Distributed-order diffusion equations and multifractality: Models and solutions

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We study distributed-order time fractional diffusion equations characterized by multifractal memory kernels, in contrast to the simple power-law kernel of common time fractional diffusion equations. Based on the physical approach to anomalous diffusion provided by the seminal Scher-Montroll-Weiss continuous time random walk, we analyze both natural and modified-form distributed-order time fractional diffusion equations and compare the two approaches. The mean squared displacement is obtained and its limiting behavior analyzed. We derive the connection between the Wiener process, described by the conventional Langevin equation and the dynamics encoded by the distributed-order time fractional diffusion equation in terms of a generalized subordination of time. A detailed analysis of the multifractal properties of distributed-order diffusion equations is provided.

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I. INTRODUCTION

The physical principles behind the Brownian motion of a test particle in a dilute solvent were popularized in a string of seminal papers by Albert Einstein [1], Marian von Smoluchowski [2], and Pierre Langevin [3]. They were guided by the early experiments by Robert Brown [4] and Louis Georges Gouy [5]. Brownian motion is typically characterized by the linear growth in time of the mean squared displacement (MSD) $\langle x^2(t)\rangle = 2Kt$, where K is the diffusion constant. One of the central theoretical results of this burst of activity at the beginning of the 20th century was the connection $K = k_BT/(m\eta)$ between the diffusion constant K, the mass m of the diffusing particle, the friction coefficient η of the liquid, and the thermal energy k_BT. As this Einstein-Smoluchowski-Sutherland relation [1,2,6] offered a way to experimentally determine Avogadro's constant it paved the way for the systematic single particle tracking experiments published by Jean Perrin [7].

Beginning with the work of Lewis Fry Richardson on the spreading of particles in turbulent flows [8], scientists soon realized that in many systems the MSD of diffusive processes deviates from the linear Brownian law. Instead, in a large variety of systems anomalous diffusion of the power-law form [9–12]

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)}t^\alpha \tag{1}$$

of the MSD is observed. Here, K_α of dimension cm^2/sec$^\alpha$ is the anomalous diffusion constant and α is the anomalous diffusion exponent. Depending on its size we distinguish subdiffusion for $0 < \alpha < 1$ and superdiffusion for $\alpha > 1$ [9–11]. Today, anomalous diffusion of the form (1) of submicron tracer particles or even individual molecules can be directly measured in single particle tracking assays, providing a wealth of information beyond the MSD (1). In such experiments, anomalous diffusion was observed in a variety of complex liquids [13–15] and in living biological cells [16–20], and it is found in large scale computer simulations of, for instance, biomembranes [21] or tracer diffusion in structured environments [22,23]. References [9–12] provide an overview of anomalous diffusion in both physical classical and biological systems.

Due to its nonuniversal character, the anomalous law (1) may have numerous physical origins corresponding to different stochastic models of anomalous diffusion [11]. To mention some of the most popular models, we refer to the closely related fractional Brownian motion [24] and fractional Langevin equation motion [25], which are driven by Gaussian power-law correlated noise. Anomalous diffusion of the form (1) is effected when a test particle moves in a fractal environment [12,26], being slowed down by bottlenecks and dead ends occurring at all scales. Markovian diffusion equations with time- or position-dependent diffusion coefficients equally give rise to the anomalous diffusion law (1). Finally, we mention the continuous time random walk (CTRW) [27], a direct generalization of the Pearson random walk. In a CTRW process, the test particle waits for a random time τ in between successive jumps, and/or the length x of individual jumps is equally random. Depending on the choice of the distributions of waiting times and jump lengths, different regimes of anomalous diffusion can be described, including Lévy flights [28] and Lévy walks [29]. Here we consider CTRW processes with a narrow distribution of jump lengths with finite variance, and different forms of the waiting time distribution. We note in passing that more general forms of CTRW processes exist, in which external noise [30] is superimposed onto the process or the renewal character of CTRW steps is broken in the form of correlated jump lengths or waiting times [31,32].

Consider a CTRW process with a finite variance $\langle \delta x^2\rangle$ of the jump lengths and a waiting time distribution of the power-law form $\psi(t) \simeq \tau^\alpha/t^{1+\alpha}$ with $0 < \alpha < 1$, for which the associated mean waiting time $\langle t\rangle = \int_0^\infty t\psi(t)dt$ diverges. The resulting motion is subdiffusive with MSD (1) [27]. The probability density function (PDF) $P(x,t)$, to find the diffusing particle at position x at time t, of this process is governed by the time fractional diffusion equation [10,33,34]

$$\frac{\partial}{\partial t}P(x,t) = K_{\alpha}\,{}_{\rm RL}D_t^{1-\alpha}\frac{\partial^2}{\partial x^2}P(x,t), \tag{2}$$

where the fractional operator of the Riemann-Liouville form is defined as [35]

$${}_{\rm RL}D_t^{1-\alpha}P(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{P(x,t')}{(t-t')^{1-\alpha}}dt'. \tag{3}$$

The formulation of anomalous diffusion processes in terms of fractional dynamic equations is particularly convenient to incorporate boundary value problems [36] and external force fields. In the presence of the latter the fractional diffusion equation turns to the fractional Fokker-Planck equation with a drift term containing the external force [10,37].

How can anomalous diffusion processes deviating from the power law (1) with the single scaling exponent α be described in terms of a dynamic equation? This question was answered in Refs. [38,39], in which so-called time and space fractional diffusion equations of distributed order were proposed. In these equations the order of the fractional operator is not fixed but may, in the most general case, be a continuous function. Distributed-order diffusion equations were shown to be useful tools to describe anomalous diffusion characterized by two or more scaling exponents in the MSD or even by logarithmic time dependencies of the MSD. Distributed-order diffusion equations can be represented in two different forms, referred to as natural and modified forms, which are not equivalent. Such equations and different techniques for finding corresponding solutions have been discussed in a range of works [38–45].

Here we provide an integral approach to distributed-order equations. By generalizing previous approaches we provide new solutions for various multifractal behaviors of the scaling exponents of the MSD. Throughout we emphasize the connection to the CTRW process and therefore the physical basis for the different forms of distributed-order diffusion equations. The paper is organized as follows. In Sec. II we review some results for the single exponent time fractional diffusion equation. Distributed-order diffusion equations in the natural form are considered in Sec. III and the connection to the CTRW model established. In Sec. IV, the modified-form distributed-order diffusion equation is studied. We draw our conclusions in Sec. V. In the Appendices we provide results for both forms of the distributed-order diffusion equations with three and N fractional exponents.

II. FRACTIONAL DIFFUSION EQUATION WITH SINGLE ORDER

We first note that there exists another popular way to write down the time fractional diffusion equation (2), namely, in the following form [46]:

$${}_{\rm C}D_t^{\alpha}P(x,t) = K_{\alpha}\frac{\partial^2}{\partial x^2}P(x,t), \tag{4}$$

using the Caputo fractional operator [35]

$${}_{\rm C}D_t^{\alpha}P(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-t')^{\alpha}}\frac{\partial}{\partial t'}P(x,t')dt'. \tag{5}$$

The Caputo fractional operator is formally related to the Riemann-Liouville operator through

$${}_{\rm C}D_t^{\alpha}P(x,t) = {}_{\rm RL}D_t^{\alpha}P(x,t) - P(x,0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{6}$$

for $0<\alpha<1$, where $P(x,0)=\lim_{t\to 0}P(x,t)$ is the initial condition of the problem. Here and in the following we consider solutions of the fractional diffusion equations in the infinite domain $-\infty<x<\infty$ with natural boundary conditions $P(\pm\infty,t)=0$ and with the Dirac delta initial condition $P(x,0)=\delta(x)$. In Eq. (4) the time fractional Caputo derivative stands on the left side of the diffusion equation—we call this the *natural* form, in comparison to the *modified* form of Eq. (2).

Equations (2) and (4) can be conveniently solved by Laplace,

$$P(x,s) = \int_0^{\infty} P(x,t)e^{-st}dt, \tag{7}$$

and Fourier,

$$P(k,t) = \int_{-\infty}^{\infty} P(x,t)e^{ikx}dx, \tag{8}$$

transforms. In what follows the transformed quantities are denoted by explicit dependence on the respective variables. The unique solution of the fractional diffusion equations (2) and (4) in Fourier-Laplace space is [10]

$$P(k,s) = \frac{s^{\alpha-1}}{s^{\alpha}+K_{\alpha}k^2}. \tag{9}$$

The inverse Laplace transform of expression (9) yields the characteristic function

$$P(k,t) = E_{\alpha}(-K_{\alpha}t^{\alpha}k^2), \tag{10}$$

where

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)} \tag{11}$$

is the one-parameter Mittag-Leffler function [47,48]. Inverse Fourier transform leads to the PDF in space-time,

$$\begin{aligned} P(x,t) &= \frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{\sqrt{K_{\alpha}t^{\alpha}}}\middle|\begin{matrix}(1,\alpha/2)\\(1,1)\end{matrix}\right] \\ &= \frac{1}{\sqrt{4\pi K_{\alpha}t^{\alpha}}}H_{1,2}^{2,0}\left[\frac{x^2}{4K_{\alpha}t^{\alpha}}\middle|\begin{matrix}(1-\alpha/2,\alpha)\\(0,1),(1/2,1)\end{matrix}\right], \end{aligned} \tag{12}$$

expressed in terms of the Fox H function [34] [see also Eq. (A1)]. The asymptotic expansion of the H function for $|x|/[\sqrt{K_{\alpha}t^{\alpha}}]\gg 1$ yields the stretched Gaussian behavior of the PDF [10],

$$\begin{aligned} P(x,t) \simeq{}& \frac{1}{2\sqrt{(2-\alpha)\pi}}\left(\frac{\alpha}{2}\right)^{(\alpha-1)/(2-\alpha)} \\ &\times |x|^{(\alpha-1)/(2-\alpha)}(K_{\alpha}t^{\alpha})^{-[1/2(2-\alpha)]}\exp\left[-\frac{2-\alpha}{2}\right. \\ &\left.\times\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}|x|^{2/(2-\alpha)}(K_{\alpha}t^{\alpha})^{-[1/(2-\alpha)]}\right]. \end{aligned} \tag{13}$$

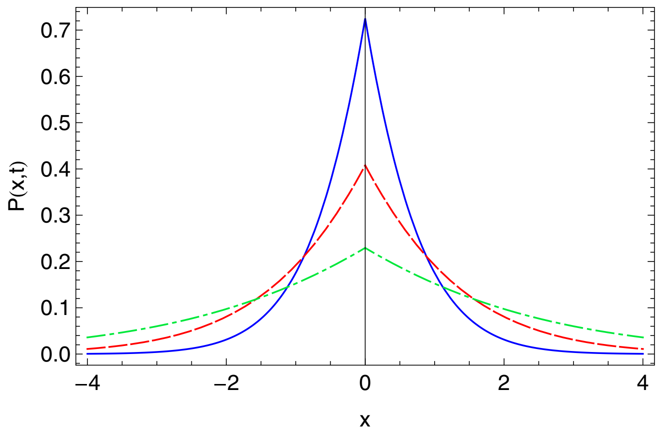

FIG. 1. (Color online) PDF (14) of the fractional diffusion equation (2) for $\alpha = 1/2$, $K_\alpha = 1$, and $t = 0.1$ (blue solid line), $t = 1$ (red dashed line), and $t = 10$ (green dot-dashed line).

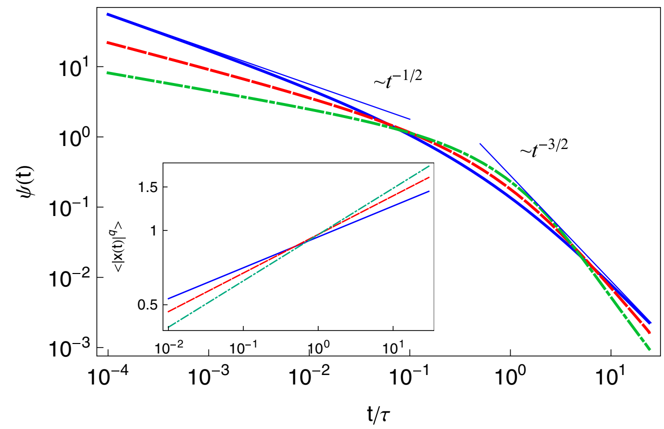

FIG. 2. (Color online) Double-logarithmic plot of the waiting-time PDF (21) for $\tau = 1$ and $\mathcal{D}_\alpha = 1$, with $\alpha = 1/2$ (blue solid line), $\alpha = 5/8$ (red dashed line), and $\alpha = 3/4$ (green dot-dashed line). Inset: qth-order moments (7) for the same parameter values and color codes.

The case $\alpha = 1$ reduces to the classical Brownian diffusion with Gaussian PDF. We note that the PDF (12) can be alternatively presented in terms of the Mainardi function $M_\alpha(y)$ [35,49],

$$P(x,t) = \frac{1}{\sqrt{4K_\alpha t^\alpha}} M_{\alpha/2}\left(\frac{|x|}{\sqrt{K_\alpha t^\alpha}}\right) \tag{14}$$

[see also Eq. (A21)].

The full solution (12) of the fractional diffusion equation (2) is plotted in Fig. 1. The cusp at the origin corresponding to the slowly decaying initial condition $P_0(x) = \delta(x)$ is distinct for this process. It arises due to the scale-free waiting time distribution $\psi(\tau)$ with its diverging characteristic time scale: an appreciable probability exists that the test particle has not moved away from its initial position even at later stages of the process (see also the discussion in terms of the CTRW process in Ref. [10]).

A. Scaling properties

The qth-order (fractional) moments of any self-affine, or monofractal, random processes $x(t)$ satisfy the scaling relation [50]

$$\langle |x(t)|^q \rangle = C(q) t^{Hq}, \tag{15}$$

where the scaling exponent H is called the Hurst exponent. As an essential feature we see that the exponent qH depends linearly on the fraction q. For instance, for ordinary Brownian motion $H = 1/2$, for fractional Brownian motion $0 < H = \alpha/2 < 1$, and for Lévy flights $H = 1/\alpha$ as long as $q < \alpha$. *Multifractal* or *multiaffine* processes $x(t)$ fulfill the generalized scaling relation [50]

$$\langle |x(t)|^q \rangle = C(q) t^{\mu(q)}, \tag{16}$$

where $\mu(q)$ is a given nonlinear function in q.

The time fractional diffusion equation (2) and its alternative form (4) constitute a monofractal process with the scaling [10,51]

$$\langle |x(t)|^q \rangle = \Gamma(q+1) \frac{(K_\alpha t^\alpha)^{q/2}}{\Gamma(1 + \alpha q/2)} = C(q) t^{\alpha q/2},$$

of the qth-order moments, where $C(q) = \Gamma(q+1) K_\alpha^{q/2} / \Gamma(1 + \alpha q/2)$. In the inset of Fig. 2 we show the qth-order moments (7) versus time on a double-logarithmic scale. Indeed the slope increases proportionally with the scaling exponent α. We note that there exists a summation rule for the even and absolute moments of the fractional diffusion equation in terms of the one-parameter Mittag-Leffler function [51]

$$\sum_{n=0}^{\infty} \frac{\langle x^{2n}(t) \rangle}{(2n)!} = E_\alpha(K_\alpha t^\alpha), \tag{17}$$

$$\sum_{n=0}^{\infty} \frac{\langle |x(t)|^n \rangle}{n!} = E_{\alpha/2}(\sqrt{K_\alpha t^\alpha}), \tag{18}$$

generalizing the corresponding relations for the regular diffusion equation with $\alpha = 1$.

B. Relation to CTRW

To establish generalization of physical laws it is favorable to have available a consistent way of introducing additional complexity. The fractional diffusion equation (2) or (4) can indeed be based on the CTRW process [10,37,52]. In the classical formulation of the CTRW the Scher-Montroll-Weiss result for the PDF in Fourier-Laplace space is given by [27]

$$P(k,s) = \frac{1 - \psi(s)}{s} \frac{1}{1 - \lambda(k)\psi(s)}, \tag{19}$$

where $\lambda(\kappa)$ is the Fourier transform of the jump length PDF $\lambda(x)$ and $\psi(s)$ denotes the Laplace transform of the waiting time PDF $\psi(t)$.

Consider first the Poissonian waiting time PDF $\psi(t) = \tau^{-1} \exp(-t/\tau)$ with the mean waiting time $\langle t \rangle = \int_0^\infty t\psi(t)dt = \tau$ and the Gaussian jump length PDF $\lambda(x) = (4\pi\sigma^2)^{-1/2} \exp(-x^2/[4\sigma^2])$, for which the variance is $\langle \delta x^2 \rangle = \int_{-\infty}^{\infty} x^2 \lambda(x) dx = 2\sigma^2$. In the diffusion limit corresponding to $s \to 0$ and $\kappa \to 0$ we find that $\psi(s) \sim 1 - s\tau$ and $\lambda(k) \sim 1 - \sigma^2 k^2$. This scaling in fact holds for any pair of waiting time and jump length PDFs with finite mean waiting time and jump length variance [53]. Thus, for the sharp initial condition $P(x,t=0)=\delta(x)$ we obtain from relation (19) the classical result [10]

$$P(k,s)=\frac{1}{s+K_1k^2}, \tag{20}$$

from where one may directly obtain the Gaussian PDF $P(x,t)=(4\pi K_1t)^{-1/2}\exp(-x^2/[4K_1t])$ with the diffusion coefficient $K_1=\sigma^2/[2\tau]$. Furthermore, if we consider the normalized power-law form $\psi(s)=(1+[s\tau]^\alpha)^{-1}$ with $0<\alpha<1$, the mean waiting time diverges. After inverse Laplace transform the waiting time PDF becomes [54–57]

$$\psi(t)=\frac{t^{\alpha-1}}{\tau^\alpha}E_{\alpha,\alpha}\left(-\left(\frac{t}{\tau}\right)^\alpha\right), \tag{21}$$

where

$$E_{\alpha,\beta}(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\alpha n+\beta)} \tag{22}$$

is the two-parameter Mittag-Leffler function [35,47]. The associated PDF in Fourier-Laplace space is then given by Eq. (9) [10,55], where $K_\alpha=\sigma^2/[2\tau^\alpha]$. The waiting time PDF (21) is shown on a double-logarithmic scale in Fig. 2. The asymptotic behavior of the waiting-time PDF follows from Eq. (21) in the form

$$\psi(t)\sim\begin{cases}\frac{1}{\tau}\frac{(t/\tau)^{\alpha-1}}{\Gamma(\alpha)}, & t\ll\tau\\ \frac{\alpha}{\tau}\frac{(t/\tau)^{-\alpha-1}}{\Gamma(1-\alpha)}, & t\gg\tau.\end{cases} \tag{23}$$

Thus, in the long-time limit the well known power-law behavior $\psi(t)\sim\tau^\alpha/t^{1+\alpha}$ emerges. Note that for $\alpha=1$ the waiting-time PDF (21) reduces to the above Poissonian form typical for normal diffusion.

C. Langevin approach and subordination

The CTRW process $x(t)$ underlying the time fractional diffusion equation can alternatively be described by the coupled pair of Langevin equations [58],

$$\frac{dx(u)}{du}=\eta(u),\qquad \frac{dt(u)}{du}=\gamma(u), \tag{24}$$

which means that the random walk $x(t)$ is parametrized in terms of the number of steps u. The connection to the real time t is given by the total $t(u)=\int_0^u\tau(u')du'$ of the individual waiting times τ for each step [58]. Here $\eta(u)$ represents white Gaussian noise with zero mean $\langle\eta(u)\rangle=0$ and correlation $\langle\eta(u)\eta(u')\rangle=2\delta(u-u')$. The term $\gamma(u)$ represents a one-sided α-stable Lévy noise with the stable index $0<\alpha<1$. In such a scheme the PDF $P(x,t)$ of the process $x(t)$ can be represented as [10,37,59,60]

$$P(x,t)=\int_0^\infty P_1(x,u)h(u,t)du, \tag{25}$$

where

$$P_1(x,u)=\frac{1}{\sqrt{4\pi uK_1\tau}}\exp\left(-\frac{x^2}{4uK_1\tau}\right) \tag{26}$$

is the PDF of the Wiener process and $h(u,t)$ is a PDF *subordinating* the random process described by the fractional diffusion equation to the Wiener process (see also Sec. V for details). The Laplace transform of this kernel reads [10,31,37,61]

$$h(u,s)=-\frac{\partial}{\partial u}\frac{1}{s}\hat{L}_\alpha(s,u)=s^{\alpha-1}e^{-us^\alpha}, \tag{27}$$

where $L_\alpha(s,u)=e^{-us^\alpha}$ is a one-sided Lévy-stable PDF. The inverse Laplace transform of Eq. (27) is given by [10,31,59,61]

$$h(u,t)=\frac{t}{\alpha u^{1+1/\alpha}}L_\alpha\left(\frac{t}{u^{1/\alpha}}\right). \tag{28}$$

III. NATURAL FORM DISTRIBUTED-ORDER DIFFUSION EQUATION

We now turn to extended forms of the fractional diffusion equation (2), which we formulate in terms of distributed-order fractional operators. The *natural form* of the distributed-order diffusion equation is hereby given by [41,62,63]

$$\int_0^1\tau^{\lambda-1}p(\lambda)\,{}_CD_t^\lambda P(x,t)d\lambda=\mathscr{K}\frac{\partial^2}{\partial x^2}P(x,t), \tag{29}$$

where $p(\lambda)$ is a non-negative normalized function, $\int_0^1p(\lambda)d\lambda=1$, τ is a parameter of dimension sec, and $\mathscr{K}$ is the diffusion coefficient of physical dimension cm^2/sec. For $p(\lambda)=\delta(\lambda-\alpha)$ relation (29) reduces to the (mono-)fractional diffusion equation in the natural form, Eq. (4) [or Eq. (2) when the Caputo operator is substituted with the Riemann-Liouville operator], and with the identification $K_\alpha=\mathscr{K}\tau^{1-\alpha}$. Using the subordination approach one can prove that the solution of Eq. (29) is a PDF, that is, a non-negative and normalizable function [38]. An alternative way to prove the non-negativity is to use the Titchmarsh theorem for the Laplace inversion, as used in Ref. [42].

The solution of Eq. (29) in the form of series and integral representations was studied in Ref. [43] for a general weight function $p(\lambda)$. Following Refs. [38,40] we here consider the generic bifractional case

$$p(\lambda)=B_1\delta(\lambda-\lambda_1)+B_2\delta(\lambda-\lambda_2), \tag{30}$$

where $0<\lambda_1<\lambda_2<1$, $B_1,B_2>0$, and $B_1+B_2=1$, in more detail.

A. Solution of the generic bifractional case

By Fourier-Laplace transform of Eq. (29) we obtain the formal solution

$$P(k,s)=\frac{1}{s}\frac{I_C(s\tau)}{I_C(s\tau)+\mathscr{K}\tau k^2} \tag{31}$$

in Fourier-Laplace space, where

$$I_C(s\tau)=\int_0^1(s\tau)^\lambda p(\lambda)d\lambda. \tag{32}$$

Using Eq. (30) we then find

$$P(k,s)=\frac{1}{s}\frac{B_2(s\tau)^{\lambda_2}+B_1(s\tau)^{\lambda_1}}{B_2(s\tau)^{\lambda_2}+B_1(s\tau)^{\lambda_1}+\mathscr{K}\tau k^2}. \tag{33}$$

Here we use the properties of the three-parameter Mittag-Leffler function to find the solution for the two fractional exponents. At first we perform an inverse Laplace transform of Eq. (33) by using the series representation approach [35,64] and Eq. (A17). We then arrive at the solution in Fourier space,

$$P(k,t)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{B_1}{B_2}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_2-\lambda_1)n}\times\left\{E^{n+1}_{\lambda_2,(\lambda_2-\lambda_1)n+1}\left(-\frac{\mathscr{K}\tau k^2}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}\right)+\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}E^{n+1}_{\lambda_2,(\lambda_2-\lambda_1)(n+1)+1}\left(-\frac{\mathscr{K}\tau k^2}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}\right)\right\}, \tag{34}$$

where $E^{\delta}_{\alpha,\beta}(z)$ is the three-parameter Mittag-Leffler function [65]. In addition to the one-parameter and two-parameter Mittag-Leffler functions, which have been widely used in different fields of science and engineering, the three-parameter Mittag-Leffler function has started to attract attention in recent years [66–71]. A summary of the properties of the three-parameter Mittag-Leffler function are provided in Appendix A.

For the inverse Fourier transform of Eq. (34) we use the relation (A20) between the three-parameter Mittag-Leffler function and the Fox H function and the Mellin cosine transform (A7). With the integral representation (A1) of the Fox H function and the duplication formula $\Gamma(z)\Gamma(z+1/2)=\Gamma(2z)\sqrt{\pi}2^{1-2z}$ [47], we find the PDF

$$P(x,t)=\frac{1}{\sqrt{4\pi[\mathscr{K}\tau/B_2](t/\tau)^{\lambda_2}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{B_1}{B_2}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_2-\lambda_1)n}\left\{H^{2,0}_{1,2}\left[\frac{x^2}{4\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}\middle|\begin{matrix}([\lambda_2-\lambda_1]n+1-\lambda_2/2,\lambda_2)\\(0,1),(n+1/2,1)\end{matrix}\right]+\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}H^{2,0}_{1,2}\left[\frac{x^2}{4\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}\middle|\begin{matrix}([\lambda_2-\lambda_1](n+1)+1-\lambda_2/2,\lambda_2)\\(0,1),(n+1/2,1)\end{matrix}\right]\right\} \tag{35}$$

in terms of an infinite series in Fox H functions. Despite their quite formal look, we now demonstrate two instructive cases showing how to deal with the Fox H functions.

At first, the normalization of expression (35) can be shown as follows. Using relation (A3) we rewrite $P(x,t)$ in the form

$$P(x,t)=\frac{1}{\sqrt{4\pi\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{B_1}{B_2}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_2-\lambda_1)n}\times\left\{\frac{1}{2}H^{2,0}_{1,2}\left[\frac{|x|}{\sqrt{4\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}}\middle|\begin{matrix}([\lambda_2-\lambda_1]n+1-\lambda_2/2,\lambda_2/2)\\(0,1/2),(n+1/2,1/2)\end{matrix}\right]+\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\frac{1}{2}H^{2,0}_{1,2}\left[\frac{|x|}{\sqrt{4\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}}\middle|\begin{matrix}([\lambda_2-\lambda_1](n+1)+1-\lambda_2/2,\lambda_2/2)\\(0,1/2),(n+1/2,1/2)\end{matrix}\right]\right\}. \tag{36}$$

By applying Eq. (A5) to (36) we obtain

$$\int_{-\infty}^{\infty}P(x,t)dx=2\int_0^{\infty}P(x,t)dx=\frac{1}{\sqrt{4\pi\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{B_1}{B_2}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_2-\lambda_1)n}\times\left\{\sqrt{4\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}\frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma([\lambda_2-\lambda_1]n+1)}+\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\sqrt{4\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}}\frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma([\lambda_2-\lambda_1](n+1)+1)}\right\}=1. \tag{37}$$

As the second case we consider the transition from Eq. (35) to the monofractional solution. Equating the fractional exponents $\lambda_1=\lambda_2=\alpha$ and using $B_2=B$ and $B_1=1-B$, Eq. (35) yields

$$P(x,t)=\frac{1}{\sqrt{4\pi\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\alpha}}}B^{-1/2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(1-\frac{1}{B}\right)^nH^{2,0}_{1,2}\left[B\frac{x^2}{4\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\alpha}}\middle|\begin{matrix}(1-\alpha/2,\alpha)\\(0,1),(n+1/2,1)\end{matrix}\right]. \tag{38}$$

With the help of Eq. (A11) and the symmetry property of the H function mentioned in Appendix A 1, Eq. (38) reduces to the solution (12) of the monofractional diffusion equation.

B. MSD and fourth-order moment

The MSD is given in Fourier space in terms of

$$\langle x^2(t)\rangle = \left\{-\frac{\partial^2}{\partial k^2}\mathscr{L}^{-1}[P(k,s)]\right\}\bigg|_{k=0} = 2\mathscr{K}\tau\mathscr{L}^{-1}\left[\frac{1}{sI_C(s\tau)}\right]. \quad (39)$$

From Laplace transform (A17) we get [38]

$$\langle x^2(t)\rangle = 2\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2} E_{\lambda_2-\lambda_1,\lambda_2+1}\left(-\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right). \quad (40)$$

This result can also be obtained by using the PDF (35) and the Mellin transform (A6) of the Fox H function.

From the series expansion (A13) of the three-parameter Mittag-Leffler function for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$ and the asymptotic expansion formula (A14) for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \gg 1$, we conclude that the MSD becomes

$$\langle x^2(t)\rangle \simeq 2\frac{\mathscr{K}\tau}{B_2}\frac{(\frac{t}{\tau})^{\lambda_2}}{\Gamma(1+\lambda_2)} \quad (41)$$

for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$, and

$$\langle x^2(t)\rangle \simeq 2\frac{\mathscr{K}\tau}{B_1}\frac{(\frac{t}{\tau})^{\lambda_1}}{\Gamma(1+\lambda_1)} \quad (42)$$

for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \gg 1$, so that the particle shows decelerating subdiffusion [38] since $\lambda_1 < \lambda_2$.

We now proceed to determine the fourth-order moment

$$\langle x^4(t)\rangle = \left\{\frac{\partial^4}{\partial k^4}\mathscr{L}^{-1}[P(k,s)]\right\}\bigg|_{k=0} = 24(\mathscr{K}\tau)^2\mathscr{L}^{-1}\left[\frac{1}{sI_C^2(s\tau)}\right], \quad (43)$$

which is a measure for the convergence of the tails of the PDF $P(x,t)$ as well as a useful diagnosis tool in the analysis of stochastic data [72]. Analogously to the derivation of the MSD for the fourth moment we find

$$\langle x^4(t)\rangle = 24\left(\frac{\mathscr{K}\tau}{B_2}\right)^2\left(\frac{t}{\tau}\right)^{2\lambda_2} \times E^2_{\lambda_2-\lambda_1,2\lambda_2+1}\left(-\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right). \quad (44)$$

In the short-time limit $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$ according to Eq. (A4) behaves as

$$\langle x^4(t)\rangle \simeq 24\left(\frac{\mathscr{K}\tau}{B_2}\right)^2\frac{(\frac{t}{\tau})^{2\lambda_2}}{\Gamma(2\lambda_2+1)}. \quad (45)$$

In the long-time limit $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \gg 1$ with Eqs. (A5) and (A7) we obtain

$$\langle x^4(t)\rangle \simeq 24\left(\frac{\mathscr{K}\tau}{B_1}\right)^2\frac{(\frac{t}{\tau})^{2\lambda_1}}{\Gamma(2\lambda_1+1)}. \quad (46)$$

The MSD (40) is shown in Fig. 3, in which the asymptotic behavior of the MSD can be clearly distinguished. In the same figure we also show the fourth moment (44) in the inset. The case of three and N different fractional exponents is considered in Appendix B.

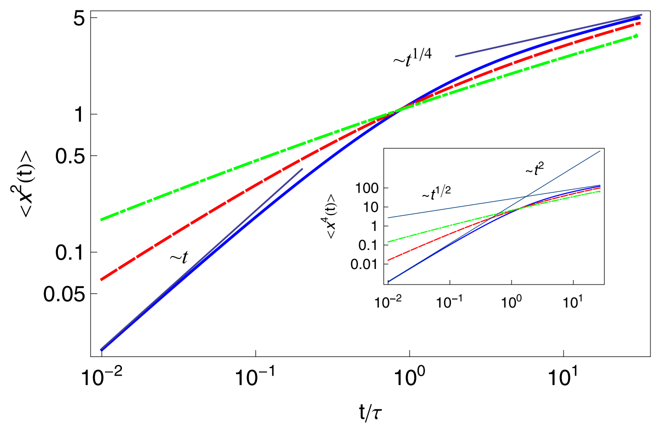

FIG. 3. (Color online) Double-logarithmic plot of the MSD (40) for $2\mathscr{K}\tau = 1$, $B_1 = B_2 = 1/2$, $\lambda_1 = 1/4$, and $\lambda_2 = 1$ (blue solid line); $\lambda_2 = 3/4$ (red dashed line); $\lambda_2 = 1/2$ (green dot-dashed line). Inset: Fourth moments (44) for the same parameters.

We note that the result (40) can also be obtained from the overdamped generalized Langevin equation with distributed-order friction memory kernel in natural form with two fractional exponents [71]. The parameters B_1 and B_2 in the distributed-order diffusion equation correspond to the weights of the power-law memory kernels decaying with the exponents λ_1 and λ_2 in the generalized Langevin equation. The PDF of the process governed by such a generalized Langevin equation is Gaussian. Thus, this is an example of two very different processes with different PDFs, which however share exactly the same MSDs.

C. Underlying CTRW model

We now establish the relation between the CTRW theory and the distributed-order diffusion equation (31) in the natural form. To this end, we use the Montroll-Weiss equation (19) for a jump length PDF with finite variance,

$$\lambda(k) = 1 - \mathscr{K}\tau k^2 \quad (47)$$

and the waiting-time PDF in Laplace space

$$\psi_C(s) = \frac{1}{1+I_C(s\tau)}, \quad (48)$$

where $I_C(s\tau)$ is given by Eq. (32). With the help of Eqs. (47), (48), and (19) we obtain exactly the same form of the PDF in Fourier-Laplace space as the one obtained from the distributed-order diffusion equation (31). We emphasize that the form (48) of the PDF is valid at all times and therefore the CTRW model corresponds exactly to the process described by the distributed-order diffusion equation in the natural form.

From relations (32) and (48) we obtain the expression

$$\psi_C(s) = \frac{1}{1+\int_0^1 \lambda(s\tau)^\lambda p(\lambda)d\lambda} \quad (49)$$

for the Laplace transform of the waiting-time PDF. Using Eq. (30) for the case of two fractional exponents we then obtain

$$\psi_C(t) = \mathscr{L}^{-1}\left[\frac{1}{1+B_1(s\tau)^{\lambda_1}+B_2(s\tau)^{\lambda_2}}\right] = \frac{1}{\tau B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{B_2^n}\left(\frac{t}{\tau}\right)^{\lambda_2 n} \times E^{n+1}_{\lambda_2-\lambda_1,\lambda_2 n+\lambda_2}\left(-\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right). \tag{50}$$

To arrive at this result, we use the series expansion approach [35] and the Laplace transform formula (A17) for the three-parameter Mittag-Leffler function. Note that if, for instance, we set $B_1 = 0$, $B_2 = 1$, and $\lambda_2 \to \alpha$ in relation (50) we arrive at Eq. (21) for the waiting-time PDF in the monofractional case. The case with $\lambda_1 = \lambda_2 = \alpha$ and $B_1 + B_2 = 1$ gives the same result for the monofractional case. The limiting behavior encoded in expression (50) yield in the form

$$\psi_C(t) \sim \frac{1}{\tau B_2}\frac{\left(\frac{t}{\tau}\right)^{\lambda_2-1}}{\Gamma(\lambda_2)} \tag{51}$$

for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$ and

$$\psi_C(t) \sim \lambda_1\frac{B_1}{\tau}\frac{\left(\frac{t}{\tau}\right)^{-\lambda_1-1}}{\Gamma(1-\lambda_1)} \tag{52}$$

for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \gg 1$. Thus the smaller exponent dominates the short-time limit and the larger one the long-time behavior. This observation is in accordance with the above conclusion from the analysis of the MSD (41) and (42) in the short- and long-time limits.

We note that qualitatively the waiting-time PDF (50) is similar to a process with two different waiting-time PDFs $\psi_1(t)$ and $\psi_2(t)$ governing the CTRW process. This may correspond to a combination of two kinds of trapping landscapes, between which the particle switches, effecting the two different waiting-time PDFs (see also [73–75]). The dominant contribution of the greater exponent in the short-time limit and of the smaller exponent in the long-time limit is observed by the numerical simulation of the MSD as shown in Fig. 4. We also note that the crossover from the short- to the long-time behavior depends on the weights B_1 and B_2, that is, on the concentration of the certain kind of traps.

D. Subordination approach

The subordination approach transforms the diffusion process from time scale t—the physical time—to the time scale u—the operational time. Rewriting the PDF in Fourier-Laplace space for the case of the natural form distributed-order diffusion equation (31) results in the form [31,38]

$$P(k,s) = \frac{I_C(s\tau)}{s}\int_0^\infty e^{-u\left(I_C(s\tau)+k^2\mathscr{K}\tau\right)}du = \int_0^\infty e^{-uk^2\mathscr{K}\tau}h(u,s)du, \tag{53}$$

where the kernel h is given by

$$h(u,s) = \frac{I_C(s\tau)}{s}e^{-uI_C(s\tau)} = -\frac{\partial}{\partial u}\frac{1}{s}L(s,u) \tag{54}$$

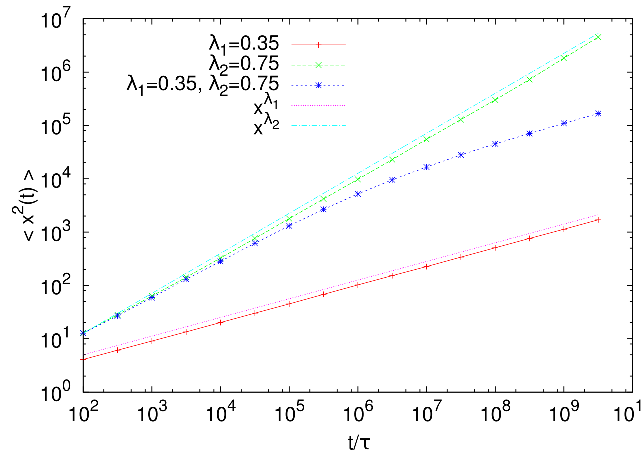

FIG. 4. (Color online) MSD of a CTRW process with mixed waiting-time PDFs (blue line). The MSD for the corresponding CTRW process with a single-power waiting-time PDF with the larger exponent and its asymptotic for the short-time limit is given by the green lines, and correspondingly with the smaller exponent and its asymptotic in the long-time limit is shown by the red lines. The fractional exponents are $\lambda_1 = 0.35$ and $\lambda_2 = 0.75$, $2\mathscr{K}\tau = 1$, and the weights are set to $B_1 = 0.01$ and $B_2 = 0.99$.

and where

$$L(s,u) = e^{-uI_C(s\tau)}. \tag{55}$$

Finally $I_C(s\tau)$ is given by Eq. (32). By inverse Fourier-Laplace transform of expression (53), we find

$$P(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi u\mathscr{K}\tau}}e^{-(x^2/4u\mathscr{K}\tau)}h(u,t)du, \tag{56}$$

which means that the function $h(u,t)$ is the PDF providing the subordination of the random process governed by the natural form to the Wiener process [38,59] by using the operational time u. With this approach we can show the non-negativity of $P(x,t)$ by proving that the function $h(u,s)$ is a completely monotonic function [38]. Note that the case of a single fractional exponent α is represented by Eqs. (27) and (28) where the function $h(u,t)$ is a one-sided Lévy stable PDF.

Plugging Eq. (32) for the bifractional case into Eq. (54) we arrive at

$$h(u,s) = -\frac{\partial}{\partial u}\frac{1}{s}e^{-u\left(B_1\tau^{\lambda_1}s^{\lambda_1}+B_2\tau^{\lambda_2}s^{\lambda_2}\right)} = -\frac{\partial}{\partial u}\frac{1}{s}L_{\lambda_1}(s,u)L_{\lambda_2}(s,u) \tag{57}$$

$$= \left(-\frac{\partial}{\partial u}\frac{1}{s}L_{\lambda_1}(s,u)\right)L_{\lambda_2}(s,u) + L_{\lambda_1}(s,u)\left(-\frac{\partial}{\partial u}\frac{1}{s}L_{\lambda_2}(s,u)\right), \tag{58}$$

where

$$L_{\lambda_i}(s,u) = e^{-uB_i\tau^{\lambda_i}s^{\lambda_i}} \tag{59}$$

for $i = 1,2$. Thus the function $h(u,t)$ is given by

$$h(u,t) = \mathscr{L}^{-1}\left[\left(-\frac{\partial}{\partial u}\frac{1}{s}L_{\lambda_1}(s,u)\right)L_{\lambda_2}(s,u) + L_{\lambda_1}(s,u)\left(-\frac{\partial}{\partial u}\frac{1}{s}L_{\lambda_2}(s,u)\right)\right]. \quad (60)$$

Via inverse Laplace transform of (60) it follows that $h(u,t)$ can be represented as a convolution of two one-sided Lévy stable PDFs since

$$\mathscr{L}^{-1}[L_{\lambda_i}(s,u)] = \frac{1}{\tau(uB_i)^{1/\lambda_i}}L_{\lambda_i}\left(\frac{t/\tau}{(uB_i)^{1/\lambda_i}}\right) \quad (61)$$

and

$$\mathscr{L}^{-1}\left[-\frac{\partial}{\partial u}\frac{1}{s}L_{\lambda_i}(s,u)\right] = \frac{B_i(t/\tau)}{\lambda_i\tau^{\lambda_i-1}(uB_i)^{1+1/\lambda_i}}L_{\lambda_i}\left(\frac{t/\tau}{(uB_i)^{1/\lambda_i}}\right), \quad (62)$$

where $i = 1,2$ and $L_\alpha(z)$ is the one-sided Lévy stable PDF (A23).

Now we turn to the Langevin description of a process with two fractional exponents. Consider the coupled Langevin equations (24) in which the noise $\gamma(u)$ is a sum of two independent one-sided stable Lévy noise sources $\gamma_i(u)$ with Lévy indices $0 < \lambda_i < 1$, for $i = 1,2$, i.e.,

$$\frac{d}{du}x(u) = \eta(u),$$
$$\frac{d}{du}t(u) = \frac{d}{du}[t_1(u) + t_2(u)] = \gamma_1(u) + \gamma_2(u). \quad (63)$$

Here $\eta(u)$ represents white Gaussian noise. The PDF $h(u,t)$ is found from relation [31,61]

$$h(u,t) = -\frac{\partial}{\partial u}\langle\Theta(t - t(u))\rangle, \quad (64)$$

where $\Theta(x)$ is the Heaviside step function. From the Laplace transform and by using that the process $t(u)$ is a sum of two independent λ_i-stable Lévy processes, for the PDF $h(u,s)$ we find

$$\begin{aligned} h(u,s) &= -\frac{\partial}{\partial u}\frac{1}{s}\left\langle\int_0^\infty \delta(t - t(u))e^{-st}dt\right\rangle \\ &= -\frac{\partial}{\partial u}\frac{1}{s}\left\langle\int_0^\infty \delta(t - [t_1(u) + t_2(u)])e^{-st}dt\right\rangle \\ &= -\frac{\partial}{\partial u}\frac{1}{s}\langle e^{-s[t_1(u)+t_2(u)]}\rangle \\ &= -\frac{\partial}{\partial u}\frac{1}{s}\langle e^{-st_1(u)}\rangle\langle e^{-st_2(u)}\rangle \\ &= -\frac{\partial}{\partial u}\frac{1}{s}L_{\lambda_1}(s,u)L_{\lambda_2}(s,u). \end{aligned} \quad (65)$$

From this result we see that it coincides with Eq. (58) obtained in the subordination approach. Generalization to a sum of N independent λ_i-stable Lévy processes $\gamma_i(u)$ with Lévy indices $0 < \lambda_i < 1$ $(i = 1,2,\ldots,N)$ is straightforward.

E. Scaling properties

As mentioned in Sec. II random processes governed by a monofractional diffusion equation belong to the class of fractal or self-affine processes. Here we consider the scaling properties of the process following the distributed-order diffusion equation in natural form.

From the PDF (31) in Fourier-Laplace space we find by inverse Fourier transform that

$$P(x,s) = \frac{1}{2s}\sqrt{\frac{I_C(s\tau)}{\mathscr{K}\tau}}\exp\left(-\sqrt{\frac{I_C(s\tau)}{\mathscr{K}\tau}}|x|\right). \quad (66)$$

The Laplace transform of the qth-order moment reads

$$\langle|x|^q(s)\rangle = \int_{-\infty}^{\infty}|x|^q P(x,s)dx. \quad (67)$$

From Eqs. (66) and (67) we arrive at

$$\langle|x|^q(s)\rangle = \Gamma(q+1)(\mathscr{K}\tau)^{q/2}\frac{1}{sI_C^{q/2}(s\tau)}. \quad (68)$$

For the generic case of two fractional exponents, we plug Eq. (32) into Eq. (68) and apply the Laplace transform formula (A17). Then the qth-order moment is expressed by the three-parameter Mittag-Leffler function as

$$\begin{aligned} \langle|x|^q(t)\rangle &= \Gamma(q+1)\left(\frac{\mathscr{K}\tau}{B_2}\right)^{q/2}\tau^{-\lambda_2 q/2} \\ &\quad\times\mathscr{L}^{-1}\left[\frac{s^{-\lambda_1 q/2-1}}{\left(s^{\lambda_2-\lambda_1} + \frac{B_1}{B_2}\tau^{-(\lambda_2-\lambda_1)}\right)^{q/2}}\right] \\ &= \Gamma(q+1)\left(\frac{\mathscr{K}\tau}{B_2}\right)^{q/2}\left(\frac{t}{\tau}\right)^{\lambda_2 q/2} \\ &\quad\times E^{q/2}_{\lambda_2-\lambda_1,\lambda_2 q/2+1}\left(-\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right). \end{aligned} \quad (69)$$

We see that the fractional moments are represented in terms of the three-parameter Mittag-Leffler function and has a more complicated form than given by Eqs. (15) and (16) for monofractal and multifractal processes, respectively. In order to analyze its behavior we find the asymptotic expansions of the three-parameter Mittag-Leffler function in the short- and long-time limits. For the short-time limit we use the first two terms of the series expansion (A13), while for the long-time limit we employ the first two terms from the asymptotic expansion (A14). Thus, the short-time limit yields

$$\begin{aligned} \langle|x|^q(t)\rangle &\sim \Gamma(q+1)\left(\frac{\mathscr{K}\tau}{B_2}\right)^{q/2}\frac{\left(\frac{t}{\tau}\right)^{\lambda_2 q/2}}{\Gamma(1+\lambda_2 q/2)} \\ &\quad\times\left[1 - \frac{q}{2}\frac{B_1}{B_2}\frac{\Gamma(\lambda_2 q/2+1)(t/\tau)^{\lambda_2-\lambda_1}}{\Gamma(\lambda_2 q/2+1+\lambda_2-\lambda_1)}\right] \end{aligned} \quad (70)$$

for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$. The long-time limit reads

$$\begin{aligned} \langle|x|^q(t)\rangle &\sim \Gamma(q+1)\left(\frac{\mathscr{K}\tau}{B_1}\right)^{q/2}\frac{\left(\frac{t}{\tau}\right)^{\lambda_1 q/2}}{\Gamma(1+\lambda_1 q/2)} \\ &\quad\times\left[1 - \frac{q}{2}\frac{B_2}{B_1}\frac{\Gamma(\lambda_1 q/2+1)(t/\tau)^{\lambda_1-\lambda_2}}{\Gamma(\lambda_1 q/2+1+\lambda_1-\lambda_2)}\right] \end{aligned} \quad (71)$$

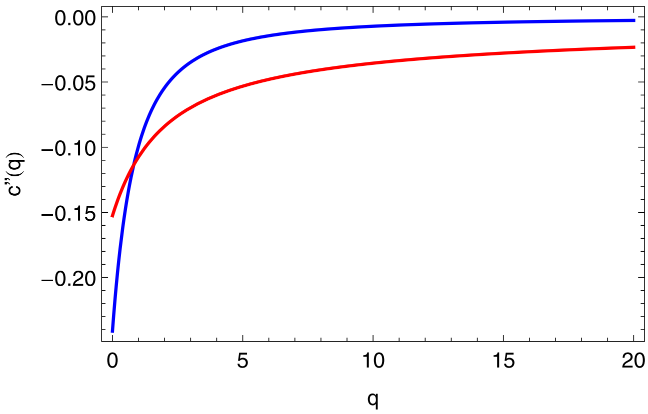

FIG. 5. (Color online) Second derivative with respect to q of $c_1(q)$ (blue line) and $c_2(q)$ (red line) for $\lambda_1 = 1/2$ and $\lambda_2 = 7/8$.

for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \gg 1$. According to the last two expressions the qth-order moments in the short- and long-time limits become

$$\langle |x|^q(t)\rangle = \langle |x|^q(t)\rangle = C(q)\left(\frac{t}{\tau}\right)^{\mu_C(q,t)}, \qquad (72)$$

where $C(q)$ are the corresponding monofractal prefactors, see expression (16) which are, of course, different at short and long times, and μ_C is time dependent. In order to obtain $\mu_C(q,t)$ we find the logarithm of Eqs. (70) and (71). In the corresponding limits of both relations we can use $\log(1-x) \simeq 1-x$ ($x \ll 1$). Thus, for $\mu_C(q,t)$ we obtain the behaviors

$$\mu_C(q,t) \sim \begin{cases} \lambda_2 q/2 + c_1(q)\frac{(t/\tau)^{\lambda_2-\lambda_1}}{\log(\tau/t)} \\ \lambda_1 q/2 + c_2(q)\frac{(t/\tau)^{\lambda_1-\lambda_2}}{\log(t/\tau)} \end{cases}, \qquad (73)$$

for $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$ and $\frac{B_1}{B_2}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \gg 1$, respectively. where

$$c_1(q) = \frac{q}{2}\frac{B_1}{B_2}\frac{\Gamma(\lambda_2 q/2+1)}{\Gamma(\lambda_2 q/2+1+\lambda_2-\lambda_1)} \qquad (74)$$

and

$$c_2(q) = -\frac{q}{2}\frac{B_2}{B_1}\frac{\Gamma(\lambda_1 q/2+1)}{\Gamma(\lambda_1 q/2+1+\lambda_1-\lambda_2)}. \qquad (75)$$

Thus, we conclude that the random processes governed by the distributed-order diffusion equation in the natural form do not belong to the class of multifractal processes whose q-th moments obey Eq. (16). Instead, it has a more general form, and in the short- and long-time limits a time dependent form; additional terms appears in the expression of the exponent μ_C. At intermediate times the qth-order moment is given by a more complicated expression involving the three-parameter Mittag-Leffler function (A13).

From relations (73), (74), and (75) we see that the sign of the second derivative of $\mu_C(q,t)$ with respect to q depends on the signs of the second derivatives of $c_1(q)$ and $c_2(q)$ with respect to q. Thus, from these signs one may conclude about the concavity or convexity of $\mu_C(q,t)$ with respect to q in the short- and long-time limits.

A graphical representation of the second derivatives of $c_1(q)$ and $c_2(q)$ is provided in Fig. 5. We see that $\mu_C(q,t)$ is a concave function with respect to q in the short- and long-time limits, since its second derivative is smaller than zero. However, it tends to zero for large values of q, as can be seen from Fig. 5 and also follows from the large argument expansion of the Γ functions entering Eqs. (74) and (75). That means that at large values of the moments the underlying process exhibits almost pure monofractality.

IV. MODIFIED-FORM DISTRIBUTED-ORDER DIFFUSION EQUATION

The distributed-order diffusion equation can be represented in the alternative, so-called modified form [40,62,63]

$$\frac{\partial}{\partial t}P(x,t) = \mathscr{K}\int_0^1 \tau^{1-\lambda}p(\lambda)_{\mathrm{RL}}D_t^{1-\lambda}\frac{\partial^2}{\partial x^2}P(x,t)d\lambda. \qquad (76)$$

Here we note that the case $p(\lambda) = \delta(\lambda-\alpha)$ leads us back to the (mono-)fractional diffusion equation (4). The PDF of the modified-form distributed-order diffusion equation is nonnegative [42]. In Ref. [43] the solution of the modified-form distributed-order diffusion equation can be given in terms of series and integral representation for a general weight function $p(\lambda)$.

A. Solution of the generic bifractional case

Analogously to the above by applying the Fourier-Laplace transform to Eq. (76) one finds that

$$P(k,s) = \frac{1}{s}\frac{I_{\mathrm{RL}}(s\tau)}{I_{\mathrm{RL}}(s\tau)+\mathscr{K}\tau\kappa^2}, \qquad (77)$$

where

$$I_{\mathrm{RL}}(s\tau) = \left[\int_0^1 d\lambda (s\tau)^{-\lambda}p(\lambda)\right]^{-1}. \qquad (78)$$

Comparing with expressions (31) and (32) we see that both forms of distributed-order diffusion equations have the same general representations of the PDF in Fourier-Laplace space with the replacement $I_C(s\tau) \to I_{\mathrm{RL}}(s\tau)$.

We again consider the case of two different fractional exponents (30). Substituting relation (30) into the general formula (77) the PDF in Fourier-Laplace space yields in the form

$$P(k,s) = \frac{s^{-1}}{1+(B_2(s\tau)^{-\lambda_2}+B_1(s\tau)^{-\lambda_1})\mathscr{K}\tau k^2}. \qquad (79)$$

With the series expansion approach [35] and the Laplace transform formula (A17) one obtains the characteristic function

$$P(k,t) = \sum_{n=0}^{\infty}(-1)^n(B_2\mathscr{K}\tau)^n k^{2n}\left(\frac{t}{\tau}\right)^{\lambda_2 n} \times E_{\lambda_1,\lambda_2 n+1}^{n+1}\left(-B_1\mathscr{K}\tau k^2\left(\frac{t}{\tau}\right)^{\lambda_1}\right). \qquad (80)$$

Additional analysis shows that in the short-time limit the smaller exponent has the dominant contribution and in the long-time limit the greater exponent is dominant. The Mellin-cosine transform formula (A7) yields the PDF in real space in terms of an infinite series of Fox H functions [44]

$$P(x,t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(B_2\mathscr{K}\tau)^n\left(\frac{t}{\tau}\right)^{\lambda_2 n}\frac{1}{|x|^{2n+1}}H_{3,3}^{2,1}\left[\frac{|x|^2}{B_1\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\lambda_1}}\middle|\begin{matrix}(1,1),(\lambda_2 n+1,\lambda_1),(n+1,1)\\(2n+1,2),(n+1,1),(n+1,1)\end{matrix}\right]. \tag{81}$$

With the definition (A1) of the Fox H function one can show [44] that this result is equivalent to that of Ref. [64]:

$$P(x,t)=\frac{1}{\sqrt{4\pi B_1\mathscr{K}\tau(t/\tau)^{\lambda_1}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{B_2}{B_1}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_2-\lambda_1)n}$$
$$\times H_{2,3}^{2,1}\left[\frac{|x|^2}{4B_1\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\lambda_1}}\middle|\begin{matrix}(1/2-n,1),([\lambda_2-\lambda_1]n-\lambda_1/2+1,\lambda_1)\\(0,1),(1/2,1),(1/2,1)\end{matrix}\right]. \tag{82}$$

When the fractional exponents are equal, $\lambda_1=\lambda_2=\alpha$ and $B_1+B_2=1$, we recover the solution (12) for the monofractional diffusion equation [44] [compare relation (A12)]. By using relation (A4) we can show that the PDF (82) is normalized to 1.

B. MSD and fourth-order moment

Since the PDFs for the natural and modified-form distributed-order diffusion equations have the same general representation, the MSD in the present case according to Eq. (39) becomes

$$\langle x^2(t)\rangle=2\mathscr{K}\tau\mathscr{L}^{-1}\left[\frac{1}{sI_{\mathrm{RL}}(s\tau)}\right]. \tag{83}$$

By inverse Laplace transform for two exponents we find the MSD [40]

$$\langle x^2(t)\rangle=2B_1\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\lambda_1}\times E_{\lambda_2-\lambda_1,\lambda_1+1}^{-1}\left(-\frac{B_2}{B_1}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right)$$
$$=2B_1\mathscr{K}\tau\frac{\left(\frac{t}{\tau}\right)^{\lambda_1}}{\Gamma(1+\lambda_1)}+2B_2\mathscr{K}\tau\frac{\left(\frac{t}{\tau}\right)^{\lambda_2}}{\Gamma(1+\lambda_2)}. \tag{84}$$

In comparison to the MSD (40) for the natural form distributed-order diffusion equation we see that there is a very similar representation in expression (84) for the modified form in terms of the Mittag-Leffler function. Note that the third parameter of the Mittag-Leffler function in Eq. (84) is equal to -1, so if one uses the series representation (A13) of the three-parameter Mittag-Leffler function only the first two terms from the series are different from zero. This is the case of accelerating subdiffusion [40], which means that at short times the process shows anomalous subdiffusion with a diffusion exponent λ_1 and at long times subdiffusion with an exponent λ_2 that fulfills $1>\lambda_2>\lambda_1>0$. The same result for the MSD can be obtained if we use the PDF (82) and the Mellin transform (A6) of the Fox H function. For a weight function of the form $p(\lambda)=B_1\delta(\lambda-\alpha)+B_2\delta(\lambda-1)$ one obtains accelerating diffusion—from subdiffusive behavior in the short-time limit to normal diffusive behavior in the long-time limit.

The fourth-order moment

$$\langle x^4(t)\rangle=24(\mathscr{K}\tau)^2\mathscr{L}^{-1}\left[\frac{1}{sI_{\mathrm{RL}}^2(s\tau)}\right] \tag{85}$$

for the modified-form distributed-order diffusion equation is given by

$$\langle x^4(t)\rangle=24(B_1\mathscr{K}\tau)^2\left(\frac{t}{\tau}\right)^{2\lambda_1}\times E_{\lambda_2-\lambda_1,2\lambda_1+1}^{-2}\left(-\frac{B_2}{B_1}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right)$$
$$=24(\mathscr{K}\tau)^2\left[B_1^2\frac{\left(\frac{t}{\tau}\right)^{2\lambda_1}}{\Gamma(2\lambda_1+1)}+2B_1B_2\frac{\left(\frac{t}{\tau}\right)^{\lambda_1+\lambda_2}}{\Gamma(\lambda_1+\lambda_2+1)}+B_2^2\frac{\left(\frac{t}{\tau}\right)^{2\lambda_2}}{\Gamma(2\lambda_2+1)}\right], \tag{86}$$

from which the short-time limit $\frac{B_2}{B_1}(\frac{t}{\tau})^{\lambda_2-\lambda_1}\ll 1$ follows as

$$\langle x^4(t)\rangle\sim 24(B_1\mathscr{K}\tau)^2\frac{\left(\frac{t}{\tau}\right)^{2\lambda_1}}{\Gamma(2\lambda_1+1)}. \tag{87}$$

The long-time limit $\frac{B_2}{B_1}(\frac{t}{\tau})^{\lambda_2-\lambda_1}\gg 1$ is

$$\langle x^4(t)\rangle\sim 24(B_2\mathscr{K}\tau)^2\frac{\left(\frac{t}{\tau}\right)^{2\lambda_2}}{\Gamma(2\lambda_2+1)}. \tag{88}$$

The case of the modified-form distributed-order diffusion equation with three and N fractional exponents is given in Appendix B.

A graphical representation of the MSD (84) is shown in Fig. 6 in which one can see the asymptotic behaviors of the MSD. The fourth-order moment (86) is presented as well.

C. CTRW model

We now consider the CTRW model (19) related to the modified-form distributed-order diffusion equation (77) corresponding to a finite variance of the jump length distribution. According to the above, the underlying waiting-time PDF in Laplace space form becomes

$$\psi_{\mathrm{RL}}(s)=\frac{1}{1+I_{\mathrm{RL}}(s\tau)}. \tag{89}$$

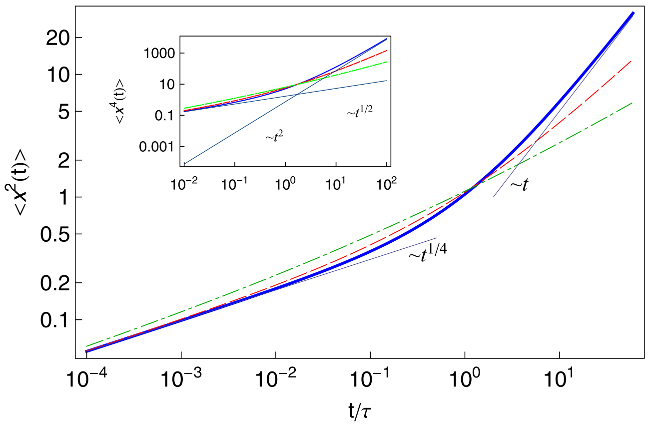

FIG. 6. (Color online) Double-logarithmic representation of the MSD (84) for $\mathscr{K}\tau = 1/2$, $B_1 = B_2 = 1/2$, $\lambda_1 = 1/4$, and $\lambda_2 = 1$ (blue solid line); $\lambda_2 = 3/4$ (red dashed line); and $\lambda_2 = 1/2$ (green dot-dashed line). Inset: Fourth-order moment (86) for the same values of parameters.

Here $I_{\rm RL}(s\tau)$ is given by Eq. (78). In Fourier-Laplace space the PDF is then the solution of the modified-form distributed-order diffusion equation. Using relations (78) and (89) we obtain the explicit form of the waiting-time PDF in Laplace space,

$$\psi_{\rm RL}(s) = \frac{1}{1+\left[\int_0^1 (s\tau)^{-\lambda} p(\lambda)\right]^{-1} d\lambda}. \tag{90}$$

For the case of two fractional exponents, from Eq. (A17) of the three-parameter Mittag-Leffler function we find that

$$\begin{aligned}\psi_{\rm RL}(t) &= \mathscr{L}^{-1}\left[\frac{1}{1+\frac{1}{B_1(s\tau)^{-\lambda_1}+B_2(s\tau)^{-\lambda_2}}}\right]\\ &= \frac{B_1}{\tau}\left(\frac{t}{\tau}\right)^{\lambda_1-1}\sum_{n=0}^{\infty}(-1)^n B_1^n\left(\frac{t}{\tau}\right)^{\lambda_1 n}\\ &\quad\times E^{-(n+1)}_{\lambda_2-\lambda_1,\lambda_1 n+\lambda_1}\left(-\frac{B_2}{B_1}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right).\end{aligned} \tag{91}$$

The waiting-time PDF (91) has a very similar representation as that for the natural form, Eq. (50). The third parameter of the Mittag-Leffler function in expression (91) is equal to $-(n+1)$ allowing us to represent $\psi_{\rm RL}(t)$ in terms of a binomial expansion and using the binomial coefficients. Again, for $B_1 = 0$, $B_2 = 1$, and $\lambda_2 \to \alpha$ by help of relation (91) we obtain the result (21) for the waiting-time PDF for a single fractional exponent. The same result follows if we use $\lambda_1 = \lambda_2 = \alpha$ and $B_1 + B_2 = 1$ in Eq. (91).

The limiting cases encoded in Eq. (91) are given in terms of

$$\psi_{\rm RL}(t) \sim \frac{B_1}{\tau}\frac{\left(\frac{t}{\tau}\right)^{\lambda_1-1}}{\Gamma(\lambda_1)} \tag{92}$$

for $\frac{B_2}{B_1}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1} \ll 1$ and

$$\psi_{\rm RL}(t) \sim \lambda_2\frac{1}{B_2\tau}\frac{\left(\frac{t}{\tau}\right)^{-\lambda_2-1}}{\Gamma(1-\lambda_2)} \tag{93}$$

for $\frac{B_2}{B_1}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1} \gg 1$. Accordingly the smaller exponent dominates the short-time behavior and the larger exponent the long-time asymptote.

What is the physical meaning of the modified-form distributed-order diffusion equation? The characteristic function of the random process governed by the monofractional diffusion equation is represented in terms of the Mittag-Leffler function (10). Thus, if we consider a process $x(t)$ which is the sum of two independent CTRW processes $x_1(t)$ and $x_2(t)$ characterized by fractional exponents λ_1 and λ_2, respectively, for the characteristic function $P(k,t) = \langle \exp[ikx(t)]\rangle$ it follows that

$$\begin{aligned}P(k,t) &= \langle \exp\{ik[x_1(t)+x_2(t)]\}\rangle\\ &= \langle \exp[ikx_1(t)]\rangle\langle \exp[ikx_2(t)]\rangle\\ &= E_{\lambda_1}\left(-\mathscr{K}_1\tau k^2\left(\frac{t}{\tau}\right)^{\lambda_1}\right)E_{\lambda_2}\left(-\mathscr{K}_2\tau k^2\left(\frac{t}{\tau}\right)^{\lambda_2}\right).\end{aligned} \tag{94}$$

To lowest order in the wave mode k we have

$$P(k,t) \sim 1 - k^2\left[\mathscr{K}_1\tau\frac{\left(\frac{t}{\tau}\right)^{\lambda_1}}{\Gamma(\lambda_1+1)} + \mathscr{K}_2\tau\frac{\left(\frac{t}{\tau}\right)^{\lambda_2}}{\Gamma(\lambda_2+1)}\right] \tag{95}$$

from which we find that after Laplace transform

$$P(k,s) \sim \frac{1}{s} - k^2[\mathscr{K}_1\tau(s\tau)^{-\lambda_1-1} + \mathscr{K}_2\tau(s\tau)^{-\lambda_2-1}]. \tag{96}$$

If we now go back to the solution (79) of the bifractional case and consider a similar expansion there, we again arrive at (96) with $\mathscr{K}_i = \mathscr{K} B_i$ for $i = 1,2$. Thus we see that at large distances—corresponding to small k—the process governed by the bifractional diffusion equation in the modified form can be viewed as a composition of two independent CTRW processes.

Analogously it can be shown that the process governed by the modified-form distributed-order diffusion equation with N delta functions can be viewed as the limiting form following a sum of N independent CTRW processes x_i with $i = 1,2,\dots,N$ with scaling exponents λ_i. Following the terminology of Cox and Smith [74] this would correspond to the *pooling* of CTRWs.

A graphical representation of the sum of two CTRWs for different values of the parameters and at different times is given in Figs. 7 and 8. The CTRW processes are independent and start at the origin $x = 0$ at $t = 0$. As can be seen from Fig. 7 for short times the smaller exponent is dominant while in the long-time limit the CTRW with the larger exponent takes over. The same domination is seen from the MSD interpolating between a power-law asymptotic with slope λ_1 at short times and a power law with exponent λ_2 in the long-time limit. Figure 8(a) shows the PDFs of the composite process on a semilogarithmic scale and Fig. 8(b) on a double-logarithmic scale calculated at different times. At short times the composite PDF is close to the PDF of the CTRW with λ_1, while in the long-time limit the transition occurs to the PDF with λ_2.

D. Subordination approach

The same relations as for the natural form distributed-order diffusion equation [Eqs. (53)–(55)] are valid for the

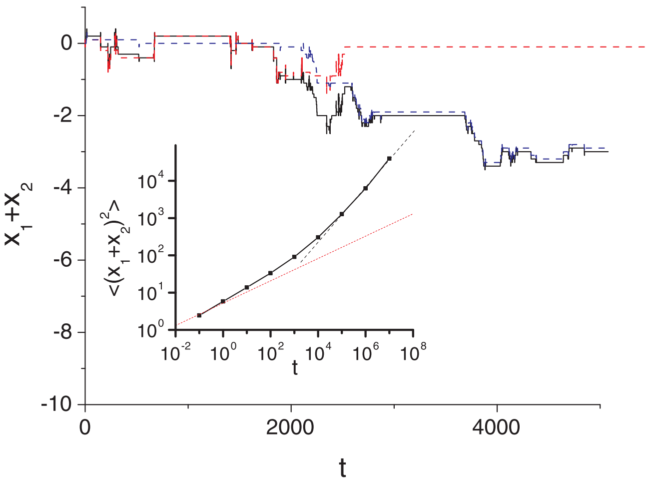

FIG. 7. (Color online) Sample trajectories of two CTRWs x_1 (red), x_2 (blue), and the composite process (black) which is the sum of x_1 and x_2. Parameters are $\lambda_1 = 0.3$, $K_{\lambda_1} = 3.06$, $\lambda_2 = 0.75$, $K_{\lambda_2} = 0.138$ [$B_1 = K_{\lambda_1}/(K_{\lambda_1} + K_{\lambda_2})$ and $B_2 = K_{\lambda_2}/(K_{\lambda_1} + K_{\lambda_2})$]. Inset: MSD of the composite process $x_1 + x_2$ (solid line) with $\lambda_1 = 0.3$, $K_{\lambda_1} = 3.06$ and $\lambda_2 = 0.75$, $K_{\lambda_2} = 0.08$. Dashed lines represent the two limits t^{λ_1} and t^{λ_2}.

modified-form distributed-order diffusion equation, if instead of $I_{\rm C}(s\tau)$ we use $I_{\rm RL}(s\tau)$ from Eq. (78). Thus, for the PDF $h(u,s)$ in double-Laplace space for the case of two fractional exponents one finds

$$h(u,s) = -\frac{\partial}{\partial u}\frac{1}{s}e^{-u\left(B_1\tau^{-\lambda_1}s^{-\lambda_1}+B_2\tau^{-\lambda_2}s^{-\lambda_2}\right)^{-1}}. \qquad (97)$$

The subordination representation for this process governed by the modified-form distributed-order diffusion equation with N fractional exponents (B7) was given in Ref. [75], where the parent process is Brownian motion, that is, the same as in the monofractal case, while the subordinator is a more general Lévy process. Thus, the PDF $P(x,t)$ has the form given by Eqs. (25) and (26), where instead of Eq. (27) we get $h(u,s) = -(\partial/\partial u)[\exp\{-u I_{\rm RL}(s\tau)\}/s]$ with $I_{\rm RL}(s\tau) = (\sum_{i=1}^{N} B_i\tau^{-\lambda_i}s^{-\lambda_i})^{-1}$, proven to be a Bernstein function [75].

E. Scaling properties

For the modified-form distributed-order diffusion equation we use the same relations (68) and (72) for the qth-order moment as for the natural form, simply changing $I_{\rm C}(s\tau)$ for $I_{\rm RL}(s\tau)$. From substitution of relation (78) for the bifractional case in Eq. (72) we find

$$\begin{aligned}\langle|x|^q(t)\rangle &= \Gamma(q+1)(B_1\mathscr{K}\tau)^{q/2}\tau^{-\lambda_1 q/2}\\ &\quad\times\mathscr{L}^{-1}\left[\frac{s^{-\lambda_2 q/2-1}}{\left(s^{\lambda_2-\lambda_1}+\frac{B_2}{B_1}\tau^{-(\lambda_2-\lambda_1)}\right)^{-q/2}}\right]\\ &= \Gamma(q+1)(B_1\mathscr{K}\tau)^{q/2}\left(\frac{t}{\tau}\right)^{\lambda_1 q/2}\\ &\quad\times E_{\lambda_2-\lambda_1,\lambda_1 q/2+1}^{-q/2}\left(-\frac{B_2}{B_1}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right), \end{aligned} \qquad (98)$$

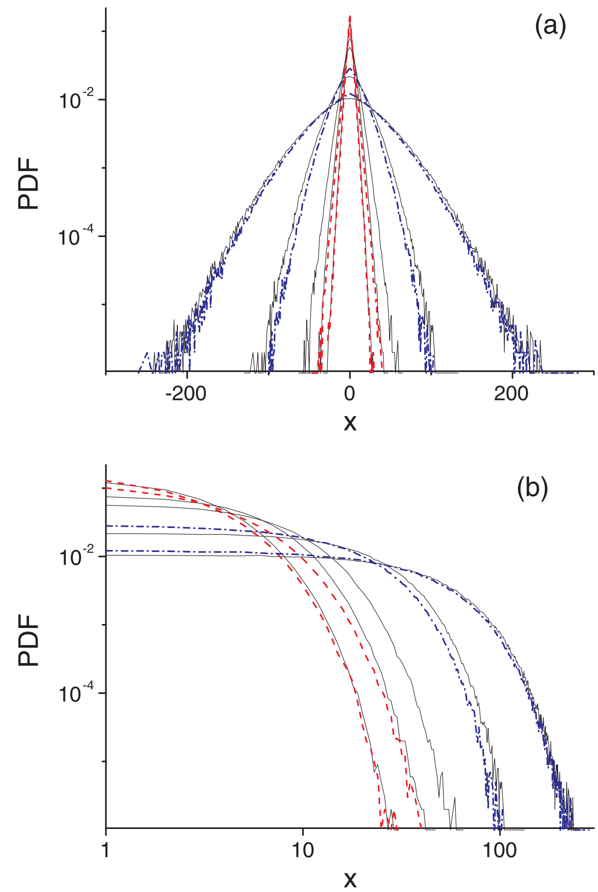

FIG. 8. (Color online) PDF of the sum of two CTRWs with $\lambda_1 = 0.3$, $K_{\lambda_1} = 3.06$ and $\lambda_2 = 0.75$, $K_{\lambda_2} = 0.138$ calculated for $t = 10$, $t = 10^2$, $t = 10^3$, $t = 10^4$, and $t = 10^5$. Dashed lines represent the PDF of the single CTRW process with $\lambda_1 = 0.3$, $K_{\lambda_1} = 3.06$ calculated for $t = 10$, $t = 10^2$; dashed-dotted lines represent the PDF of the single CTRW with $\lambda_2 = 0.75$, $K_{\lambda_2} = 0.138$ calculated for $t = 10^4$ and $t = 10^5$ [$B_1 = K_{\lambda_1}/(K_{\lambda_1} + K_{\lambda_2})$, $B_2 = K_{\lambda_2}/(K_{\lambda_1} + K_{\lambda_2})$]. (b) Same as in (a) in double-logarithmic scale.

by virtue of the Laplace transform formula (A17) for the three-parameter Mittag-Leffler function. This relation is very similar to expression (45) for the natural form distributed-order diffusion equation. The limiting behaviors become

$$\begin{aligned}\langle|x|^q(t)\rangle &\sim \Gamma(q+1)(B_1\mathscr{K}\tau)^{q/2}\frac{\left(\frac{t}{\tau}\right)^{\lambda_1 q/2}}{\Gamma(1+\lambda_1 q/2)}\\ &\quad\times\left[1+\frac{q}{2}\frac{B_2}{B_1}\frac{\Gamma(\lambda_1 q/2+1)(t/\tau)^{\lambda_2-\lambda_1}}{\Gamma(\lambda_1 q/2+1+\lambda_2-\lambda_1)}\right] \end{aligned} \qquad (99)$$

for $\frac{B_2}{B_1}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$ and

$$\begin{aligned}\langle|x|^q(t)\rangle &\sim \Gamma(q+1)(B_1\mathscr{K}\tau)^{q/2}\frac{\left(\frac{t}{\tau}\right)^{\lambda_2 q/2}}{\Gamma(1+\lambda_2 q/2)}\\ &\quad\times\left[1+\frac{q}{2}\frac{B_1}{B_2}\frac{\Gamma(\lambda_2 q/2+1)(t/\tau)^{\lambda_1-\lambda_2}}{\Gamma(\lambda_2 q/2+1+\lambda_1-\lambda_2)}\right] \end{aligned} \qquad (100)$$

for $\frac{B_2}{B_1}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \gg 1$. Thus, we see that similar to the natural form, in the limit cases the qth-order moments can be represented in the form of Eq. (72). Following the same

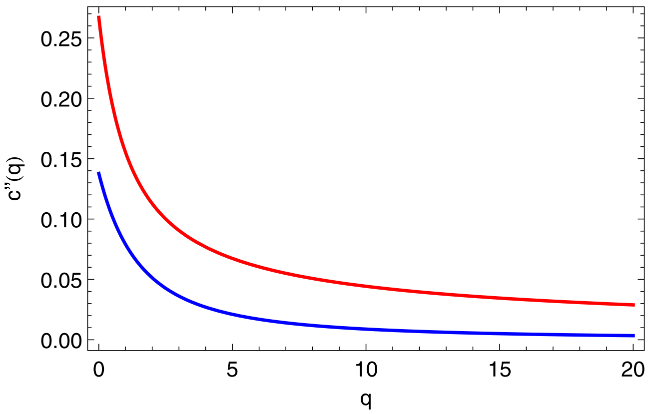

FIG. 9. (Color online) Second derivative with respect to q of $c_3(q)$ (blue line) and $c_4(q)$ (red line), for $\lambda_1 = 1/2$ and $\lambda_2 = 7/8$.

procedure as in Sec. III E for $\mu_{\mathrm{RL}}(q,t)$ we find that

$$\mu_{\mathrm{RL}}(q,t) \sim \begin{cases} \lambda_1 q/2 + c_3(q)\frac{(t/\tau)^{\lambda_2-\lambda_1}}{\log(\tau/t)} \\ \lambda_2 q/2 + c_4(q)\frac{(t/\tau)^{\lambda_1-\lambda_2}}{\log(t/\tau)} \end{cases} \tag{101}$$

for $\frac{B_2}{B_1}(\frac{t}{\tau})^{\lambda_2-\lambda_1} \ll 1$ and $\gg 1$, respectively. Here

$$c_3(q) = -\frac{q}{2}\frac{B_2}{B_1}\frac{\Gamma(\lambda_1 q/2+1)}{\Gamma(\lambda_1 q/2+1+\lambda_2-\lambda_1)}, \tag{102}$$

as well as

$$c_4(q) = \frac{q}{2}\frac{B_1}{B_2}\frac{\Gamma(\lambda_2 q/2+1)}{\Gamma(\lambda_2 q/2+1+\lambda_1-\lambda_2)}, \tag{103}$$

whose form is quite similar to those of the natural form [compare with Eqs. (73)–(75)].

The difference with the natural form is seen from a comparison of Fig. 5 with the graphical representation of the second derivatives of $c_3(q)$ and $c_4(q)$ with respect to q in Fig. 9. The signs of $c_3''(q)$ and $c_4''(q)$ are the same as the sign of the second derivative of $\mu_{\mathrm{RL}}(q,t)$ with respect to q for fixed time t. From the figure we find that $\mu_{\mathrm{RL}}(q,t)$ is concave, in contrast to the natural form for which it is a convex function. However, similarly to the natural form, the second derivative of $\mu_{\mathrm{RL}}(q,t)$ with respect to q tends to zero for large values of q, which implies that as in the natural case, the underlying process becomes more monofractional with increasing order of the moment.

V. DISCUSSION

Conventional CTRW processes with scale-free waiting-time distributions characterized by a single scaling exponent are equivalent to the fractional diffusion equation and can alternatively be expressed through the subordination approach in the limit of long times or small Laplace variables. For more general processes leading to distributed-order diffusion equations, two different physical scenarios arise, leading to the natural and modified forms of the resulting dynamic equations. For the bifractional case considered here the natural form distributed-order diffusion equation is equivalent to a waiting-time distribution, which features different scaling exponents λ_1 and λ_2 for short and long waiting times. Conversely, a process $x(t)$ representing the sum of two independent CTRW processes $x_1(t)$ and $x_2(t)$ with fractional exponents λ_1 and λ_2 leads to the modified form.

We here presented a comparative study of natural and modified forms of the distributed-order diffusion equation underlining the relevance of the three-parameter Mittag-Leffler function in the analysis of the PDFs, MSDs, waiting-time PDFs, and qth moments for natural and modified distributed-order diffusion equations. We provided analytical results for the PDF and MSD in terms of this three-parameter Mittag-Leffler function and the Fox H function for the case of distributed-order diffusion equations with two and three fractional exponents. We also discussed that by using the same approach and the properties of three parameter and multinomial Mittag-Leffler functions one may analyze distributed-order diffusion equations with N fractional exponents. We analyzed the short- and long-time limits and showed that the considered models are useful to describe subdiffusion processes which as function of time are accelerating and decelerating in terms of the anomalous diffusion exponent. New results were provided for the CTRW model corresponding to the distributed-order diffusion equations in both natural and modified forms, and the corresponding waiting-time PDFs in terms of an infinite series in three-parameter Mittag-Leffler functions were obtained. Connections to previously considered CTRW models were drawn.

The subordination scheme for processes described by distributed-order diffusion equations was analyzed in detail. This technique is important in the construction of solutions from the Brownian case, in particular, for numerical approaches. Analytical results for the subordinators for natural and modified-form distributed-order diffusion equations with respect to the Wiener process were presented. The Langevin picture of the natural form distributed-order diffusion equation with two fractional exponents and the stochastic representation of the processes governed by the modified-form distributed-order diffusion equation are discussed. Details of the multiscaling properties of the distributed-order diffusion equations were presented. In particular, we found the functional dependence of the exponent μ on the moment's order q and time t.

Distributed-order diffusion equations have appeared in an increasing number of studies involving more elaborate stochastic phenomena. We hope that this unifying and systematic study of distributed-order diffusion equations will find wide applications.

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APPENDIX A: SOME DETAILS ON SPECIAL FUNCTIONS

In this Appendix we provide the definitions and some useful relations for the special functions used in the main text.

1. Fox H function

The Fox H function is defined in terms of the Mellin-Barnes integral [76]

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi\imath}\int_{\Omega}\theta(s)z^s\,ds, \tag{A1}$$

where

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=n+1}^{p}\Gamma(a_j-A_js)}, \tag{A2}$$

with $0\leqslant n\leqslant p$, $1\leqslant m\leqslant q$, $a_i,b_j\in C$, $A_i,B_j\in R^+$, $i=1,\ldots,p$, and $j=1,\ldots,q$. The contour Ω, starting at $c-i\infty$ and ending at $c+i\infty$, separates the poles of the function $\Gamma(b_j+B_js)$, $j=1,\ldots,m$ from those of the function $\Gamma(1-a_i-A_is)$, $i=1,\ldots,n$.

The H function is symmetric in the pairs $(a_1,A_1),\ldots,(a_n,A_n)$, likewise $(a_{n+1},A_{n+1}),\ldots,(a_p,A_p)$; in $(b_1,B_1),\ldots,(b_m,B_m)$ and $(b_{m+1},B_{m+1}),\ldots,(B_q,B_q)$ [76]. This property is used to reduce Eq. (38) to the solution (12) of the monofractional diffusion equation.

The Fox H function has the scaling property [76]

$$H_{p,q}^{m,n}\left[z^{\delta}\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{\delta}H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p,A_p/\delta)\\(b_q,B_q/\delta)\end{matrix}\right.\right], \tag{A3}$$

where $\delta>0$.

The Mellin transform of the H- function is given by [76]

$$\int_0^{\infty}x^{\xi-1}H_{p,q}^{m,n}\left[ax\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]dx=a^{-\xi}\theta(-\xi), \tag{A4}$$

where $\theta(-\xi)$ is defined in relation (A1). This property is used to show the normalization of the PDFs and for the calculation of the MSDs (40), (84), (B4), and (B12). For instance, we have

$$\int_0^{\infty}H_{1,2}^{2,0}\left[ax\left|\begin{matrix}(a_1,A_1)\\(b_1,B_1),(b_2,B_2)\end{matrix}\right.\right]dx =a^{-1}\theta(-1)=a^{-1}\frac{\Gamma(b_1+B_1)\Gamma(b_2+B_2)}{\Gamma(a_1+A_1)} \tag{A5}$$

and

$$\int_0^{\infty}x^2H_{1,2}^{2,0}\left[ax\left|\begin{matrix}(a_1,A_1)\\(b_1,B_1),(b_2,B_2)\end{matrix}\right.\right]dx=a^{-3}\theta(-3)=a^{-3}\frac{\Gamma(b_1+3B_1)\Gamma(b_2+3B_2)}{\Gamma(a_1+3A_1)}. \tag{A6}$$

The Mellin-cosine transform of the Fox H function is given by [76]

$$\int_0^{\infty}k^{\rho-1}\cos(kx)H_{p,q}^{m,n}\left[ak^{\delta}\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]dk=\frac{\pi}{x^{\rho}}H_{q+1,p+2}^{n+1,m}\left[\frac{x^{\delta}}{a}\left|\begin{matrix}(1-b_q,B_q),\left(\frac{1+\rho}{2},\frac{\delta}{2}\right)\\(\rho,\delta),(1-a_p,A_p),\left(\frac{1+\rho}{2},\frac{\delta}{2}\right)\end{matrix}\right.\right], \tag{A7}$$

where

$$\mathrm{Re}\left(\rho+\delta\min_{1\leqslant j\leqslant m}\left(\frac{b_j}{B_j}\right)\right)>1,\quad x^{\delta}>0, \tag{A8}$$

$$\mathrm{Re}\left(\rho+\delta\max_{1\leqslant j\leqslant n}\left(\frac{a_j-1}{A_j}\right)\right)<\frac{3}{2},\quad |\arg(a)|<\pi\theta/2, \tag{A9}$$

$$\theta=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j>0. \tag{A10}$$

The following formulas hold true [76]:

$$H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]=\eta^{b_1/B_1}\sum_{r=0}^{\infty}\frac{(1-\eta^{1/B_1})^r}{r!}H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1+r,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right] \tag{A11}$$

and

$$H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]=\eta^{(a_1-1)/A_1}\sum_{r=0}^{\infty}\frac{(1-\eta^{-(1/A_1)})^r}{r!}H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(a_1-r,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]. \tag{A12}$$

2. Three-parameter and multinomial Mittag-Leffler functions

The three-parameter Mittag-Leffler function is defined by [65]

$$E_{\alpha,\beta}^{\delta}(z)=\sum_{k=0}^{\infty}\frac{(\delta)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \tag{A13}$$

where $(\delta)_k=\Gamma(\delta+k)/\Gamma(\delta)$ is the Pochhammer symbol. The one-parameter Mittag-Leffler function $E_{\alpha}(z)$ and the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ are special cases of the three-parameter Mittag-Leffler function if we use $\beta=\delta=1$ and $\delta=1$, respectively. For the three-parameter Mittag-Leffler function one can use the following formula

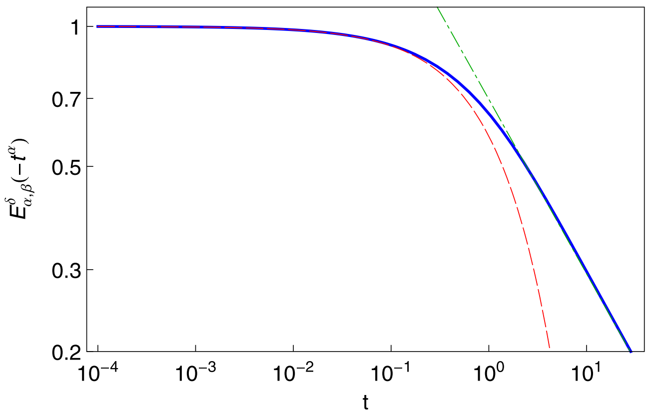

FIG. 10. (Color online) Double-logarithmic representation of the three-parameter Mittag-Leffler function (A13) for $\alpha = 3/4$, $\beta = 1$, and $\delta = 1/2$ (blue solid line). The red dashed line is the stretched exponential function (A15) and the green dot-dashed line is the power-law function (A16) for the same values of parameters.

[66,71,77] (see also [78–80] for the two-parameter Mittag-Leffler function):

$$E^{\delta}_{\alpha,\beta}(-z) = \frac{z^{-\delta}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta+n)}{\Gamma(\beta-\alpha(\delta+n)} \frac{(-z)^{-n}}{n!} \tag{A14}$$

for $|z| > 1$. For the three-parameter Mittag-Leffler function appearing in the solutions of fractional relaxation and diffusion equations [67–69] as well as the generalized Langevin equation (GLE) with different memory kernels [66,70,71,81], the short-time behavior is of stretched exponential form [see Eq. (A13)]

$$E^{\delta}_{\alpha,\beta}(-t^{\alpha}) \sim \frac{1}{\Gamma(\beta)} - \delta \frac{t^{\alpha}}{\Gamma(\alpha+\beta)} \sim \frac{1}{\Gamma(\beta)} \exp\left(-\delta \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha}\right). \tag{A15}$$

Conversely, at long times from relation (A14) we obtain the power-law decay

$$E^{\delta}_{\alpha,\beta}(-t^{\alpha}) \sim \frac{t^{-\alpha\delta}}{\Gamma(\beta-\alpha\delta)}. \tag{A16}$$

Figure 10 shows the behavior of the three-parameter Mittag-Leffler function and its transition from stretched exponential to power-law behavior. Here we note that the convergence of the series representation of the three-parameter Mittag-Leffler function is shown in [70] and further elaborated in [82].

The Laplace transform of the three-parameter Mittag-Leffler function (A13) is given by [65]

$$\mathscr{L}\left[t^{\beta-1} E^{\delta}_{\alpha,\beta}(\pm a t^{\alpha})\right](s) = \frac{s^{\alpha\delta-\beta}}{(s^{\alpha} \mp a)^{\delta}} \tag{A17}$$

for $\mathrm{Re}(s) > |a|^{1/\alpha}$. In Ref. [68] a numerical approximation of the three-parameter Mittag-Leffler function, which is useful for the description of nonexponential relaxation laws [67,69], is presented. Reference [83] introduces the two-parameter Mittag-Leffler function with negative α.

There are further generalizations of the Mittag-Leffler functions. One of those is the multinomial or multivariate Mittag-Leffler function [84,85] defined by

$$E_{(a_1,a_2,\ldots,a_N),b}(z_1,z_2,\ldots,z_N) = \sum_{j=1}^{\infty} \sum_{k_1\geqslant 0,k_2\geqslant 0,\ldots,k_N\geqslant 0}^{k_1+k_2+\cdots+k_N=j} \begin{pmatrix} & & j & \\ k_1 & k_2 & \ldots & k_N \end{pmatrix} \frac{\prod_{i=1}^{N}(z_i)^{k_i}}{\Gamma\left(b+\sum_{i=1}^{N} a_i k_i\right)}, \tag{A18}$$

where

$$\begin{pmatrix} & & j & \\ k_1 & k_2 & \ldots & k_N \end{pmatrix} = \frac{j!}{k_1!k_2!\ldots k_N!} \tag{A19}$$

are the so-called multinomial coefficients. It was recently shown [71] that the multinomial Mittag-Leffler function plays an important role in the representation of the MSD, the time-dependent diffusion coefficient, and the velocity autocorrelation function of a GLE of a free particle driven by a mixture of internal independent Dirac delta, power-law, and Mittag-Leffler noises.

The three-parameter Mittag-Leffler function (A13) is a special case of the Fox H function [76]

$$E^{\delta}_{\alpha,\beta}(-z) = \frac{1}{\Gamma(\delta)} H^{1,1}_{1,2}\left[z \left| \begin{matrix} (1-\delta,1) \\ (0,1),(1-\beta,\alpha) \end{matrix} \right.\right]. \tag{A20}$$

3. Wright-type functions

The auxiliary functions of the Wright-type or Mainardi function is defined by

$$M_{\alpha}(y) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(-\alpha n+1-\alpha)} \frac{(-y)^n}{n!}. \tag{A21}$$

It is related to the Fox H function in the following way [49]:

$$M_{\alpha}(y) = H^{1,0}_{1,1}\left[y \left| \begin{matrix} (1-\alpha,\alpha) \\ (0,1) \end{matrix} \right.\right]. \tag{A22}$$

The one-sided Lévy stable probability density $L_{\alpha}(y)$ can be represented through the Mainardi function as [86]

$$L_{\alpha}(t) = \frac{\alpha}{t^{\alpha+1}} M_{\alpha}\left(\frac{1}{t^{\alpha}}\right), \tag{A23}$$

which has the Laplace transform

$$L_{\alpha}(t) = \mathscr{L}^{-1}[e^{-s^{\alpha}}]. \tag{A24}$$

APPENDIX B: FURTHER GENERALIZATIONS OF DISTRIBUTED-ORDER DIFFUSION EQUATIONS

In what follows we provide additional results for the natural and modified-form distributed-order diffusion equations with three and N fractional exponents.

1. Natural form

Consider the case of three different fractional exponents,

$$p(\lambda) = B_1\delta(\lambda-\lambda_1) + B_2\delta(\lambda-\lambda_2) + B_3\delta(\lambda-\lambda_3), \quad \text{(B1)}$$

where $0<\lambda_1<\lambda_2<\lambda_3<1$, $B_1,B_2,B_3>0$, and $B_1+B_2+B_3=1$. For the PDF in the Fourier-Laplace space one finds

$$P(k,s) = \frac{1}{s}\frac{B_3(s\tau)^{\lambda_3}+B_2(s\tau)^{\lambda_2}+B_1(s\tau)^{\lambda_1}}{B_3(s\tau)^{\lambda_3}+B_2(s\tau)^{\lambda_2}+B_1(s\tau)^{\lambda_1}+\mathscr{K}\tau k^2}. \quad \text{(B2)}$$

Application of the inverse Laplace transform to expression (B2), where we use the series expansion method given in Ref. [35] and the Laplace transform formula (A17) for the three-parameter Mittag-Leffler function, and then employing inverse Fourier transform [see Eq. (A7)] one finds

$$\begin{aligned} P(x,t) = {} & \frac{1}{\sqrt{4\pi\frac{\mathscr{K}\tau}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{B_2}{B_3}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_3-\lambda_2)n+\lambda_3}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{B_1}{B_2}\right)^j\left(\frac{t}{\tau}\right)^{(\lambda_2-\lambda_1)j} \\ & \times\left\{\frac{B_3}{B_3}\left(\frac{t}{\tau}\right)^{-\lambda_3}H_{1,2}^{2,0}\left[\frac{x^2}{4\frac{\mathscr{K}\tau}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3}}\left|\begin{matrix}([\lambda_3-\lambda_2]n+[\lambda_2-\lambda_1]j+1-\lambda_3/2,\lambda_3)\\(0,1),(n+1/2,1)\end{matrix}\right.\right]\right. \\ & +\frac{B_2}{B_3}\left(\frac{t}{\tau}\right)^{-\lambda_2}H_{1,2}^{2,0}\left[\frac{x^2}{4\frac{\mathscr{K}\tau}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3}}\left|\begin{matrix}([\lambda_3-\lambda_2]n+[\lambda_2-\lambda_1]j+\lambda_3/2-\lambda_2+1,\lambda_3)\\(0,1),(n+1/2,1)\end{matrix}\right.\right] \\ & \left.+\frac{B_1}{B_3}\left(\frac{t}{\tau}\right)^{-\lambda_1}H_{1,2}^{2,0}\left[\frac{x^2}{4\frac{\mathscr{K}\tau}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3}}\left|\begin{matrix}([\lambda_3-\lambda_2]n+[\lambda_2-\lambda_1]j+\lambda_3/2-\lambda_1+1,\lambda_3)\\(0,1),(n+1/2,1)\end{matrix}\right.\right]\right\}. \end{aligned} \quad \text{(B3)}$$

Using the PDF (B3) or relation (39) by employing a Mellin transform of the H function (A4) we represent the MSD in terms of the convergent [70] infinite series in three-parameter Mittag-Leffler functions

$$\langle x^2(t)\rangle = 2\frac{\mathscr{K}\tau}{B_3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{B_1}{B_3}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_3-\lambda_1)n+\lambda_3}E_{\lambda_3-\lambda_2,(\lambda_3-\lambda_1)n+\lambda_3+1}^{n+1}\left(-\frac{B_2}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3-\lambda_2}\right). \quad \text{(B4)}$$

Note that in the limit $B_3\to 0$, using the asymptotic expansion formula for the three-parameter Mittag-Leffler function (A14), we obtain the MSD (40) in the case of two fractional exponents.

The short-time limit of Eq. (B4) yields

$$\begin{aligned}\langle x^2(t)\rangle &\sim 2\frac{\mathscr{K}\tau}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3}\times E_{\lambda_3-\lambda_2,\lambda_3+1}\left(-\frac{B_2}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3-\lambda_2}\right) \\ &\simeq 2\frac{\mathscr{K}\tau}{B_3}\frac{\left(\frac{t}{\tau}\right)^{\lambda_3}}{\Gamma(1+\lambda_3)},\end{aligned} \quad \text{(B5)}$$

and the long-time limit is

$$\begin{aligned}\langle x^2(t)\rangle &\sim 2\frac{\mathscr{K}\tau}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2}E_{\lambda_2-\lambda_1,\lambda_2+1}\left(-\frac{B_1}{B_2}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right) \\ &\sim 2\frac{\mathscr{K}\tau}{B_1}\frac{\left(\frac{t}{\tau}\right)^{\lambda_1}}{\Gamma(1+\lambda_1)}.\end{aligned} \quad \text{(B6)}$$

In the same way one can derive the PDF and the MSD for distributed-order diffusion equations with N different fractional exponents,

$$p(\lambda) = \sum_{i=1}^{N}B_i\delta(\lambda-\lambda_i), \quad \text{(B7)}$$

where $0<\lambda_1<\lambda_2<\cdots<\lambda_N<1$, $B_i>0$, $i=1,2,\ldots,N$, and $\sum_{i=1}^{N}B_i=1$. For this case, one uses multinomial and three-parameter Mittag-Leffler functions [35,71]. For the MSD we obtain

$$\langle x^2(t)\rangle = 2\frac{\mathscr{K}\tau}{B_N}\left(\frac{t}{\tau}\right)^{\lambda_N}E_{(\lambda_N-\lambda_1,\lambda_N-\lambda_2,\ldots,\lambda_N-\lambda_{N-1}),\lambda_N+1}\left(-\frac{B_1}{B_N}\left(\frac{t}{\tau}\right)^{\lambda_N-\lambda_1},-\frac{B_2}{B_N}\left(\frac{t}{\tau}\right)^{\lambda_N-\lambda_2},\ldots,-\frac{B_{N-1}}{B_N}\left(\frac{t}{\tau}\right)^{\lambda_N-\lambda_{N-1}}\right), \quad \text{(B8)}$$

where $E_{(a_1,a_2,\ldots,a_N),b}(z_1,z_2,\ldots,z_N)$ is the multinomial Mittag-Leffler function (A18).

The waiting-time PDF in the case of the three fractional exponents can be obtained in the same way as above for two fractional exponents from the series expansion method [35]. One may apply the method of multinomial Mittag-Leffler functions [71,84,85] as well.

For three fractional exponents the qth-order moments can be represented through infinite series in the three-parameter Mittag-Leffler functions,

$$\langle|x|^q(t)\rangle = \Gamma(q+1)\left(\frac{\mathscr{K}\tau}{B_3}\right)^{q/2}\sum_{n=0}^{\infty}(-1)^n \times\binom{n+q/2-1}{n}\left(\frac{B_1}{B_3}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_3-\lambda_1)n+\lambda_3 q/2} \times E^{n+q/2}_{\lambda_3-\lambda_2,(\lambda_3-\lambda_1)n+\lambda_3 q/2+1}\left(-\frac{B_2}{B_3}\left(\frac{t}{\tau}\right)^{\lambda_3-\lambda_2}\right), \quad \text{(B9)}$$

where the series expansion approach [35] and the Laplace transform formula (A17) were applied. The qth-order moment in case of two fractional exponents (45) can be obtained from relation (B9) in the limit $B_3 \to 0$. Then we apply the asymptotic expansion formula (A14) to relation (B9) and use the known relation

$$\binom{n+q/2-1}{n} = \frac{(q/2)_n}{n!} \quad \text{(B10)}$$

for the Pochhammer symbol. The same type of analysis can be pursued here in order to show that the qth-order moments of natural form distributed-order diffusion equations with three fractional exponents satisfy the relation (72), which is a generalization of the qth-order moment for the monofractional diffusion equation.

2. Modified form

The case of three different fractional exponents (B1) yields the following result obtained by Saxena and Pagnini [44]:

$$P(x,t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(B_2\mathscr{K}\tau)^n\left(\frac{t}{\tau}\right)^{\lambda_2 n}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{B_3}{B_2}\right)^j\left(\frac{t}{\tau}\right)^{(\lambda_3-\lambda_2)j} \times\frac{1}{|x|^{2n+1}}H^{2,1}_{3,3}\left[\frac{|x|^2}{B_1\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\lambda_1}}\middle|\begin{matrix}(1,1),(\lambda_2 n+[\lambda_3-\lambda_2]j+1,\lambda_1),(n+1,1)\\(2n+1,2),(n+1,1),(n+1,1)\end{matrix}\right]$$
$$= \frac{1}{\sqrt{4\pi B_1\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\lambda_1}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{B_2}{B_1}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_2-\lambda_1)n}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{B_3}{B_2}\right)^j\left(\frac{t}{\tau}\right)^{(\lambda_3-\lambda_2)j} \times H^{2,1}_{2,3}\left[\frac{x^2}{4B_1\mathscr{K}\tau\left(\frac{t}{\tau}\right)^{\lambda_1}}\middle|\begin{matrix}(1/2-n,1),([\lambda_2-\lambda_1]n+[\lambda_3-\lambda_2]j+1-\lambda_1/2,\lambda_1)\\(0,1),(1/2,1),(1/2,1)\end{matrix}\right]. \quad \text{(B11)}$$

From here we find the MSD, which is given by

$$\langle x^2(t)\rangle = 2\mathscr{K}\tau\sum_{i=1}^{3}B_i\frac{\left(\frac{t}{\tau}\right)^{\lambda_i}}{\Gamma(1+\lambda_i)}. \quad \text{(B12)}$$

These results can be generalized to the case of N different fractional exponents (B7) using the method given in [35,71]. For this case, from relation (83) for the MSD we obtain

$$\langle x^2(t)\rangle = 2\mathscr{K}\tau\sum_{i=1}^{N}B_i\frac{\left(\frac{t}{\tau}\right)^{\lambda_i}}{\Gamma(1+\lambda_i)}, \quad \text{(B13)}$$

from where we conclude that accelerating subdiffusion appears, as was expected. Here we mentioned that in [75] the corresponding Fokker-Planck equation in the presence of a constant external force is studied.

The waiting-time PDF can be obtained by using either the series expansion method given in Ref. [35] or the method of multinomial Mittag-Leffler functions [84,85].

As previously we derive the qth-order moment in the case of the three fractional exponents, given in terms of an infinite series in three-parameter Mittag-Leffler functions,

$$\langle|x|^q(t)\rangle = \Gamma(q+1)(B_1\mathscr{K}\tau)^{q/2}\sum_{n=0}^{\infty}(-1)^n \times\binom{n-q/2-1}{n}\left(\frac{B_3}{B_1}\right)^n\left(\frac{t}{\tau}\right)^{(\lambda_3-\lambda_1)n+\lambda_1 q/2} \times E^{n-q/2}_{\lambda_2-\lambda_1,(\lambda_3-\lambda_1)n+\lambda_1 q/2+1}\left(-\frac{B_2}{B_1}\left(\frac{t}{\tau}\right)^{\lambda_2-\lambda_1}\right). \quad \text{(B14)}$$

We see that if $B_3 = 0$ only the first term in the series is different from zero and equals relation (87) for the case of two fractional exponents. Using $n = 0$ in result (B14) obtains the result for two fractional exponents. One can show in the same way as above that the modified-form distributed-order diffusion equation with three fractional exponents satisfies relation (72) as well. We conclude that the qth-order moment of the modified-form distributed-order diffusion with three fractional exponents has a more complicated behavior than that given by (15) and (16).

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