}{\sqrt{\left(\frac{t}{2\tau}\right)^2-r^2}}=e^{-\frac{t}{2\tau}}\sinh\left(\frac{t}{2\tau}\right) \quad (80)$$

For the second integral, we first introduce $z=-x$ and then $p\,z^{1/p}=y$ and $k=y+p\,\mathrm{sgn}(x_0)|x_0|^{1/p}$, to find

$$\mathcal{J}_2=\frac{e^{-\frac{t}{2\tau}}}{2v_p\tau}\int_0^{\infty}\theta\left(v_pt-\left|y+p\,\mathrm{sgn}(x_0)|x_0|^{1/p}\right|\right)\times I_0\left(\frac{\sqrt{v_p^2t^2-\left[y+p\,\mathrm{sgn}(x_0)|x_0|^{1/p}\right]^2}}{2v_p\tau}\right)dy = \frac{e^{-\frac{t}{2\tau}}}{2v_p\tau}\int_0^{\infty}\theta\left(v_pt-|k|\right)I_0\left(\frac{\sqrt{v_p^2t^2-k^2}}{2v_p\tau}\right)dk=e^{-\frac{t}{2\tau}}\sinh\left(\frac{t}{2\tau}\right). \quad (81)$$

Therefore, $\mathcal{J} = 2e^{-\frac{t}{2\tau}}\sinh\left(\frac{t}{2\tau}\right) = 1 - e^{-\frac{t}{\tau}}$. Thus, we finally obtain

$$\int_{-\infty}^{\infty} P(x,t)\,dx = \left(1 - e^{-\frac{t}{\tau}}\right) + \tau\frac{\partial}{\partial t}\left(1 - e^{-\frac{t}{\tau}}\right) = 1 - e^{-\frac{t}{\tau}} + e^{-\frac{t}{\tau}} = 1, \quad (82)$$

which completes the proof.

Appendix C. Calculation of the integral in the renewal equation

Let us analyze the renewal equation

$$P_r(x,t) = e^{-rt}P(x,t) + \int_0^t r\,e^{-rt'}P(x,t')\,dt'. \quad (83)$$

For large time t the integral term is dominant, and thus

$$P_r(x,t) \sim \int_0^t r\,e^{-rt'}P(x,t')\,dt'. \quad (84)$$

We introduce

$$\xi' = \frac{\sqrt{t'^2 - \frac{X^2}{v_\alpha^2}}}{2\tau}, \quad (85)$$

where

$$X = p\left|\mathrm{sgn}(x)|x|^{1/p} - \mathrm{sgn}(x_0)|x_0|^{1/p}\right|.$$

We also use $t' = t\tau_0$ ($dt' = t\,d\tau_0$), from where it follows

$$\xi' = \frac{t}{2\tau}\sqrt{\tau_0^2 - \frac{w^2}{v_\alpha^2}}, \quad (86)$$

where $w^2 = X^2/t^2$. Since when t is large then ξ' is also large, and the Bessel function behaves as $I_\nu(\xi') \sim \frac{e^{\xi'}}{\sqrt{2\pi\xi'}}$. Thus, for the integral we obtain

$$\begin{aligned}\int_0^t r\,e^{-rt'}P(x,t')\,dt' &\sim \int_0^t r\,e^{-rt'}\frac{|x|^{1/p-1}}{4v_\alpha\tau}e^{-\frac{t'}{2\tau}}\left[I_0(\xi') + \frac{t'}{2\tau}\frac{I_1(\xi')}{\xi'}\right]dt' \\ &\sim r\frac{|x|^{1/p-1}}{4v_\alpha\tau}\int_0^t e^{-\left(r+\frac{1}{2\tau}\right)t'}\left[1 + \frac{t'}{2\tau}\frac{1}{\xi'}\right]\frac{e^{\xi'}}{\sqrt{2\pi\xi'}}dt' \\ &= r\frac{|x|^{1/p-1}}{4v_\alpha\tau}\int_0^t\left[1 + \frac{t'}{2\tau}\frac{1}{\xi'}\right]\frac{1}{\sqrt{2\pi\xi'}}e^{-\left(r+\frac{1}{2\tau}\right)t'+\xi'}dt' \\ &= r\frac{|x|^{1/p-1}}{4v_\alpha\sqrt{\pi\tau}}\int_0^t\left[1 + \frac{t'}{\sqrt{t'^2 - \frac{X^2}{v_\alpha^2}}}\right] \\ &\quad\times\frac{e^{-\left(r+\frac{1}{2\tau}\right)t'+\frac{1}{2\tau}\sqrt{t'^2-\frac{X^2}{v_\alpha^2}}}}{\left(t'^2 - \frac{X^2}{v_\alpha^2}\right)^{1/4}}dt' \\ &= r\frac{|x|^{1/p-1}}{4v_\alpha\sqrt{\tau\pi}}\int_0^1\left[1 + \frac{t\tau_0}{\sqrt{t^2\tau_0^2 - \frac{X^2}{v_\alpha^2}}}\right] \\ &\quad\times\frac{e^{-\left(r+\frac{1}{2\tau}\right)t\tau_0+\frac{t}{2\tau}\sqrt{\tau_0^2-\frac{w^2}{v_\alpha^2}}}}{\left(t^2\tau_0^2 - \frac{X^2}{v_\alpha^2}\right)^{1/4}}t\,d\tau_0 \\ &= r\frac{|x|^{1/p-1}}{4v_\alpha\sqrt{\tau\pi}}\sqrt{t}\int_0^1\left[1 + \frac{\tau_0}{\sqrt{\tau_0^2 - \frac{w^2}{v_\alpha^2}}}\right] \\ &\quad\times\frac{e^{-t\Phi(\tau_0,w)}}{\left(\tau_0^2 - \frac{w^2}{v_\alpha^2}\right)^{1/4}}d\tau_0, \end{aligned} \quad (87)$$

where

$$\Phi(\tau_0,w) = \left(r + \frac{1}{2\tau}\right)\tau_0 - \frac{1}{2\tau}\sqrt{\tau_0^2 - \frac{w^2}{v_\alpha^2}}. \quad (88)$$

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