Fractional diffusion, waiting-time distributions, and Cattaneo-type equations

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We discuss a generalized diffusion equation resulting from an additive two-state process, in combination with an asymptotically fractal (asymptotic power-law) waiting-time distribution. The obtained equation is an extension to previously discussed fractional diffusion equations. Our description leads to a mean squared displacement which describes enhanced, subballistic transport for long times. The short time behavior, however, is of a ballistic nature. This separation into two domains results from the introduction of a time scale through the asymptotically fractal waiting-time distribution. This is also mirrored by the observation that, for small times, our generalized diffusion equation reduces to the standard Cattaneo equation. The asymptotic probability density is of compressed Gaussian type, and thus differs from the Lévy tail generally found for these kinds of processes. [S1063-651X(98)01606-7]

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I. INTRODUCTION

Fractional differential equations have been extensively shown to be a well-suited tool for the description of anomalous relaxation and diffusion processes in complex systems [1–3]. On the other hand, continuous time random walk (CTRW) theory has been extraordinarily successful in modeling similar processes, in an even wider range of physical parameters [4–9]. It is thus an interesting question whether there exists a connection between both models. Compte [8] established fractional diffusion equations where the fractional time derivative mirrors the anomalous waiting-time distribution, and the fractional spatial derivative refers to a power-law jump length distribution.

Recently, West *et al.* [10] considered an additive stochastic differential equation (see van Kampen [11])

$$\dot{x}(t)=\xi(t) \tag{1}$$

for the two-state process $\xi(t)$, which takes on the values $\pm w$. In phase space, the evolution equation corresponding to the stochastic Eq. (1) is given by

$$\frac{\partial\varphi(x,\xi,t)}{\partial t}=\left(-\hat{\xi}\frac{\partial}{\partial x}+\hat{\Gamma}\right)\varphi(x,\xi,t), \tag{2}$$

where $\varphi(x,\xi,t)dx\,d\xi$ is the probability that the dynamical variables $x(t)$ and $\xi(t)$ take on values in the intervals $(x,x+dx)$ and $(\xi,\xi+d\xi)$, respectively. (For this discussion, also see Ref. [5].) The operator $\hat{\Gamma}$ defines the actual dynamics of the stochastic process x, which is described through ξ. The two-state operator $\hat{\xi}$, according to the basic assumption, has the eigenvalues $\pm w$, i.e., transforms between the ''up'' and ''down'' states. Following a projection operator formalism, the integrodifferential equation

$$\frac{\partial\sigma_0(x,t)}{\partial t}=\int_0^t dt\mathcal{P}\hat{\xi}\mathcal{Q}e^{-\hat{\Gamma}(t'-t)}\mathcal{Q}\hat{\xi}\frac{\partial^2\sigma_0(x,t')}{\partial x^2} \tag{3}$$

for the spatial distribution function $\sigma_0(x,t)$ of the random walk, described by the stochastic variable $x(t)$, is recovered [see Eq. (7) of Ref. [10]]. Here, $\mathcal{P}^2=\mathcal{P}$ and $\mathcal{P}+\mathcal{Q}=1$ map φ onto distributions in (x,t) space. This means that $\mathcal{P}$ averages φ over the configurational space of $\hat{\xi}$.

It can also be proven that $\mathcal{P}\hat{\xi}\mathcal{Q}e^{-\hat{\Gamma}(t'-t)}\mathcal{Q}\hat{\xi}\equiv\langle\xi(t)\xi(t')\rangle$, namely, the kernel in Eq. (3) is the two-time correlation function of $\xi(t)$. We do not hesitate to remark that, due to the left hand side of this identity, the correlation function depends on the time difference, exclusively, i.e., $\langle\xi(t)\xi(t')\rangle=f(t-t')$. One finally arrives at the evolution equation

$$\frac{\partial\sigma_0(x,t)}{\partial t}=\int_0^t dt'\langle\xi(t)\xi(t')\rangle\frac{\partial^2\sigma_0(x,t')}{\partial x^2} \tag{4}$$

of the two-state process (1). Introducing the normalized equilibrium correlation function

$$\Phi_\xi(t)=\frac{\langle\xi(0)\xi(t)\rangle}{\langle\xi^2\rangle}, \tag{5}$$

the integral in Eq. (4) is a Laplace convolution. After a Laplace transform, Eq. (4) becomes

$$u\sigma_0(x,u)-\sigma_0(x,0)=\langle\xi^2\rangle\Phi_\xi(u)\frac{\partial^2\sigma_0(x,u)}{\partial x^2}, \tag{6}$$

where $\sigma_0(x,0)=\delta(x)$ is the initial value. For a finite characteristic time, which is defined via the equilibrium correlation function according to

$$\tau=\int_0^\infty dt\ \Phi_\xi(t), \tag{7}$$

one is led to a Gaussian diffusion process [5,6,8].

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The generalization to anomalous diffusion processes is based on the connection

$$\Phi_\xi(t)=\frac{\int_t^\infty dt'(t-t')\psi(t')}{\int_0^\infty dt' t' \psi(t')}, \tag{8}$$

established by Geisel and co-workers [12], relating the correlation function $\Phi_\xi(t)$ to the waiting-time distribution $\psi(t)$ used in CTRW theory. The probability that $\xi(t)$ makes a transition at a given time t is given by $\psi(t)$. Thus one is able to model the two-state stochastic process (1) in terms of CTRW theory. A comparison to Ref. [13] shows that Φ_ξ corresponds to the velocity-velocity correlation function.

II. FRACTIONAL DIFFUSION EQUATIONS

Due to Eq. (8), one can now introduce a waiting-time distribution which will, through inserting Φ_ξ into Eqs. (4) or (6), determine the structure of the resulting equation for σ_0. In Ref. [10], the power-law waiting-time distribution

$$\psi(t)\sim t^{-(\gamma+1)}, \quad 1<\gamma<2 \tag{9}$$

is considered. The restriction of γ to be larger than 1 is due to the convergence of the integrals in Eq. (8). Comparing the resulting equation with the definition of fractional operators (see the Appendix), one recovers the fractional diffusion equation

$$\frac{\partial\sigma_0(x,t)}{\partial t}=C\frac{\partial^{\gamma-2}}{\partial t^{\gamma-2}}\frac{\partial^2\sigma_0(x,t)}{\partial x^2}, \tag{10}$$

which, after applying $\partial^{2-\gamma}/\partial t^{2-\gamma}$, leads to the generalized equation

$$\frac{\partial^{3-\gamma}\sigma_0(x,t)}{\partial t^{3-\gamma}}=C\frac{\partial^2\sigma_0(x,t)}{\partial x^2}. \tag{11}$$

This equation is similar to equations derived from CTRW theory by Compte [8]. There, however, we encounter a derivative of order between 0 and 1, whereas in Eq. (11) we have $1<(3-\gamma)<2$; this gives rise to the intermediate transport $\langle x^2\rangle(t)\sim t^{3-\gamma}$, i.e., enhanced sub-ballistic transport. [We note in passing that Eq. (11) corresponds to Eq. (20) in Ref. [10], where on the right hand side β should be replaced by $\beta-1$, a trivial misprint which does not affect the further procedure there.] Of course, $\psi(t)\sim t^{-(\gamma+1)}$ diverges for $t\to 0$, and so does $\Phi_\xi(t)\sim t^{-(\gamma-1)}$.

III. ASYMPTOTICALLY FRACTAL WAITING-TIME DISTRIBUTION (1)

To overcome the divergence of Φ_ξ in the model, West *et al.* introduced the asymptotically fractal waiting-time distribution [10]

$$\psi(t)=\frac{A}{(B+t)^{\gamma+1}}, \quad B>0, \tag{12}$$

and the space-time coupling via a δ function as constraint:

$$\sigma_0(x,t-t')=\frac{1}{2w}\int_{-\infty}^{\infty}dx'\,\delta\left(t'-\frac{|x-x'|}{w}\right)\sigma_0(x',t). \tag{13}$$

Some comments concerning Eq. (12) are in order. Pure fractals or power laws do not possess a characteristic length or time: $f(\lambda t)=\lambda^\mu f(t)$, i.e., the change of length or time scale by a factor of λ simply changes the scale of the result, characterized by a power of μ. This is, of course, fulfilled for the waiting-time distribution given in Eq. (9). In the case of Eq. (12), however, this is only true for $t\gg B$. Thus the parameter B which has the dimension of time limits the fractal region for small times. For such a behavior, which is quite common in nature, Rigaut coined the notion of asymptotic fractals [14]; see also Refs. [15,16].

Let us briefly discuss the consequence of the step of introducing the asymptotically fractal waiting-time distribution (12) and the constraint (13). Equation (11), which is intimately connected to the integrodifferential Eq. (4), includes a second-order spatial derivative leading to a modified exponential decrease of the solution $\sigma_0(x,t)$. This can be seen most easily in its Fourier-Laplace transform

$$\sigma_0(k,u)=\frac{1}{u+A\langle\xi^2\rangle u^{\gamma-2}k^2}, \tag{14}$$

where the second power of k occurs. Reminding the calculation of the moments via the characteristic function through

$$\langle x^n\rangle(u)\lim_{k\to 0}\frac{d^n\sigma_0(k,u)}{i^n dk^n}, \tag{15}$$

one recognizes immediately that all odd moments vanish while all even moments are finite, thus causing the modified exponential decay [11]. On the other hand, the fractional derivative in Eq. (11) leads to a divergence of the characteristic time τ in Eq. (7) [6,8].

The modifications introduced in Ref. [10], i.e., the waiting-time distribution (12) and the constraint (13), on the other hand, lead to the characteristic function [see Eq. (54) in Ref. [10]]

$$\sigma_0(k,u)=\frac{1}{u+b|k|^\gamma}. \tag{16}$$

Relation (16) gives rise to a finite characteristic time τ, but an infinite variance $\langle x^2\rangle$, see the discussion in Ref. [6]. In other words, Eq. (16) leads to the well-known Lévy tail asymptotics for enhanced sub-ballistic transport [17,18], see below.

Equation (14), i.e., the original model, describes a physical situation where very large jumps of the random walker are ‘‘penalized’’ by a longer time cost. [Compare Eq. (14) with Ref. [6].] This can, in CTRW theory, be achieved via the coupling $\psi(x,t)=p(t|x)\lambda(x)$ in the jump probability $\psi(x,t)$, where the conditional probability $p(t|x)$ selects the waiting time in accordance to a given jump length; see Ref. [6]. In the constant velocity model, this is equivalent to the choice $\psi(x,t)=p(x|t)\psi(t)$ due to the δ function. Such a classification according to the different CTRW categories listed in Ref. [6] is unequivocal. Equation (16), on the other hand, is equivalent to a decoupled model $\psi(x,t)=\psi(t)\lambda(x)$, with a power-law jump length distribution. In this model, even very large jumps are allowed to occur, a situation which is often referred to as Lévy flights, in contrast to Lévy walks like Eq. (14).

Moreover, there is no longer a fractional diffusion equation of the type (4). Instead, as Compte showed, this leads to a Riesz derivative substituting the second-order spatial derivative, see [8].

IV. ASYMPTOTICALLY FRACTAL WAITING-TIME DISTRIBUTION (2)

In the following, we show that the introduction of the asymptotically fractal waiting-time distribution (12) suffices to correct the unphysical behavior for $\Phi_\xi(t)$. We calculate an exact expression for $\Phi_\xi(u)$, and discuss the consequences for the mean squared displacement. An important result is that the modified Gaussian decay is preserved, which is typical for anomalous diffusion phenomena; see Refs. [7,19,20] and the modeling in Ref. [3]. Thus the asymptotic fractal (12) does not alter the overall shape of the solution. This is, of course, due to the fact that, in Eq. (4), the second-order differentiation in space is not altered, again leading to a Fourier-Laplace transform of the form (14) in the asymptotic limit. $\langle x^2\rangle$ of this process is finite.

On the other hand, West *et al.* remarked that both Zumofen and Klafter [17] and Trefàn *et al.* [18] ended up in processes that are characterized by Lévy statistics, not by an exponential in the asymptotic limit in (x,t) space. This is a point where the description through a fractional diffusion equation and the CTRW theory differ. Both lead to a stretched Gaussian for dispersive transport. But CTRW theory shows a Lévy tail in the probability distribution of the enhanced sub-ballistic case, whereas our model presented here has a compressed exponential as its asymptotic behavior. However, both approaches have the same scaling of the variance, i.e., $\langle x^2\rangle(t)\sim t^{3-\gamma}$ in this range. This observation leads us to the consideration of expression (12), without coupling (13), as an alternative solution, preserving the existence of $\langle x^2\rangle$.

Now taking Eq. (12) and inserting it into Eq. (8), we arrive at $\langle\xi^2\rangle=(AB^{1-\gamma})/(\gamma^2-\gamma)$. The new correlation function is given by

$$\Phi_\xi(t)=\frac{B^{\gamma-1}}{(B+t)^{\gamma-1}}. \tag{17}$$

We observe that, for $t\gg B$, we recover the original result $\Phi_\xi(t)\sim t^{1-\gamma}$ of West *et al.* [10]. On the other hand, for $t\ll B$, the correlation is unity, as it should be.

To compute the equivalent of Eq. (6), we need the Laplace transform of $\Phi_\xi(t)$. The result is (Refs. [21–23])

$$\Phi_\xi(u)=B^{\gamma-1}e^{Bu}u^{\gamma-2}\Gamma(2-\gamma,Bu), \tag{18}$$

where $\Gamma(2-\gamma,Bu)$ denotes the incomplete γ function $\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}dt$. The asymptotic expansions of $\Gamma(a,t)$ are given in the Appendix. This result is to be compared with Ref. [13].

From the characteristic function (14), we compute the mean squared displacement

$$\langle x^2\rangle(u)=2\langle\xi^2\rangle u^{-2}\Phi_\xi(u) \tag{19}$$

in Laplace space. For the limiting cases of large and small times, it follows (see the Appendix for the expansions) that

$$\langle x^2\rangle(t)\sim 2\langle\xi^2\rangle B^{\gamma-1}\Gamma(2-\gamma)\times\left(\frac{t^{3-\gamma}}{\Gamma(4-\gamma)}-\frac{B^{2-\gamma}}{\Gamma(3-\gamma)}t+\cdots\right),\quad t\gg B, \tag{20a}$$

$$\langle x^2\rangle(t)\sim 2\langle\xi^2\rangle\left(\frac{t^2}{2}-\frac{\gamma-1}{6B}t^3+\cdots\right),\quad t\ll B. \tag{20b}$$

Thus, in the long-time limit, we recover the former result, Eq. (44) in Ref. [10], in the first order of the expansion in Eq. (20b), as it should be. The next term gives a negative correction. We observe that $\langle x^2\rangle\sim t^{3-\gamma}$ corresponds to enhanced transport between normal and ballistic transport, i.e., intermediate transport; see above. For small times, however, we find purely ballistic transport, $\sim t^2$, as it is normally recovered for Cattaneo-type approaches where a modified constitutive equation is assumed; see Ref. [24]. A further characteristic of Eq. (20b) is that γ appears only in the coefficients of the higher-t powers. Thus the short-time region is not affected by the anomalous behavior, as is to be expected for the choice of $\psi(t)$ according to Eq. (12), which is constant for small times.

V. CATTANEO-TYPE EQUATION

A typical feature for the diffusion equation, be it standard or of fractional order in time, is the infinitely fast propagation; i.e., even for very small times, there is a finite portion of the probability density for very large $|x|$. This is due to the Gaussian (or modified Gaussian) structure of the solution. Mathematically speaking, this is due to the fact that the diffusion equation is a *parabolic* partial differential equation. To avoid this, Cattaneo [25] proposed in 1948 his *hyperbolic* modified diffusion equation, which is of a telegrapher's equation type. His modified constitutive equation introduces a relaxation of the flux. A detailed discussion is found in Refs. [24,26].

To underline the relation to the Cattaneo formalism, let us introduce the extension of Eq. (11) for the waiting-time distribution (12). In order to obtain a differential equation where the spatial and temporal derivatives are separated, we divide by the correlation function Φ_ξ in Eq. (6), and expand the fraction. For small times (i.e., large u), we end up, to second order, with the Cattaneo equation

$$\frac{\partial^2\sigma_0(x,t)}{\partial t^2}+\frac{\gamma-1}{B}\frac{\partial\sigma_0(x,t)}{\partial t}=\langle\xi^2\rangle\frac{\partial^2\sigma_0(x,t)}{\partial x^2}, \tag{21}$$

leading to the short-time mean squared displacement (20b), as would be expected [24]. The solution to Eq. (21) is given by Ref. [27] (t and x are rescaled to keep the notation as simple as possible):

$$\sigma_0(x,t)=\frac{e^{-t}}{2}\left[\left\{I_0(\sqrt{t^2-x^2})+\frac{t}{\sqrt{t^2-x^2}}I_1(\sqrt{t^2-x^2})\right\}\times\chi_{-t,t}(x)+\delta(x+t)+\delta(x-t)\right], \tag{22}$$

where $\chi_{-t,t}(x)=\Theta(x+t)-\Theta(x-t)$ is the rectangle function. Note the occurrence of the wave variable $x\pm t$ and the typical variable $\sqrt{t^2-x^2}$ of the telegrapher's equation. For Eq. (21), one can show that, for small times, the propagation velocity is finite, namely.

$$v=\sqrt{\langle\xi^2\rangle} \tag{23}$$

in our notation. (For the definition of v, see Refs. [24,26].)

To end up with an equation for large t, we neglect $\sigma_0(x,0)$ in Eq. (6), divide by Φ_ξ, and finally expand this expression for small u, i.e., large t. The resulting equation reads

$$\frac{\partial^{3-\gamma}\sigma_0(x,t)}{\partial t^{3-\gamma}}+\frac{B^{2-\gamma}}{\Gamma(3-\gamma)}\frac{\partial^{5-2\gamma}\sigma_0(x,t)}{\partial t^{5-2\gamma}}+\cdots+B\frac{\partial^2\sigma_0(x,t)}{\partial t^2}=K\frac{\partial^2\sigma_0(x,t)}{\partial x^2}. \tag{24a}$$

The constant on the right hand side is given by $K=B^{\gamma-1}\Gamma(2-\gamma)\langle\xi^2\rangle$. Compare this equation with the GCE II model in Ref. [26]. The dots in Eq. (24a) indicate the higher orders of the expansion, which we do not write down explicitly. In the limit $\gamma\to 2$, both of the first terms reduce to a first-order temporal derivative, as do the many higher-order terms symbolized by the dots. However, at next higher order, the second-order derivative remains unchanged. Thus for $\gamma\to 2$, calculating all the occurring summations, one recovers a standard Cattaneo equation. This complicated behavior is due to the series expansion of $\Phi_\xi(t)$ involved.

Finally, rewriting Eq. (24a) in terms of $\beta=\gamma-1$, i.e., $0<\beta<1$,

$$\frac{\partial^{2-\beta}\sigma_0(x,t)}{\partial t^{2-\beta}}+\frac{B^{1-\beta}}{\Gamma(2-\beta)}\frac{\partial^{3-2\beta}\sigma_0(x,t)}{\partial t^{3-2\beta}}+\cdots+B\frac{\partial^2\sigma_0(x,t)}{\partial t^2}=K\frac{\partial^2\sigma_0(x,t)}{\partial x^2}, \tag{24b}$$

it becomes clear that the lowest order of the time derivatives, $2-\beta$, describes enhanced transport in the intermediate region. Equations (24a) and (24b) may be regarded as anomalous extensions of the Cattaneo equation. Generalized Cattaneo equations of fractional order were extensively discussed in Ref. [26]. Both limiting Cattaneo equations (21) and (24a) reconstitute the standard Cattaneo equation for $\gamma\to 2$. We note in passing that Eqs. (24a) and (24b) reduce to the result of Ref. [10] for $B\to 0$.

VI. CONCLUSION

We observe that a nondiverging (''nonpathological'') choice of the waiting-time distribution [Eq. (12)], leads to a distribution function for which all moments are finite. We cannot calculate an exact solution for Eqs. (24a) and (24b). The asymptotic shape in the extreme diffusion limit $t\gg B$, however, can be calculated from a truncation in the $1/\Phi_\xi$ expansion, leading to the well-known result

$$\sigma_0(x,t)\sim\frac{2t^{1/2-1/\beta}}{\sqrt{K}(2-\beta)}\left(\frac{|x|}{\sqrt{K}}\right)^{1/\beta-1}\times\exp\left\{-c(\beta)\left(\frac{|x|}{\sqrt{K}t^{1-\beta/2}}\right)^{2/\beta}\right\}, \tag{25}$$

where $c(\beta)=2^{-2/\beta}\beta(2-\beta)^{2/\beta-1}$, matching Eq. (39) in Ref. [10], as it should. Of course, Eq. (25) leads back to a standard Gaussian for the limiting case $\beta\to 1$.

We have demonstrated that the asymptotically fractal waiting-time distribution (12) gives rise to a description of enhanced sub-ballistic diffusion for $t\gg B$. For small times $t\ll B$, however, the correlation function approaches unity which causes the ballistic transport, i.e., the Cattaneo-type behavior. Generalized Cattaneo equations (21), (24a), and (24b) can be deduced, valid for small and large time, respectively. This behavior, i.e., separation into $t\ll B$ and $t\gg B$, is due to the introduction of a microscopic time scale B, which separates two different domains on the time arrow.

The Cattaneo picture found for small times can be understood in physical terms as the movement of gas particles moving ballistically and independently of each other, before collisions come into play (which is the case for times of the order of the mean collision time and longer), see the connections to the Boltzmann equation [28].

This result contrasts with the alternative Lévy approach of West *et al.* in Ref. [10], as in our case the mean squared displacement is kept finite and the modified exponential solution is preserved. Our finding is asymptotically equivalent to the coupled case describing enhanced transport in Ref. [6], and it can still be described by a modified diffusion equation. It thus refers to physical situations where very large jumps are ''punished'' (Lévy walks). Thus, while the mean squared displacement of both approaches leads to the same result, the Lévy tail of the CTRW dynamical approach differs from the compressed exponential found here.

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APPENDIX: FRACTIONAL CALCULUS

Fractional calculus ideas date back to the days when classical calculus came of age. These ideas were first expressed in some letters between de l'Hospital and Leibniz in 1695.

Today's definitions are mainly basing on the works of Liouville and Riemann published in the last century [29–32].

Today, there exists an immense variety of definitions of fractional operators, see the compendium of Samko, Kilbas, and Marichev (Ref. [31]). The most common definition, however, goes back to the Riemann definition

$$ {}_{t_0}D_t^{-p}f(t)=\frac{1}{\Gamma(p)}\int_{t_0}^{t}d\tau\,\frac{f(\tau)}{(t-\tau)^{1-p}}, \tag{A1}$$

extending Cauchy's multiple integral for arbitrary complex p with $\mathrm{Re}(q)>0$, by use of the γ function. Herein, we are led to the Riemann–Liouville fractional calculus, i.e., $t_0=0$:

$$ {}_{0}D_t^{-p}f(t)=\frac{1}{\Gamma(p)}\int_{0}^{t}d\tau\,\frac{f(\tau)}{(t-\tau)^{1-p}} \tag{A2}$$

for $\mathrm{Re}(p)>0$. A derivative of order q, $q>0$, is consequently established via the definition

$$\frac{d^q}{dt^q}f(t)\equiv{}_0D_t^q f(t)=\frac{d^n}{dt^n}{}_0D_t^{q-n}f(t), \tag{A3}$$

where $n\geqslant q$, $n\in\mathbb{N}$ is a natural number. Here, also, we introduce the shorthand notation d^q/dt^q used in the text which we use for both $q<0$ and $q>0$, in the above spirit.

The Laplace transform of a fractional integral expression is very convenient:

$$\int_0^\infty dt\ e^{-ut}\frac{d^{-q}}{dt^{-q}}f(t)=u^{-q}f(u), \tag{A4}$$

where $f(u)$ is the Laplace transform of $f(t)$ [29].

Often, linear fractional equations can be solved by use of Fox functions, or transformations can be given exactly in terms of Fox functions. In the case of generalized Cattaneo equations, no exact solutions can be given. Even the solution of the standard Cattaneo equation becomes quite intricate [27]. We are thus limited to the discussion of $\sigma_0(k,u)$ and asymptotic cases in (x,t) space, and $\langle x^2\rangle$.

For the calculations in this paper, we need the asymptotic expansion of the incomplete γ function $\Gamma(a,t)$:

$$\Gamma(a,x)=\Gamma(a)-\gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}dt \tag{A5}$$

$$=\begin{cases} x^{a-1}e^{-x}\left[1+\dfrac{a-1}{x}+\dfrac{(a-1)(a-2)}{x^2}+\cdots\right], & x\to\infty\\ \Gamma(a)\left[1-e^{-x}\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(a+n+1)}\right], & |x|<\infty, \end{cases} \tag{A6}$$

where the last expression is obtained via the connection

$$\Gamma(a,x)=\Gamma(a)[1-x^a\gamma^*(a,x)] \tag{A7}$$

relating $\Gamma(a,x)$ and $\gamma^*(a,x)$. $\gamma^*(a,x)$ is single valued in a and x, and shows no finite singularities [23].

For completeness, we finally mention the well-known Taylor series

$$\frac{1}{1-x}=1+x+x^2+O(x^3). \tag{A8}$$

[1] C. Friedrich and H. Braun, Rheol. Acta **31**, 309 (1992); W. G. Glöckle and T. F. Nonnenmacher, Macromolecules **24**, 6426 (1991); J. Stat. Phys. **71**, 741 (1993); Rheol. Acta **33**, 337 (1994); R. Metzler, W. Schick, H.-G. Kilian, and T. F. Nonnenmacher, J. Chem. Phys. **103**, 7180 (1995); H. Schiessel, R. Metzler, A. Blumen, and T. F. Nonnenmacher, J. Phys. A **28**, 6567 (1995).

[2] R. Metzler, W. G. Glöckle, T. F. Nonnenmacher, and B. J. West, Fractals **5**, 597 (1997).

[3] W. R. Schneider and W. Wyss, J. Math. Phys. **30**, 134 (1989); M. Giona and H. E. Roman, Physica A **185**, 87 (1992); R. Metzler, W. G. Glöckle, and T. F. Nonnenmacher, *ibid.* **211**, 13 (1994); R. Metzler and T. F. Nonnenmacher, J. Phys. A **30**, 1089 (1997).

[4] E. W. Montroll and G. H. Weis, J. Math. Phys. **6**, 167 (1995); E. W. Montroll and H. Scher, J. Stat. Phys. **9**, 101 (1973); H. Scher and E. W. Montroll, Phys. Rev. B **7**, 4491 (1973); M. F. Shlesinger, J. Stat. Phys. **10**, 421 (1974).

[5] J. Klafter and R. Silbey, Phys. Rev. Lett. **44**, 55 (1980).

[6] J. Klafter, A. Blumen, and M. F. Shlesinger, Phys. Rev. A **35**, 3081 (1987).

[7] J. Klafter, G. Zumofen, and A. Blumen, J. Phys. A **25**, 4835 (1991).

[8] A. Compte, Phys. Rev. E **53**, 4191 (1996).

[9] A. Compte, Phys. Rev. E **55**, 6821 (1997); A. Compte, R. Metzler, and J. Camacho, Phys. Rev. E **56**, 1445 (1997).
[10] B. J. West, P. Grigolini, R. Metzler, and T. F. Nonnenmacher, Phys. Rev. E **55**, 99 (1997).
[11] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1983).
[12] T. Geisel, J. Heldstab, and H. Thomas, Z. Phys. B **55**, 165 (1984); T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984); T. Geisel, J. Nierwetberg, and A. Zacherl, *ibid.* **54**, 616 (1985).
[13] G. Zumofen and J. Klafter, Physica D **69**, 436 (1993).
[14] J. P. Rigaut, J. Microsc. **133**, 41 (1984).
[15] J. Dollinger, R. Metzler, and T. F. Nonnenmacher, J. Phys. A (to be published).
[16] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).
[17] G. Zumofen and J. Klafter, Phys. Rev. E **47**, 851 (1993).
[18] G. Trefàn, E. Floriani, B. J. West, and P. Grigolini, Phys. Rev. E **50**, 2564 (1994).
[19] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).
[20] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[21] F. Oberhettinger and L. Badii, *Tables of Laplace Transforms* (Springer-Verlag, Berlin, 1973).
[22] *Higher Transcendental Functions*, edited by A. Erdélyi, Bateman Manuscript Project Vol. I (McGraw-Hill, New York, 1974).
[23] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
[24] J. Casas-Vàzquez, D. Jou, and G. Lebon, *Extended Irreversible Thermodynamics* (Springer-Verlag, Berlin, 1996).
[25] G. Cattaneo, Atti. Sem. Mat. Fis. Univ. Modena **3**, 83 (1948).
[26] A. Compte and R. Metzler, J. Phys. A **30**, 7277 (1997).
[27] P. C. de Jagher, Physica A **101**, 629 (1980).
[28] T. F. Nonnenmacher, J. Non-Equilib. Thermodyn. **9**, 171 (1984).
[29] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic, New York, 1974).
[30] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993).
[31] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integral and Derivatives—Theory and Applications* (Gordon and Breach, New York, 1993).
[32] T. F. Nonnenmacher and R. Metzler, Fractals **3**, 557 (1995); *Fractal Geometry and Analysis, The Mandelbrot Festschrift, Curaçao 1995*, edited by C. J. G. Evertsz, H.-O. Peitgen, and R. F. Voss (World Scientific, Singapore, 1996).