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Physica A 268 (1999) 454–468

PHYSICA A

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Stochastic foundation of normal and anomalous Cattaneo-type transport

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Received 1 November 1998

Abstract

We investigate the connection of the Cattaneo equation and the stochastic continuous time random walk (CTRW) theory. We show that the velocity model in a CTRW scheme is suited to derive the standard Cattaneo equation, and allows, in principle, for a generalisation to anomalous transport. As a result for a broad waiting time distribution with diverging mean, we find a strong memory to the initial condition of the system: The ballistic behaviour subsists also for long times. Only if a characteristic waiting time exists, a non-ballistic, enhanced motion is found in the limit of long times. No transition to subdiffusion can be found. © 1999 Elsevier Science B.V. All rights reserved.

PACS: 05.60.+w; 05.40.+j; 44.30.+i; 05.70.Ln

Dedicated to Prof. José Casas-Vázquez on his 60th birthday

1. Introduction

The *Cattaneo equation* for the probability density function $P(x,t)$ representing particle densities, concentrations, or heat profiles, reads [1,2]

$$\tau\ddot{P}(x,t) + \dot{P} = KP'' \tag{1}$$

and is usually derived by combining the continuity equation of particle (or probability) conservation

$$\dot{P}(x,t) = -J'(x,t)\,, \tag{2}$$

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0378-4371/99/$ - see front matter
PII: S0378-4371(99)00058-8

and the Cattaneo constitutive equation [1]

$$J(x,t) + \tau \dot{J} = -KP' . \tag{3}$$

This last equation introduces a relaxation of the flux J with the characteristic time τ, and K denotes the diffusion coefficient. We indicate derivatives in respect to time with a dot, and in respect to space with a prime. As a consequence of the introduction of the term $\tau\ddot{P}$ in Eq. (1), for short times one obtains a *finite velocity of propagation* $v = \sqrt{K/\tau}$ for the diffusion process, corresponding to a $\langle x^2 \rangle \sim v^2 t^2$ behaviour of the mean squared displacement. In a particle picture, this corresponds to an initial condition where all particles move mutually in one direction. After the characteristic time τ, the effects of random velocity changes become dominant. Thus, for long times $t \gg \tau$, one recovers the standard Brownian diffusion behaviour characterised by Fick's second law and a linear time dependence of the second moment, where the propagator is given by the Gaussian distribution. The Cattaneo equation (1) is therefore a phenomenological extension of Fick's second law accounting for the physical requirement of a constant velocity of propagation. Note that the Cattaneo equation is of the telegrapher's type, and can be derived from the Boltzmann equation [3,4]. The solution of the Cattaneo equation (1) can be calculated analytically [5], and it is readily verified that this solution reduces to the standard Gaussian for long times. The Cattaneo equation (1) finds application in a broad diversity of systems, including heat and particle transport in irreversible thermodynamics [2], heat transfer in Bénard convection [6,7], cosmological models [8], shock waves in rigid heat conductors [9], and the diffusion in crystalline solids [10]. Recent reviews of the Cattaneo equation are given in Refs. [11,12].

Anomalous diffusion is characterised by the occurrence of a mean-squared displacement of the power-law dependence [13–16]

$$\langle x^2 \rangle \propto K_\alpha t^\alpha , \tag{4}$$

deviating from the normal linear dependence on time found for the Fickean description of Brownian motion. Anomalous diffusion is encountered in a large variety of systems [13–16], and it is often intimately related to Lévy-type statistics, i.e. the validity of the Generalised Central Limit Theorem [15,17,18]. Eq. (4) describes subdiffusion for $0 < \alpha < 1$, or enhanced diffusion (superdiffusion) for $\alpha > 1$. In Eq. (4), K_α is the anomalous diffusion coefficient with dimension $[K_\alpha] = \mathrm{cm}^2\,\mathrm{s}^{-\alpha}$. In the following, we will work with reduced variables, i.e. $\tau = 1$ and $K_\alpha = 1$, to simplify the notation. Among other approaches fractional diffusion equations are often used to describe anomalous diffusion [19–21]. The occurrence of the fractional operator within the fractional diffusion equations can be connected with broad waiting time distributions in continuous time random walk theory [22–27].

It is our goal to extend the Cattaneo picture basing on Eq. (1) in order to include anomalous transport features as defined in Eq. (4). To this end, we introduced in Ref. [26] possible generalisations of the Cattaneo equation in terms of *fractional derivatives*. In the framework of a non-local transport theory [28], the fractional expression

$$\dot{P}(x,t) + {}_0D_t^{\beta-1}\ddot{P} = {}_0D_t^{\beta-1}P'' , \tag{5}$$

arises naturally from the introduction of a broad memory ($0 < \beta < 1$) in the constitutive equation for the flux [26], causing a slowly decaying dependence on the previous history of the diffusion process. The definitions of the *fractional Riemann–Liouville operators* which appear in Eq. (5) are given in Appendix A. For Eq. (5), the asymptotic behaviour of the mean squared displacement goes like [26]

$$\langle x^2 \rangle \sim \begin{cases} t^2, & t \ll 1\,, \\ \dfrac{2}{\Gamma(3-\beta)} t^{2-\beta}, & t \gg 1\,. \end{cases} \tag{6}$$

Note that the ballistic short-time behaviour turns over into an intermediate enhanced diffusion for $0 < \beta < 1$, which reduces to the standard Fickean behaviour for $\beta = 1$. On the other hand, using a generalised definition of the flux in a continuous time random walk (CTRW) *jump* picture, the fractional order in the generalised Cattaneo equation came about through a broad waiting time distribution [26], whereas the general form of the underlying master equation remained unchanged. In this model, however, the short-time behaviour was actually only a correction to the long-time behaviour, and it was not ballistic. Generalised Cattaneo equations were derived in Ref. [29] from a stochastic two-state model and the connection of the velocity–velocity correlation function of the two-state variable with a long-tailed waiting time distribution, an approach which is closely related to the Lévy walk jump model mentioned below.

Here we discuss a generalisation of the Cattaneo picture for situations of anomalous transport through a stochastic random walk approach, the so-called *velocity model* in the CTRW framework [30]. We show that for waiting time distributions which do not possess a finite characteristic waiting time, i.e. a first moment, the ballistic motion subsists also for long times. If the first moment exists, we find enhanced, intermediate transport, similar to Eq. (6). Thus, we find a microscopic footing for enhanced, sub-ballistic Cattaneo-type transport.

In the next section we introduce the velocity model and discuss it for three different cases of waiting time distributions, these being of Poissonian and asymptotic power-law nature. For the power-laws, the cases of diverging and existing first moment are differentiated. We are led to a schematic parametric diagram connecting the anomalous diffusion coefficient with the index of the waiting time distribution, when we draw our conclusions in Section 3. The appendix defines some basic properties of fractional calculus and briefly introduces two models to derive the Cattaneo equation, which are not well suited for generalisation to anomalous diffusion.

2. Random walk approach

Here we are interested in the question of how the Cattaneo equation (1) can be connected to a statistical master equation approach, and thus be generalised towards the description of anomalous diffusion. There are a number of ways to derive the Cattaneo equation, but many are not well suited to generalise it for anomalous diffusion (see

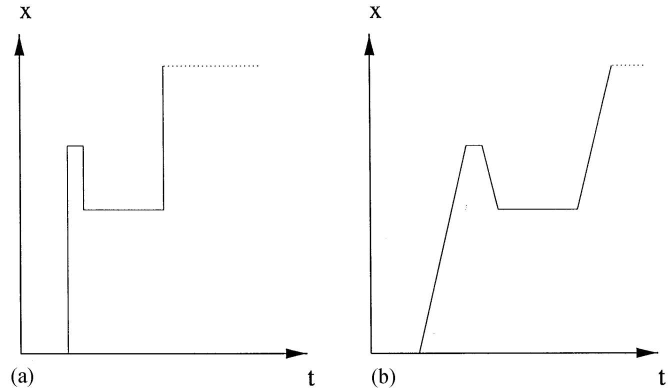

Fig. 1. Schematic diagram for (a) the jump model, and (b) the velocity model. Whereas jumps occur instantaneously in (a), the finite slope in (b) mirrors the finite propagation speed in the velocity model. Both diagrams are graphed for the same sequence of waiting times.

Appendices B and C). We report here how the velocity model in a CTRW scheme does provide an appropriate footing for carrying out this generalisation.

2.1. The velocity model

The original CTRW was developed as a so-called jump model, featuring instantaneous jumps from one site to another [27]. More recently, the velocity and the two-state model have been added [30]. As the two-state model can be mapped upon the velocity model [30], we neglect it in the following. The jump model and the velocity model are compared in Fig. 1. Both are drawn for a so-called coupled case including a finite velocity v [27]. In the jump model displayed in Fig. 1(a), the particle moves instantaneously to a new site, only longer jumps are less probable due to the introduced time cost $p(t|x)$, see below. In Fig. 1(b) the velocity model is shown, where the particle moves at a constant velocity to the new site. The slope in the (x,t) diagram gives v.

In CTRW terms, the propagator can be calculated through the probability that the particle arrives in x at time t

$$\eta(x,t) = \int \mathrm{d}x' \int_0^t \mathrm{d}t'\, \eta(x',t')\psi(x-x',t-t') + \delta(t)\delta(x)\,. \tag{7}$$

The second term on the right-hand side describes the initial condition of starting in the origin. In Eq. (7), $\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$ denotes the jump probability distribution, from which the waiting time $t\ldots t+\mathrm{d}t$ elapsing from the last jump to the next, and the jump length $x\ldots x+\mathrm{d}x$ of the next jump is drawn. Now the propagator in the jump model is simply given by being at a certain point x at some time t, this is, having arrived in x at t, and not having moved since

$$P_J(x,t) = \int_0^t \mathrm{d}t'\, \eta(x,t')\Phi(t-t')\,, \tag{8a}$$

where the cumulative "sticking probability" of not moving during the time interval $0\ldots t$ is given by

$$\Phi(t) = 1 - \int_0^t \mathrm{d}t' \psi(t) \, . \tag{8b}$$

In Eq. (8b), the waiting time probability density function[2] $\psi(t) = \int \mathrm{d}x \, \psi(x,t) = \psi(k = 0, t)$ is introduced, which defines the probability density that the particle jumps after a waiting time t. On the other hand, in the velocity model, the propagator should describe the probability to stop or pass by a location x at some time t [30]

$$P_V(x,t) = \int \mathrm{d}x' \int_0^t \mathrm{d}t' \, \eta(x',t') \Psi(x - x', t - t') \, , \tag{8c}$$

where now the probability to pass a given location x at time t in a single motion event is expressed through

$$\Psi(x,t) = p(t|x) \int_t^\infty \int_{|x|}^\infty \mathrm{d}x' \, \psi(x',t') \, . \tag{9}$$

Here, we assumed a jump probability density of the form $\psi(x,t) = p(t|x)\psi(t)$, where the conditional probability $p(t|x)$ determines the time cost for a jump of length x [or the 'permitted' jump length x for a given waiting time t] [30].

In Fourier–Laplace space, the propagators of both models are thus given by

$$P_J(k,u) = \frac{\Phi(u)}{1 - \psi(k,u)} \, , \tag{10a}$$

$$P_V(k,u) = \frac{\Psi(k,u)}{1 - \psi(k,u)} \, , \tag{10b}$$

respectively. We note that the exact expressions for the (k,u) space propagators can be given in terms of an arbitrary waiting time distribution and the conditional probability $p(t|x) = \frac{1}{2}\delta(|x| - t)$, i.e. $\psi(x,t) = \frac{1}{2}\delta(|x| - t)\psi(t)$, via

$$P_J(k,u) = \frac{1 - \psi(u)}{u\{1 - \frac{1}{2}\,[\psi(u - \mathrm{i}k) + \psi(u + \mathrm{i}k)]\}} \, , \tag{11a}$$

$$P_V(k,u) = \frac{u}{u^2 + k^2} - \frac{\mathrm{i}k[\psi(u - \mathrm{i}k) - \psi(u + \mathrm{i}k)]}{(u^2 + k^2)[2 - \psi(u - \mathrm{i}k) - \psi(u + \mathrm{i}k)]} \, . \tag{11b}$$

Therefore, the propagator is always real valued, as it should. Note that P_V naturally splits up in a purely ballistic part and a second term dominated by the actual jump distribution.

[2] In the following, we will speak loosely of the waiting time distribution, and similarly of the jump length distribution.

2.2. Poissonian waiting time distribution

Choosing for the waiting time distribution a Poissonian law $\psi(t)=e^{-t}$, i.e.

$$\psi(x,t)=\frac{1}{2}\delta(|x|-t)e^{-t}\,, \tag{12}$$

we find for the propagators

$$P_J(k,u)=\frac{(1+u)^2+k^2}{(1+u)(u+u^2+k^2)}\,, \tag{13a}$$

$$P_V(k,u)=\frac{1+u}{u+u^2+k^2}\,. \tag{13b}$$

Via multiplication by the denominator and inverting the obtained equation into the (x,t) domain using the differentiation theorems of Laplace and Fourier transform [for instance $\mathscr{L}^{-1}\{uP(x,u)-P(x,t=0)\}=\dot{P}(x,t)$ and $\mathscr{F}^{-1}\{-k^2P(k,t)\}=P''$], we find the corresponding differential equations:

$$\frac{\partial^3 P_J}{\partial t^3}+2\frac{\partial^2 P_J}{\partial t^2}+\frac{\partial P_J}{\partial t}=\frac{\partial^2 P_J}{\partial x^2}+\frac{\partial^3 P_J}{\partial t\partial x^2}\,, \tag{14a}$$

$$\frac{\partial^2 P_V}{\partial t^2}+\frac{\partial P_V}{\partial t}=\frac{\partial^2 P_V}{\partial x^2}\,. \tag{14b}$$

Eqs. (13a and 13b) and (14a and 14b) are valid on the whole (k,u) or, equivalently, (x,t) domains. Eqs. (13b) and (14b) are equivalent to the Cattaneo equation in (k,u) and (x,t) space, respectively.

Finally, calculating the mean squared displacement from the characteristic function via the relation $\langle x^n\rangle(u)=\mathrm{i}^n\lim_{k\to 0}\mathrm{d}^nP(k,u)/\mathrm{d}k^n$, we arrive at

$$\langle x^2\rangle_J(u)=\frac{2}{u^2(1+u)^2}\,, \tag{15a}$$

$$\langle x^2\rangle_V(u)=\frac{2}{u^2(1+u)}\,. \tag{15b}$$

In the case of the jump model we find diffusive behaviour for long times, but a short-time behaviour $\langle x^2\rangle_J(t)\sim\frac{1}{3}t^3$, which is physically not meaningful, and arises from the third-order derivatives in Eq. (14a). These are caused by the coupling $p(t|x)$. On the other hand, the velocity model reveals the standard Cattaneo picture, this is

$$\langle x^2\rangle_V(t)\sim\begin{cases} t^2, & t\ll\tau\,,\\ 2t, & t\gg\tau\,.\end{cases} \tag{16}$$

Thus, the velocity model is the natural extension of the CTRW jump model for the description of Cattaneo-type transport. Our forthcoming considerations are therefore based on the velocity model.

2.3. Broad waiting time distribution with no characteristic waiting time

Let us now address the problem of introducing anomalous transport behaviour. To this end, we consider a broad waiting time distribution with a diverging first moment in the velocity model. We will see that we encounter a well-defined and finite velocity of propagation for short times, but the long-time limit does not turn over to the solution of an anomalous walker.

Another result will be that we cannot recover an extension of the Cattaneo equation, which is unique over the whole time range: As the standard Cattaneo equation (1) splits up into a wave equation

$$\ddot{P}(x,t) = P'' , \tag{17a}$$

for short times $t \ll 1$, and a diffusion equation

$$\dot{P}(x,t) = P'' , \tag{17b}$$

in the diffusion limit $t \gg 1$, we will recover wave and fractional diffusion equations in the corresponding limits. This is typical for a coupled model, and was already encountered in Ref. [29]. Only in the Poissonian case, both limiting equations reduce to the Cattaneo equation valid for all t.

Let us at first investigate the solution for the asymptotically fractal waiting time distribution

$$\psi(u) = \frac{1}{1+u^{\gamma}} , \tag{18}$$

in Laplace space, which has the asymptotic behaviours

$$\psi(u) \sim \begin{cases} 1 - u^{\gamma}, & u \ll 1 , \\ u^{-\gamma}, & u \gg 1 . \end{cases} \tag{19}$$

For the propagator we find:

$$P_V = \frac{(u-\mathrm{i}k)(u+\mathrm{i}k)^{\gamma} + (u+\mathrm{i}k)(u-\mathrm{i}k)^{\gamma} + 2u(u-\mathrm{i}k)^{\gamma}(u+\mathrm{i}k)^{\gamma}}{(u^2+k^2)[(u+\mathrm{i}k)^{\gamma} + (u-\mathrm{i}k)^{\gamma} + 2(u+\mathrm{i}k)^{\gamma}(u-\mathrm{i}k)^{\gamma}]} , \tag{20}$$

the second moment of which can be evaluated analytically so that the mean-squared displacement in Laplace space reads

$$\langle x^2 \rangle_V(u) = \frac{2}{u^3} \frac{(1-\gamma) + (2-\gamma)u^{\gamma} + u^{2\gamma}}{(1+u^{\gamma})^2} , \tag{21}$$

from which finally the asymptotic behaviour in time space can be deduced:

$$\langle x^2 \rangle_V(t) \sim \begin{cases} t^2, & t \ll \tau , \\ (1-\gamma)t^2, & t \gg \tau . \end{cases} \tag{22}$$

This is an a priori surprising behaviour. Whereas both P_V from Eq. (20) and therefore also $\langle x^2 \rangle_V(u)$ from Eq. (21) reduce to the usual Cattaneo behaviour for $\gamma \to 1$, the subsisting ballistic behaviour for the case $0 < \gamma < 1$ has to be understood.

This subsistence of the ballistic behaviour for the case of the waiting time distribution (18) is also reflected in the corresponding generalised quasi-Cattaneo equations. Starting off from the propagator P_V from Eq. (20), we find in the usual $k \to 0$ approximation the following two limiting equations: In the short-time limit $t \ll 1$ the wave equation

$$\ddot{P}(x,t) = P'' , \tag{23a}$$

results as for the standard Cattaneo equation, see Eq. (17a), whereas in the diffusion régime $t \gg 1$ we find the equation

$$\ddot{P}(x,t) = \frac{(1-\gamma)}{2} P'' , \tag{23b}$$

which is just the re-scaled wave equation. Fractional corrections occurring in the limit equations are proportional to ${}_0D_t^{2-\gamma}P$ in Eq. (23a), similar to the non-local transport result (5), and proportional to ${}_0D_t^{2+\gamma}P$ in Eq. (23b). Eq. (23b) corresponds to the asymptotic behaviour of the propagator (20) in the diffusion limit according to

$$P_V \sim \frac{u - \frac{1}{2}(1-\gamma)(2-\gamma)u^{-2}k^2}{u^2 + \frac{1}{2}(1-\gamma)k^2} . \tag{24}$$

To check whether the above obtained result is depending on the actual shape of the waiting time distribution, let us introduce the following distribution:

$$\psi(t) = \frac{\gamma}{(1+t)^{1+\gamma}} , \tag{25}$$

the Laplace transform of which can be calculated analytically [30,31]

$$\psi(u) = \begin{cases} \gamma u^{\gamma} e^{u} \Gamma(-\gamma, u), & \gamma \notin \mathbb{N} , \\ 1 - u e^{u} E_1(u), & \gamma = 1 . \end{cases} \tag{26}$$

Here, $\Gamma(-\gamma, u)$ denotes the incomplete Gamma function and $E_1(u)$ the exponential integral $E_1(x) = -\mathrm{Ei}(-x) = \Gamma(0,x)$ [31]. The asymptotic behaviour for $u \ll 1$ of (25) in Laplace space reads [30,32]

$$\psi(u) \sim \begin{cases} 1 - \Gamma(1-\gamma)u^{\gamma}, & 0<\gamma<1 , \\ 1 + u\log u, & \gamma = 1 , \\ 1 - \dfrac{u}{\gamma - 1} - \Gamma(1-\gamma)u^{\gamma}, & 1<\gamma<2 . \end{cases} \tag{27}$$

Note that for $1 < \gamma < 2$, this asymptotic behaviour is different from the corresponding limit given in Eq. (19) for the waiting time distribution (18). For the discussion in the next subsection we will therefore use the functional behaviour of Eq. (25), as this waiting time distribution was already established in Refs. [30,33,34].

Thus, with Eq. (26), the propagator becomes

$$P_V = \frac{2u - \gamma(u+\mathrm{i}k)(u-\mathrm{i}k)^{\gamma}\Gamma(-\gamma, u-\mathrm{i}k) - \gamma(u-\mathrm{i}k)(u+\mathrm{i}k)^{\gamma}\Gamma(-\gamma, u+\mathrm{i}k)}{(u^2+k^2)[2 - \gamma(u-\mathrm{i}k)^{\gamma}e^{u-\mathrm{i}k}\Gamma(-\gamma, u-\mathrm{i}k) - \gamma(u+\mathrm{i}k)^{\gamma}e^{u+\mathrm{i}k}\Gamma(-\gamma, u+\mathrm{i}k)]} . \tag{28}$$

The result for the mean-squared displacement reads:

$$\langle x^2 \rangle_V(u) = \frac{2}{u^3} \frac{(1-\gamma)[1 - \gamma u^{\gamma} e^{u}\Gamma(-\gamma,u)] + \gamma u^{1+\gamma} e^{u}\Gamma(-\gamma,u)}{1 - \gamma u^{\gamma} e^{u}\Gamma(-\gamma,u)} \tag{29}$$

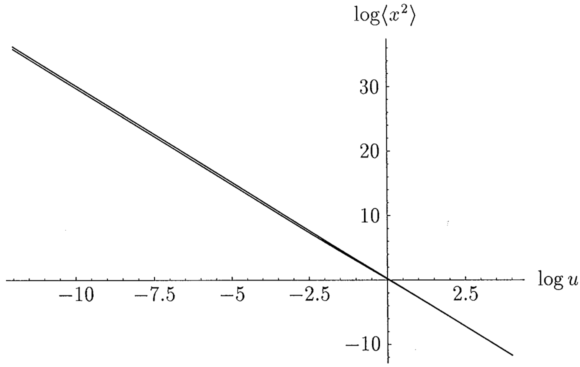

Fig. 2. Decadic, double-logarithmic plot of the mean-squared displacement (29). The power-law behaviour over the whole range of the plot is clear. The estimated slope is -3, corresponding to the ballistic behaviour proportional to t^2. The parameters were chosen as $\gamma_1 = 1/5$ and $\gamma_2 = 2/3$.

and corresponds to the limiting cases

$$\langle x^2\rangle_V(t) \sim \begin{cases} t^2, & t \ll 1\,, \\ (1-\gamma)t^2, & t \gg 1\,, \end{cases} \tag{30}$$

in time, in full agreement with Eq. (22). [Correction terms and higher-order moments are however different for the waiting time distributions (18) and (25).] Fig. 2 shows a double-logarithmic plot of the function in Eq. (29). The slight discrepancy between the two values for γ for small u in Fig. 2 does not widen up for even smaller u, as can be checked using `Mathematica`. The estimated slope is -3 in the u space, which takes over to a subsisting $\propto t^2$ behaviour in time space. Thus, also for this model, the memory of the initial behaviour does not die out. Therefore, it is an inherent feature of the broad Lévy walk velocity model.

In fact, this behaviour is to be expected, as was shown in Ref. [34]. For our case, when even the first moment of the waiting time distribution does not exist, there will always be a finite number of initial motion events with a walk time as long as or longer than the finite observation time. Thus, the overall behaviour is dominated by the ballistic motion, corresponding to a subsisting motion in one direction. As was seen from Eq. (22), the prefactor of the ballistic term varies in between the two limiting cases of short and long times. Comparing Eq. (30) with Eq. (18), both waiting time distributions lead to the same coefficient $(1-\gamma)$ for the long-time ballistic behaviour, indicating that the occurrence of new motion events reduces the overall time spent in long ballistic movements. For $\gamma = 1$, the term $\propto t^2$ vanishes, the direction is now changed with the finite rate τ^{-1} typical for the Brownian random walk, and the usual Cattaneo equation (1) is recovered. In the following, we concentrate on the analytical form (25).

We finally remark that in the analogous jump model, we recover the mean squared displacement

$$\langle x^2\rangle_J(u) = \frac{\gamma}{u^3}\frac{(1-\gamma)-u}{1-\gamma u^\gamma e^u \Gamma(-\gamma,u)} + \frac{[2\gamma u + u^2 - (1-\gamma)\gamma]u^\gamma e^u \Gamma(-\gamma,u)}{1-\gamma u^\gamma e^u \Gamma(-\gamma,u)}\,, \tag{31}$$

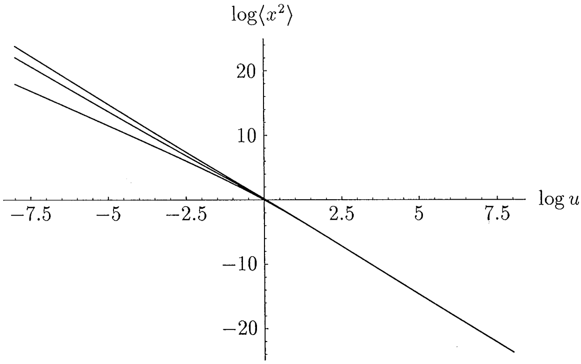

Fig. 3. Mean-squared displacement for $\gamma_1 = 15/8$, $\gamma_2 = 9/8$ and $\gamma_3 = 7/8$ (from bottom to top) in a decadic double-logarithmic plot. The bend in the slope for the two cases γ_1, γ_2 is obvious. Note the slow turnover to the small-u behaviour according to $u^{-(4-\gamma)}$ for the lowest curve corresponding to $\gamma_1 = 15/8$. As expected, for $\gamma_3 = 7/8$ the u^{-3} behaviour is present over the whole plot range.

for the waiting time distribution (25), which also leads to a ballistic behaviour for long times

$$\langle x^2\rangle_J(t) \sim \frac{\gamma(1-\gamma)}{2}t^2, \quad t \gg 1 . \tag{32}$$

But again, the short time behaviour is dominated by the unphysical t^3-proportionality, as seen in Eq. (15a).

2.4. Broad waiting time distribution with a finite characteristic waiting time

In this subsection, we again use the waiting time distribution (25), but for the parameter range $1<\gamma<2$. In this range, one would expect to find a non-ballistic long-time behaviour, as now a characteristic waiting times exists, so that for long observation times virtually all particles have undergone collisions. In Fig. 3 we graph the mean squared displacement over the whole range of u. For large u, corresponding to small times, the slope is -3, this is equivalent to ballistic motion. On the other hand, for small u, we see a transition to another power law with a smaller slope, which can be estimated to be $-(4-\gamma)$, i.e. which corresponds to a sub-ballistic behaviour $\propto t^{3-\gamma}$.

Thus indeed, the existence of the characteristic waiting time separates a microscopic and a macroscopic time scale.

Taking into account the asymptotic expansion for the waiting time distribution (25), Eq. (27), we recover for the propagator (28) in the diffusion limit $k \to 0$, $u \to 0$

$$P_V \sim \frac{u^{2-\gamma} - \frac{1}{2}(\gamma-1)\Gamma(3-\gamma)u^{-1}k^2}{u^{3-\gamma} + \frac{1}{2}\gamma(\gamma-1)\Gamma(2-\gamma)k^2} , \tag{33}$$

and for the mean squared displacement the relation

$$\langle x^2\rangle_V(u) \sim 2(\gamma-1)\Gamma(2-\gamma)u^{\gamma-4}, \quad u \ll 1 , \tag{34}$$

corresponding to the long-time behaviour

$$\langle x^2\rangle_V(t) \sim \frac{2(\gamma-1)}{(3-\gamma)(2-\gamma)}t^{3-\gamma}, \quad t \gg 1 . \tag{35}$$

Thus, the analytical calculation indeed reveals the expected turnover behaviour towards enhanced, sub-ballistic motion. This behaviour is mirrored in the fractional diffusion equation, corresponding to the propagator (33) for long times,

$${}_0D_t^{3-\gamma}P(x,t) = \frac{\gamma(\gamma-1)}{2}\Gamma(2-\gamma)P'' . \tag{36}$$

We have thus found that in the velocity model, if a characteristic waiting time exists, a transition from the initial ballistic motion to an enhanced anomalous diffusion behaviour comes about. The basic behaviour thus corresponds to the non-local transport model extended to a broad waiting time distribution in Ref. [26], only the prefactor of the long-time motion differs [note that the parameter γ occurring there ranges in the interval $(0,1)$!]. This coefficient can be absorbed by a scaling of the variables. The close connection of the CTRW velocity result and the non-local transport model is an interesting result. Therefore, the fractional Cattaneo equation

$${}_0D_t^{3-\gamma}P(x,t) + \ddot{P} = \frac{\gamma(\gamma-1)}{2}P'' , \tag{37}$$

corresponding to Eq. (5) is a good interpolation to the exact CTRW velocity result, revealing the desired behaviour of Eq. (30), in consistency with the limiting equations (35) and (23b) derived from the propagator (28) with $1<\gamma<2$.

3. Conclusions

We have considered the velocity model continuous time random walk approach which is well-established in diffusion theory [34] to a generalised Cattaneo scheme describing the transition from the ballistic motion to anomalous transport behaviour. This connection of the continuous time random walk framework with the Cattaneo picture, one of the fundamental ideas in extended irreversible thermodynamics, to our best knowledge, has not been discussed in literature.

For a broad waiting time distribution of the type (25) with $0<\gamma<1$, the velocity model leads to a subsisting ballistic motion, independent of the index γ of the underlying waiting time distribution. The reason is that in the velocity picture, the broad waiting time distribution causes long periods of walks in one direction before changing direction, irrespectively of the size of the interval one considers. This consequently leads to an overall, averaged ballistic motion. In the long-time limit, the scaling factor $(1-\gamma)$ indicates that a certain number of collisions take place, but that the overall motion is still dominated by long motion events in one direction, both features being intimately connected to the anomalous diffusion behaviour resulting from the power-law asymptotics of $\psi(t)$ from Eqs. (18) and (25), i.e. the non-existence of a characteristic waiting time separating microscopic and macroscopic time scales. On the other hand, for $1<\gamma<2$, a characteristic waiting time exists, which makes it possible to distinguish

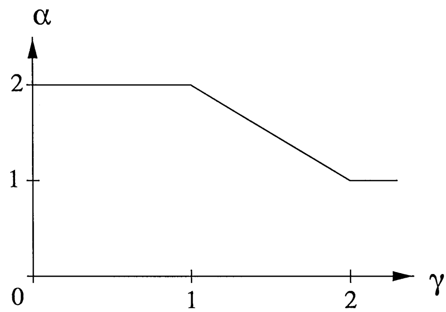

Fig. 4. Schematic diagram for the anomalous diffusion exponent α defined in Eq. (4), and the index γ of the waiting time distribution (25). For $\gamma > 2$, even the second moment exists, and the diffusion is always normal.

such time scales. Now, the change in direction occurs often enough for the motion to become sub-ballistic, i.e. we find the desired transition to anomalous diffusion. For $\gamma > 2$, also the variance exists, and therefore the long-time limit will become equivalent to the Brownian diffusion, where the mean-squared displacement grows linearly with time. This parametric dependence of the long-time, anomalous diffusion exponent is graphed in Fig. 4.

Neither in the range $0<\gamma<1$ nor in the interval $1<\gamma<2$, a closed-form generalisation of the Cattaneo equation can be found. On the other hand, we could show that it is possible to establish a microscopic random walk picture for enhanced anomalous Cattaneo behaviour. If exact results for the propagator, or the knowledge of higher-order moments is not of interest, the fractional Cattaneo equation (37), equivalent to Eq. (5) derived from a non-local transport model, is an alternative formulation.

Summarising, the derivation of a consistent generalised Cattaneo equation leading to anomalous diffusion in the long-time limit and preserving the ballistic behaviour for short times, is a difficult task. Adding to the generalised Cattaneo equation found for non-local transport theory (5) in Ref. [26], we have here discussed a stochastic approach to Cattaneo-type anomalous transport. Both models however, lead to enhanced, intermediate motion with an anomalous diffusion exponent in between one and two. The turnover to subdiffusion is not found, moreover we have argued that it cannot exist within the Lévy walk velocity concept.

As anomalous transport is a typical feature for a rich variety of systems, we believe that our findings contribute to the field of extended irreversible thermodynamics, especially in the modelling of reaction kinetics and other fields, where the finite velocity contribution in the short-time behaviour cannot be neglected. The approximate interpolation through Eqs. (37), or (5), and their connections to non-local transport theory makes this approach attractive for future research.

Acknowledgements

We thank José Casas-Vázquez and David Jou for their support, helpful comments, and especially for inviting R.M. to UAB and thus making this work possible. We also thank Yossi Klafter for instigating discussions. R.M. is Feodor-Lynen fellow of the Alexander von Humboldt Stiftung, Bonn am Rhein, Germany. Part of this work was sponsored by MINERVA through an Amos-de-Shalit fellowship. A.C. acknowledges financial support from the DGR of the Generalitat de Catalunya. Financial support from the German Israeli Foundation (GIF) and from the DGICyT of the Spanish Ministry of Education under grant No. PB94-0718 is acknowledged as well. R.M. thanks Eli Barkai for discussions on the collision model, and Israela Becker for pointing out a problem in the long-time behaviour.

Appendix A: Fractional calculus

Fractional calculus ideas date back to the days when classical calculus came of age. These ideas were first expressed in some letters between the French and German mathematicians de l'Hospital and Leibniz in 1695. Later, contributions came from Grünwald, Krug, Heaviside, Laurent, Hadamard, Lévy, Hardy, Weyl, and others. Today's definitions are mainly based on the works of Liouville and Riemann published in the last century [37,38].

There exists a variety of definitions of fractional operators, see the compendium of Samko et al. [38]. The most widely used definition is the Riemann–Liouville operator

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t \mathrm{d}\tau \frac{f(\tau)}{(t-\tau)^{1-p}} , \tag{A.1} $$

extending Cauchy's multiple integral for arbitrary complex p with $\mathrm{Re}(p) > 0$, by use of the Gamma function. A derivative of order q, $q > 0$, is consequently established via the definition

$$ {}_0D_t^{q} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\, {}_0D_t^{q-n} f(t) , \tag{A.2} $$

where $n \geqslant q$, $n \in \mathbb{N}$. The Laplace transform of a fractional integral expression $[p > 0]$ is very convenient:

$$ \int_0^\infty \mathrm{d}t\, e^{-ut} \frac{\mathrm{d}^{-p}}{\mathrm{d}t^{-p}} f(t) = u^{-p} f(u) \tag{A.3} $$

where $f(u)$ is the Laplace transform of $f(t)$ [37].

Appendix B: Derivation of the Cattaneo equation from a local model

Here we consider a way of deriving the Cattaneo equation from a discrete and local jump model, i.e. from statistical considerations [11]. To this end, we start off from the discrete random walk step

$$ p_j(t+\tau) = a p_{j-1}(t) + b p_{j+1}(t) , \tag{B.1} $$

where a and b with $a+b=1$ describe the probability to go left and right, and introducing a continuum limit for small time steps τ and small step lengths Δx in the form:

$$p_j(t+\tau) \sim p(x,t) + \tau \dot{p}(x,t) + \frac{\tau^2}{2}\ddot{p}(x,t)\,, \tag{B.2a}$$

$$ap_{j-1}(t) \sim ap(x,t) - (\Delta x)ap' + \frac{(\Delta x)^2}{2}ap''\,, \tag{B.2b}$$

and a similar expression for $bp_{j+1}(t)$. Taking into account second-order expansions in both time and space, the extended Cattaneo equation

$$\dot{p} - \frac{\tau}{2}\ddot{p} = \left[-f\frac{\partial}{\partial x} + K\frac{\partial^2}{\partial x^2}\right] p(x,t)\,, \tag{B.3}$$

is recovered straightforwardly, and includes the force and diffusion coefficients

$$f = \lim_{\Delta x \to 0} \frac{\Delta x}{\tau}(a-b)\,, \tag{B.4a}$$

$$K = \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{\tau}\,. \tag{B.4b}$$

Of course, in the symmetric case $a=b$, the force term vanishes, and we are led back to the standard Cattaneo equation (1). This persistent random walk approach is not suited for the introduction of broad waiting time distributions however, as it contains a local formulation. Also, the usual way of taking the limit and considering second-order expansions in time may seem artificial.

Appendix C: Derivation of the Cattaneo equation from a Fokker–Planck equation

Another approach goes back to a model discussed by Davies [35], where a two-variable Fokker–Planck equation is introduced and a Cattaneo equation derived. Considering the two-variable equivalent of a recently introduced fractional Fokker–Planck equation [36] for the distribution function $W(x,v,t)$ in coordinate-velocity phase space

$$\dot{W}(x,v,t) = {}_0D_t^{1-\gamma}\left[-vW' + \frac{\partial}{\partial v}\left[\beta v - K(x)\right]W + \frac{k_B T\beta}{m}\frac{\partial^2}{\partial v^2}W\right]\,, \tag{C.1}$$

and following Davies' steps, we derive a generalised continuity equation $\dot{P}(x,t) = -{}_0D_t^{1-\gamma}\bar{v}P$, and consequently the generalised Cattaneo equation

$${}_0D_t^{2\gamma}P + \beta\,{}_0D_t^{\gamma}P - \frac{t^{-\gamma}\delta(x)}{\Gamma(1-\gamma)} - \frac{t^{-2\gamma}\delta(x)}{\Gamma(1-2\gamma)} = -\frac{\partial}{\partial x}K(x)P + \frac{\partial^2}{\partial x^2}P\langle v^2\rangle\,. \tag{C.2}$$

However, this equation does not lead to a finite velocity of propagation for small times, as can be readily verified. In the force-free, the mean-squared displacement reads

$$\langle x^2\rangle \sim \begin{cases} \dfrac{2t^{2\gamma}}{\Gamma(1+2\gamma)}, & t \ll 1\,, \\[2ex] \dfrac{2t^{\gamma}}{\Gamma(1+\gamma)}, & t \gg 1\,. \end{cases} \tag{C.3}$$

Therefore, this approach is not suitable for the description we have in mind, as well, as it does not lead to a finite velocity of propagation for small times, compare the discussion of the model GCE I in Ref. [26].

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