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"FRACTIONAL TUNING" OF THE RICCATI EQUATION

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Abstract

The dynamics of a free–falling body in complex materials such as a polymer fluid is phenomenologically modeled using a fractional generalization of the Riccati equation. The solution exhibits a rich behavior in its parametric dependence, and unlike normal free–fall there is no terminal velocity, instead a power–law increase in time is obtained. Within this approach the fractional order allows to tune the resulting equation.

1. INTRODUCTION

Every person who has jumped from an aeroplane with a parachute recognizes that he reaches a terminal velocity after falling for some interval of time. The hydrodynamic drag of the air being quadratic in the speed of the jumper eventually balances the attractive force of gravity, and the jumper descends at constant speed. This phenomenon is described by the Riccati equation.[1]

Riccati–type equations play a prominent role in the mathematical modeling of non–linear phenomena. They arise, for example, in the context of ecological systems and chemical reactions, in addition to jumping from aeroplanes. Many non–linear partial differential equations — via similarity reductions — can be mapped on a Riccati–type equation.[2] Because of its simple structure and widespread applications, the Riccati equation represents one of the most often studied non–linear differential equations.

Suppose, however, that we wanted to study bungee cord jumping rather than parachuting. One thing we would want to know is whether the cord would exceed its elastic limit and "creep" to an increased length. For this phenomenon we would expect to see a modification of the quadratic velocity dependence observed in the Riccati equation. Herein we use the fractional calculus to generalize the Riccati equation.

The fractional calculus has become a powerful tool in mathematical physics, generalizing well–known standard linear equations like the relaxation,[3,4] diffusion[5–7] or wave equations.[5] There, the involved convolution in time significantly enriches the class of solutions to be appropriate to describing dynamics in complex systems.[8] Also, non–linear Abel–Volterra type equations have been discussed recently.[9]

Here, we study the purely phenomenological introduction of a fractional operator into the Riccati equation. However, we do not solve a non–linear equation of fractional order. Instead, it is possible to generalize the ordinary differential equation, onto which the Riccati equation can be mapped, to fractional order, and solve it. The introduced fractional order is a kind of a tuning parameter that allows the transition from the description of parachuting over plastic creep to the free fall problem.

2. FALL IN RESISTANT MEDIUM

For the motion of a body of mass m free–falling in a homogeneous potential, exerting a constant acceleration g on the body, in a medium where the resistance varies as the square of the velocity with the proportionality constant K, the equation of motion is

$$m\frac{\mathrm{d}^2 s(t)}{\mathrm{d}t^2} = mg - K\left(\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right)^2 . \tag{1}$$

The distance coordinate is $s(t)$ and Eq. (1) is usually called the Riccati equation. In terms of the particle velocity, Eq. (1) can be re–written:

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = mg - Kv(t)^2 . \tag{2}$$

The solution to Eq. (2) is readily obtained in the form

$$v(t) = V\frac{v_0^* + \tanh rt}{1 + v_0^* \tanh rt} \tag{3}$$

where $V = \sqrt{mg/K}$, $r = g/V$ and $v_0^* = v_0/V$ is the limit speed of the body for large t.[1]

We are now interested in generalizing the Riccati equation [Eq. (2)] to account for the anomalous dynamics caused by the structure of the material through which the body is possibly falling. Following a purely phenomenological approach we do this through the introduction of a fractional differential operator. To accomplish this generalization we re–write Eq. (2) in a reduced form

$$\dot{v}(t) + v(t)^2 = M \tag{4}$$

where $M = mg/K$, and the factor m/K before $\dot{v}$ is incorporated into a scaled time.

Consider now the well–known transformation

$$w(t) = \exp y(t) \tag{5}$$

so that the combination of Eq. (5) with the ordinary differential equation

$$\ddot{w}(t) - Mw(t) = 0 \tag{6}$$

is equivalent to Eq. (4) with the identification

$$v(t) = \dot{y}(t) \qquad \text{or} \qquad y(t) = \frac{\mathrm{d}^{-1}v(t)}{\mathrm{d}t^{-1}} . \tag{7}$$

By use of this, Eqs. (6,7) can then be generalized by the introduction of a fractional operator

$$y(t) = \frac{\mathrm{d}^{-\alpha}v(t)}{\mathrm{d}t^{-\alpha}} \tag{8}$$

for $\alpha \in (0,1)$, see also the Appendix. Thus for $\alpha = 1$ we have the standard Riccati equation [Eq. (4)], and the corresponding fractional Riccati equation, obtained by substituting Eq. (5) into Eq. (6) and using the generalization Eq. (8), reads:

$$\frac{\mathrm{d}^{2-\alpha}v(t)}{\mathrm{d}t^{2-\alpha}} + \left(\frac{\mathrm{d}^{1-\alpha}v(t)}{\mathrm{d}t^{1-\alpha}}\right)^2 = M . \tag{9}$$

Equation (9) is of a purely phenomenological nature. The interpretation as a Newton equation with a quadratic force term is no longer valid. However, we show in the subsequent discussion that it provides an interesting description of the anomalous creep region in between the known Riccati approach to resistive fall, and the standard free fall problem.

3. SOLUTION OF THE FRACTIONAL RICCATI EQUATION

The solution of Eq. (9) can be obtained by first solving Eq. (6) in terms of

$$w(t) = A\exp\sqrt{M}\,t + B\exp -\sqrt{M}\,t \tag{10}$$

and recovering $y(t) = \log w(t)$. Thus, to obtain the velocity of the falling body, one inverts Eq. (8) and is confronted with the task of computing

$$v(t) = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \log\left(A \exp\sqrt{M}t + B\exp -\sqrt{M}t\right) . \quad (11)$$

By factoring Eq. (11) we arrive at the three separate terms

$$\log\left(A \exp\sqrt{M}t + B\exp -\sqrt{M}t\right) = \log B - \sqrt{M}t + \log\left(1 + \frac{A}{B}\exp 2\sqrt{M}t\right) . \quad (12)$$

We now use the property of fractional derivatives,

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} t^\gamma = \frac{t^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}, \quad (13)$$

to write the velocity in Eq. (11) as

$$v(t) = \frac{\log B}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{\sqrt{M}}{\Gamma(2-\alpha)} t^{1-\alpha} + \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \log\left(1 + \frac{A}{B}\exp 2\sqrt{M}t\right) \quad (14)$$

which cannot be reduced to a closed form. Taking into account the chain rule for fractional derivatives (see the Appendix) one arrives at the formal result

$$v(t) = \frac{\log B}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{\sqrt{M}}{\Gamma(2-\alpha)} t^{1-\alpha} + \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{t^{k-\alpha}}{\Gamma(1+k-\alpha)} \Lambda_k(t) , \quad (15)$$

where $\Lambda_k(t)$ is defined via

$$\Lambda_k(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} \log\left(1 + \frac{A}{B}\exp 2\sqrt{M}t\right) , \quad k = 0, 1, 2, \ldots \quad (16)$$

and the generalized binomial coefficient is defined in the Appendix. Eq. (15) can be discussed in the limiting cases of long times, i.e. $t \gg 1/(2\sqrt{M})$, and short times, i.e. $t \ll 1/(2\sqrt{M})$. In between a numerical evaluation shows excellent convergence.

In the limiting case $t \ll 1/(2\sqrt{\mathrm{M}})$ an expansion of the exponential functions in Eq. (11) to first order in time, $\log(a + bt) \cong \log a + b/at$, yields

$$v(t) \sim \frac{\log(A+B)}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{(A-B)\sqrt{M}}{(A+B)\Gamma(2-\alpha)} t^{1-\alpha} . \quad (17)$$

We require a non–diverging expression for $t \to 0$ so that the coefficients must satisfy the relation $A + B = 1$. Thus the velocity at short times is an algebraic function of time:

$$v(t) \sim (A - B)\sqrt{M}/\Gamma(2-\alpha)t^{1-\alpha} . \quad (18)$$

On the other hand, for $t \gg 1/(2\sqrt{\mathrm{M}})$ the argument of the logarithm is dominated by the exponential and one finds

$$v(t) \sim \frac{\log A}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{\sqrt{M}}{\Gamma(2-\alpha)} t^{1-\alpha} \quad (19)$$

where the first term may be neglected in comparison to the second as $t \to \infty$. One thus observes a power–law increase of the velocity in contrast to the familiar case of approaching a maximum velocity for a free–falling body in air. The $t^{1-\alpha}$ increase (creep) in time is the fractional counterpart of the standard stationarity behavior.

Comparing Eqs. (17) and (19) one observes a transition from the power law $\sqrt{M}(A - B)/\Gamma(2 - \alpha)t^{1-\alpha}$ to $\sqrt{M}/\Gamma(2-\alpha)t^{1-\alpha}$ which are both of the same universality class — thus bear the same power $(1-\alpha)$ — but have a different magnitude (see Fig. 1 for the complete solution). As $B = 1-A$ the two coefficients differ by a factor $1 \geq (2A-1) \geq 0$. In the special case $A = 1$ implying $B = 0$, one finds "fractional stationarity",[10] i.e. $v = \sqrt{M}/\Gamma(2-\alpha)t^{1-\alpha}$ for all times $t > 0$.

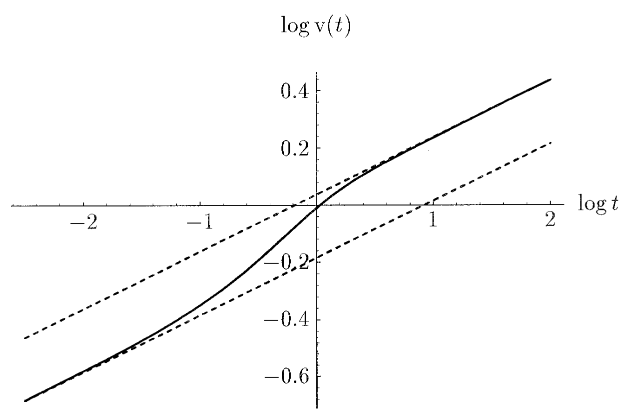

Fig. 1 The velocity of the free–falling body $v(t)$ given by Eq. (15) with the parameter values $M = 1$, $A = 0.8$, $B = 0.2$, $\alpha = 0.8$ is graphed. The dashed lines indicate the asymptotes given by Eqs. (17) and (19). Clearly, the solution [Eq. (15)] interpolates between both of the power laws of the same order.

4. DISCUSSION AND CONCLUSIONS

The case $\alpha = 1$ reveals the standard Riccati equation discussed in Sec. 2 whereas $\alpha = 0$ leads to the equation

$$\ddot{v}(t) + \dot{v}(t)^2 = M \tag{20}$$

for which the constant acceleration $\dot{v}(t) = \sqrt{M}$ is a solution. Thus for $B = 0$ one recovers $v(t) = \log A + \sqrt{M}t$, the case of a free falling mass with normalized initial speed $v_0 = \log A$ and normalised acceleration $\sqrt{M}$. On the other hand, for $A = 0$ the result is $\mathrm{v}(t) = \log B - \sqrt{M}t$, the case of free fall with an initial velocity opposite to the direction of the gravitational force (rocket problem).

For $\alpha \in (0,1)$ the fall is slower than a free falling mass ($\alpha = 0$) but still faster than in a medium causing a v^2–proportional counterforce ($\alpha = 1$). It is a characteristic issue that for $\alpha \in (0,1)$, v does not reach a limit value but is ever–increasing, as a power law. For three different values of α the solution [Eq. (15)] is displayed in Fig. 2. The fractional order of the generalized Eq. (9) rotates and stretches the standard tanh profile [Eq. (3)] to the straight line of the free fall problem, in the log–log plot of Fig. 2.

For the physical interpretation of above result [Eq. (15)], recall that in complex systems like polymers, polymer solutions or bubble gum, one often encounters "slow creep", i.e. power-law flows with the time dependence $t^{-\beta}$, in some time range.[4] A gedanken example for a process appropriate for Eq. (15) would be a heavy mass affixed to a polymer woven cord that has exceeded its elastic limit. If the mass is heavy enough it will cause a plastic flow of the material, like a too heavy bungee cord jumper.

This is the first time, to our knowledge, that a physical problem modeled by a fractional Riccati–type equation of arbitrary order $\alpha \in (0,1)$ has been discussed and solved analytically. Starting off from a standard Riccati equation a generalized equation for the fall in a "fractionally" resisting medium has been proposed, discussed and solved. In this example the speed does not reach a limit value but increases with a power–law tail. Therefore it would be wise to prefer a parachute to a bungee cord if your weight is too much.

Despite the relatively simple phenomenological generalization of Eq. (4) to Eq. (9) by a fractional differential operator $\mathrm{d}^{-\alpha}/\mathrm{d}t^{-\alpha}$ the solution issues a rather complicated behavior. The procedure similar to the one developed herein should be possible for other non–linear reduction transforms which reveal ordinary linear differential equations with constant coefficients.

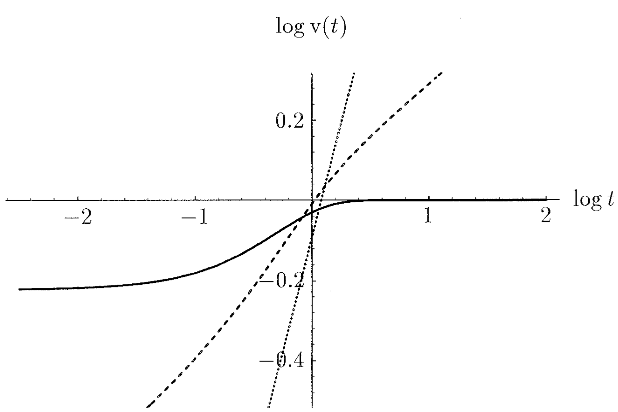

Fig. 2 The velocity of the free–falling body $v(t)$ given by Eq. (15) with the parameter values $M = 1$, $A = 0.8$, $B = 0.2$, is graphed for the three fractional parameters $\alpha = 0$ (free fall, dotted line), $\alpha = 3/4$ (plastic creep, dashed line) and $\alpha = 1$ (counterforce quadratic in v, full line).

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REFERENCES

1. H. Davis, *Introduction to nonlinear differential and integral equations* (Dover Publications, Inc., New York, 1962).
2. *Soliton theory: a survey of results*, ed. A. P. Fordy (Manchester University Press, Manchester, 1990).
3. W. G. Glöckle and T. F. Nonnenmacher, *Macromolecules* **24**, 6426 (1991).
4. H. Schiessel, R. Metzler, A. Blumen and T. F. Nonnenmacher, *J. Phys. A: Math. Gen.* **28**, 6567 (1995).
5. W. R. Schneider and W. Wyss, *J. Math. Phys.* **30**, 134 (1989).
6. R. Metzler, W. G. Glöckle and T. F. Nonnenmacher, *Physica* A **211**, 13 (1994).
7. B. J. West, P. Grigolini, R. Metzler and T. F. Nonnenmacher, *Phys. Rev.* E **55**, 99 (1997).

8. T. F. Nonnenmacher and R. Metzler, *Fractals* **3**, 557 (1995), and in *Fractal Geometry and Analysis, The Mandelbrot Festschrift, Curaçao 1995*, eds. C. J. G. Evertsz, H.–O. Peitgen and R. F. Voss (World Scientific, Singapore, 1996).
9. R. Gorenflo and A. A. Kilbas, *J. Fract. Calc.* **8**, 103 (1995).
10. R. Hilfer, *Fractals* **3**, 549 (1995).
11. K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).

APPENDIX A: FRACTIONAL CALCULUS

Fractional calculus ideas are by no means new; they were exchanged in some letters between de l'Hospital and Leibniz in 1695. Today's definitions are mainly based on the works of Liouville and Riemann published in the last century.[11]

Herein, we use the Riemann–Liouville fractional calculus defined as a generalization of Cauchy's multiple integral with the lower limit $t_0 = 0$:

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t \mathrm{d}\tau \, \frac{f(\tau)}{(t-\tau)^{1-p}} \qquad \text{(A.1)} $$

for $p > 0$. A derivative of order q, $q > 0$, is then established via the definition

$$ \frac{\mathrm{d}^q}{\mathrm{d}t^q} f(t) \equiv {}_0D_t^q f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\, {}_0D_t^{q-n} f(t) \qquad \text{(A.2)} $$

where $n \geq q$, $n \in \mathbb{N}$ is a natural number. Here, also, we introduce the short–hand notation $\mathrm{d}^q/\mathrm{d}t^q$ used in the text.

The chain rule for differentiation involves an infinite summation[11]

$$ \frac{\mathrm{d}^p}{\mathrm{d}t^p} \phi\left(g(t)\right) = \sum_{k=0}^{\infty} \binom{p}{k} \left(\frac{\mathrm{d}^{p-k}}{\mathrm{d}t^{p-k}} 1 \right) \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} \phi\left(g(t)\right) \right) \qquad \text{(A.3)} $$

where the generalized binomial symbol is given in terms of Gamma functions

$$ \binom{p}{k} = \frac{\Gamma(1+p)}{\Gamma(1+k)\Gamma(1-k+p)} \qquad \text{(A.4)} $$

and the derivative of a constant c is readily recovered to be (see [11]):

$$ \frac{\mathrm{d}^{p-k}}{\mathrm{d}t^{p-k}} c = \frac{c t^{k-p}}{\Gamma(1+k-p)} \,. \qquad \text{(A.5)} $$

Thus the problem is reduced to an infinity of standard chain rule problems.