# Clustering of Granular Assemblies with Temperature Dependent Restitution under Keplerian Differential Rotation

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The clustering of granular assemblies is studied under the influences of (i) a temperature-dependent coefficient of restitution and in (ii) a central gravitational force field. Our stability analyses of the constitutive equations as well as numerical experiments show that in case (i) clusters are still formed even though collisions become more elastic as the temperature decreases. In case (ii) the clusters appear as a transient phenomenon during the establishment of a quasiequilibrium. These transient patterns rotate driven by the shear and then they "melt" away with elapsed time. [S0031-9007(97)02381-8]

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Granular material is very common in nature, from terrestrial sands and gravels to dust and planetary rings in space. The dissipative nature of the particle collisions makes such systems quite different from usual gases where interactions are conservative. This dissipation enables granular systems to form clusters, a process studied intensively under force-free conditions [1–3] or under shear [4–6]. However, in most of these investigations a constant coefficient of restitution, which measures the damping of the impact velocity, was applied. This results in stable clusters if the dissipation is strong enough.

However, laboratory experiments [7] show that this coefficient depends sensitively on the relative velocities of the granules, i.e., on the temperature. This dependence together with a Keplerian differential rotation was taken into account in studies of the dynamics/kinetics of planetary rings. The differential rotation is caused by a central gravitational field ($\propto r^{-2}$) which acts as a disrupting "tidal force" (in the following this term will be used) for extended clusters. Then it was found that the viscous (cluster) instability [8,9] does not work [10–12]. This provokes the question: Does the (i) *variable restitution* or the (ii) *tidal force*—or both in combination—prevent the cluster formation?

In this Letter we investigate these problems by applying stability analyses and numerical test-particle experiments.

At first the dynamics of collisions between two granular spheres is briefly sketched. Recent related investigations [13] have yielded an extended Hertzian law $\ddot{\xi} \propto -(\xi + C\dot{\xi})^{3/2}$ for the collisional dynamics, where $\xi$ and $C$ denote the deformation of either bodies during the contact and a dissipative material constant, respectively. Numerical solutions of this collision dynamics have yielded the restitution coefficient $\epsilon(v_{\mathrm{imp}}) \approx A/(v_{\mathrm{imp}} + v_*)^{\beta}$ in dependence on the impact velocity $v_{\mathrm{imp}}$. It reproduces fairly well the results of laboratory experiments [7] if one uses $A \approx 0.2 \cdots 0.4$ (dissipation parameter) and $\beta \approx 1/4$. We use this relation in the following numerical experiments. Assuming a Gaussian velocity distribution [14], the mean restitution becomes $\epsilon(T) \approx A/(T + T_*)^{\beta/2}$ ($T$ denotes temperature), which will be used for the theoretical analysis. The parameters $v_*$ and $T_*$ ensure that $\epsilon(T, v_{\mathrm{imp}} \to 0) \to 1$. The dissipation rate will be varied via the dissipation parameter $A(C)$.

Now, we investigate the clustering instability in a small box corotating in an orbit around a central body (e.g., planetary rings). The motion of a particle in such a region is governed by Hill's equations, and hence, the hydrodynamic approximation [1,4,15] for this case reads

$$\rho\left\{\frac{d\vec{u}}{dt} + 2\vec{\Omega}_0 \times \vec{u} - 3\Omega_0^2 y \vec{e}_y\right\} = -\rho\nabla\Phi - \nabla \cdot \hat{P}\,, \tag{1}$$

$$\frac{d\rho}{dt} = -\rho\nabla \cdot \vec{u}\,, \tag{2}$$

$$\frac{3}{2}\,\rho\,\frac{dT}{dt} = -\nabla \cdot \vec{Q} - \hat{P} : \nabla \circ \vec{u} - \gamma\,. \tag{3}$$

The substantial derivative is given by $\frac{d}{dt} \to \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$. The shear tensor is denoted by $\hat{D} = \frac{1}{2}(\nabla \circ \vec{u} + \vec{u} \circ \nabla - \frac{2}{3}\nabla \cdot \vec{u}\hat{\mathbf{I}}) = \frac{3}{2}\,\Omega_0\,\vec{e}_x \circ \vec{e}_y$, where a mean circular motion around the planet $\vec{u} = \frac{3}{2}\,y\Omega_0\vec{e}_x$ is assumed ($-y$ denotes direction to the planet, $-x$ denotes direction of the orbital motion of the box, $\Omega_0$ denotes orbital frequency, ":" denotes double inner product). The deviations from this circular motion—the inclinations and eccentricities—account for the granular temperature $T$. The values $\Phi$, $\rho$, $\hat{P}$ and $\vec{Q}$ are force fields, the mass density, the stress tensor, and the heat flux, respectively. The stress tensor and the heat flux read $\hat{P}\ (\hat{D}) \approx P\hat{\mathbf{I}} - 2\eta(T)\hat{D}$; $\vec{Q} = -\kappa(T)\nabla T$, where the constitutive relations [1,4,15] in the dilute limit yield for the viscosity $\eta \approx (5/24)\sqrt{\pi T}/[3 - \epsilon(T)]/[1 + \epsilon(T)]$, for the heat conductivity $\kappa \approx (25/4)\sqrt{\pi T}/[1 + \epsilon(T)]/[49 - 33\epsilon(T)]$, for the pressure $P \approx \rho T$, and for the cooling rate $\gamma \approx 24[1 - \epsilon(T)^2]\rho^2\sqrt{T^3/\pi}$. All quantities are scaled to the particle diameter $D_p = 1$, density $\rho_p = 1$, Boltzmann constant $k_B = 1$.

Next, we consider the force-free case $\Omega_0 = \Phi = 0$, already discussed for constant restitution $\epsilon = \mathrm{const}$ by

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Goldhirsch and Zanetti [1], to judge whether the relation $\epsilon(T)$ alone can hinder the clustering. Afterwards we study the clustering in orbit $\Omega_0 \neq 0$.

*(i) The force-free case* ($\Omega_0 = \Phi = 0$).—Equations (1)–(3) have the homogeneous solution [1] $\vec{u}_0 = 0$, $\nabla T_0 = \nabla\rho_0 = 0$, $T_0(t)$ and its stability is judged by linearizing these equations (2D problem for simplicity: $\vec{u} \cdot \vec{e}_z = w = \text{const}$). Equation (3) reduces to a mere cooling in this case: $\dot{T}_0 = -2(3\rho_0)^{-1}\gamma(T_0)$. For $\beta = 1/4$ one finds $t + t_0 \propto A^{-4}\{\epsilon(T_0)^2 + \ln[1 - \epsilon(T_0)^2]\}$ as an expression for the basic solution $T_0(t)$. If $\epsilon(T_0) \ll 1$ its expansion yields the expression for constant restitution $T_0(t) = T_0(0)/(1 + t/t_0)^2$ (e.g., [1]).

As usual, all values are expanded in a ground state and small perturbations $X = X_0(t) + X'(\vec{r}, t)$; $(|X'/X_0| \ll 1;\ X' \Rightarrow \vec{u}' = (u', v'), \rho', T')$. In a periodic (infinite) system the eigenfunctions of the linearized Eqs. (1)–(3) are plane waves $X'(\vec{r}, t) \propto \exp(\alpha t + i\vec{k} \cdot \vec{r})$ ($\vec{k}$ denotes wave vector; $\alpha$ denotes growth rate). The resulting eigenvalue problem gives a system of linear equations $\{\hat{\mathbf{A}}(X_0(t), k_i, \ldots, k_i^2) - \alpha\hat{\mathbf{I}}\}\vec{X}' = 0$ ($i = x, y$; $\hat{\mathbf{A}}$ denotes coefficient matrix) with the solvability condition $|\hat{\mathbf{A}} - \alpha\hat{\mathbf{I}}| = 0$ which determines the characteristic equation $\sum_{i=0}^{4} a_i\alpha^i = 0$. If one of its roots takes $\text{Re}(\alpha) > 0$, an instability is found. Because all coefficients in the characteristic equation are $a_i > 0$ ($\forall\ i > 0$), Vieta's root rules indicate instability for $a_0 < 0$, which is fulfilled for

$$\gamma_0 + \frac{\partial P}{\partial\rho}\bigg|_0 \frac{\partial\gamma}{\partial T}\bigg|_0 - \rho\,\frac{\partial\gamma}{\partial\rho}\bigg|_0 + \kappa_0 k^2\,\frac{\partial P}{\partial\rho}\bigg|_0 \leq 0\,, \tag{4}$$

$$k_c \propto \frac{4}{5}\,\rho\left\{\frac{3}{\pi}\,[1 + \epsilon(T_0)][49 - 33\epsilon(T_0)] \times \left[1 - \epsilon(T_0)^2 + \frac{2T_0\beta\epsilon(T_0)^2}{T_0 + T_*}\right]\right\}^{\frac{1}{2}}, \tag{5}$$

where $k_c$ is a critical wave number [equality in relation (4). The relations for constant restitution are obtained for $\epsilon(T_0, \beta = 0) = \text{const}$. In our case $\beta > 0$ the expression in the curly brackets never gets negative, provided that $\partial P_0/\partial\rho_0 < 0$, but approaches zero for $T \to 0$. That means at any time there are typical length scales $L > k_c^{-1}$ which can form clusters. However, in the course of the cooling process, $k_c$ gets smaller and hence only larger scales remain unstable. In the case of constant restitution [1] becomes $k_c(\beta = 0) = \text{const}$, and all scales are unstable in a stationary fashion. The differences between both cases are illustrated in Fig. 1 where the difference $\Delta k = k_c(\beta \neq 0) - k_c(\beta = 0)$ is plotted. At high temperatures ($t = 0$) there may well be positive $\Delta k$, but with increasing time the values get negative indicating that the instability for the case $\beta \neq 0$ gets weaker compared to the case $\beta = 0$. The right part of Fig. 1 shows a numerical experiment with 20 000 particles and with $A = 0.1$ and $\beta = 1/4$. One recognizes clusters isotropically oriented and having wavelengths quite below the size of the box.

*Bottom line*.—The variable restitution cannot prevent the cluster formation; it slows the process down in the same way as the slope of the cooling curve is smaller for $\epsilon(T)$ because $[1 - \epsilon(T)^2]$ is monotonically dropping to zero.

*(ii) Clustering in the orbit* ($\Omega_0 \neq 0, \Phi = 0$).—Here, the temperature evolution [Eq. (3)] $\dot{T} \approx (a\eta\Omega_0^2 - b\gamma)/\rho$ is mainly determined by a viscous heating $\propto \eta\Omega_0^2/\rho$ and the collisional cooling: ($a$ and $b$ constant). This balance ensures the establishment of a "quasi"-equilibrium related to a steady temperature $T_0(t \to \infty, \rho_0) = T_\infty(\rho_0)$ ($\propto \rho_0^{-2}$ for $\beta = 0$) where $\dot{T}_0 \to 0$.

This quasiequilibrium reduces the analysis to Eqs. (1) and (2). The same procedure as in the force-free case (i) can be applied if all values depend merely on $y$ corresponding to pattern rotation in the shear [4]. It should be noted that this assumption is valid only if self-gravity does not play a role, i.e., for dilute (low mass) systems. Then the stationary state becomes $\vec{u}_0 = 1.5\Omega_0 y\vec{e}_x$; $\rho_0 = \text{const}$ and the criterion of its instability is found

$$3\,\Omega_0^2\,\frac{\partial\eta}{\partial\rho}\bigg|_{\rho_0} + k^2\,\frac{\eta}{\rho}\,\frac{\partial P}{\partial\rho}\bigg|_{\rho_0} < 0\,. \tag{6}$$

Relation (6) combines the pressure instability [1] with the viscous instability, already discussed in course of the explanation of the irregular fine structures in planetary rings [8,9]. In the dilute limit [see case (i)] one gets $\eta(\rho) = \text{const}$, and thus, $\partial_\rho\eta = 0$. However, taking into account the nonlocal part of $\eta$ at higher packing fractions [6,15,16]—one obtains $\partial_\rho\eta > 0$ ([10–12,17]). Simultaneously, the dynamic dependence $\epsilon(T)$ counteracts the pressure instability as mentioned in the former case. Thus, at least for larger scales a fragmentation of granular agglomerates must be expected.

In order to prove this theoretical conclusion we carry out numerical particle experiments (2D), where cases of different dissipation—according to a variation of $A$—will be considered. $N = 20\,000$ particles have been initially distributed homogeneously over a box which resembles the conditions in Saturn's B ring. The area-filling factor (optical depth) is $\tau = N\pi R_p^2/F_{\text{box}} = 1/10$ with a particle radius $R_p = 1$ cm, a surface area $F_{\text{box}} \approx 8\text{ m} \times 8\text{ m}$ in all numerical experiments [cases (i) and (ii). Different initial conditions have been chosen for the rm velocity: equal distribution $\Theta(v_i - v_0)\Theta(v_i - v_0)$; as well as a delta function $\delta(v - v_0)$ ($v_0 = 1\text{ cm s}^{-1}$; isotropy). In either case the velocity distribution approaches an approximate Gaussian form [14] after a few collisions per particle and follow comparable evolutions during two orbital periods watched numerically. In our runs we have in average $10 \cdots 30$ collisions per orbital period, respectively. These values are higher than the $1 \cdots 10$ collisions per orbit obtained by Salo [12], which is due to the difference between the collision frequencies in 2D (this study) and 3D.

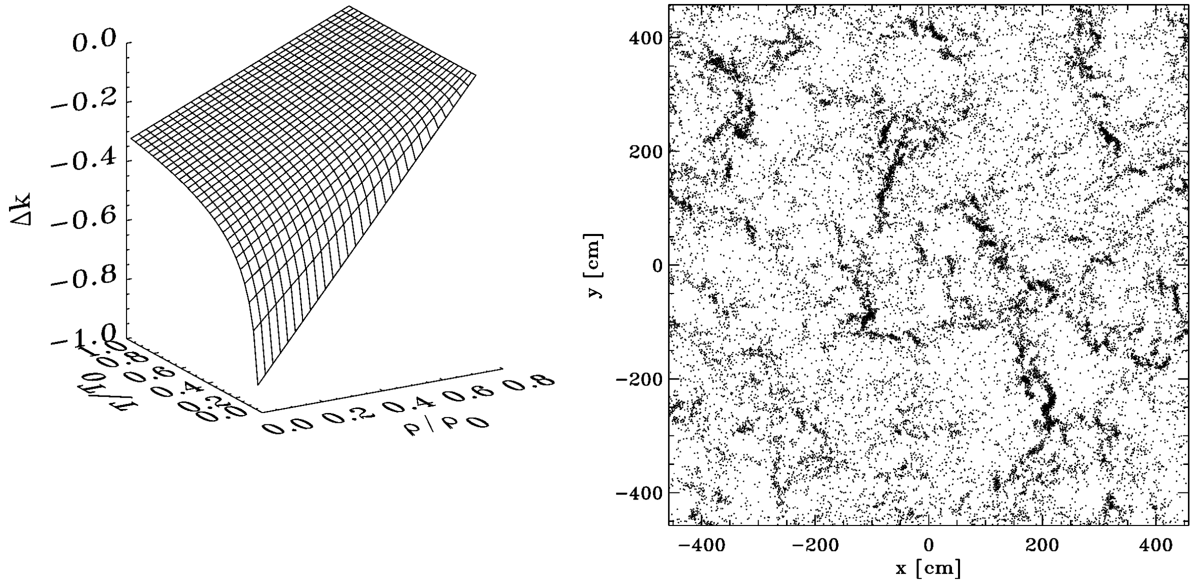


FIG. 1. Left: Difference of the critical wave number of $\Delta k$ vs density and temperature. Right: Snapshot (after 1 h) of spatial structure without tidal force ($A = 0.1$); 20 000 particles (radius = 1 cm); $T_0(0) = 1\ \mathrm{cm}^2\,\mathrm{s}^{-2}$.

Figure 2 shows the snapshots $[t = 0.3(2\pi/\Omega_0)]$ of the particle configuration (top) in the box and a grey level representation of the temperature (bottom) of simulations for a quite inelastic $A = 0.1$ (left), and a more conservative case $A = 0.5$ (right). In the former we observe phases with rather different "temperatures." In the clusters the relative motion is almost completely damped which corresponds to $T \to 0$. There is a correlation between low temperature spots and the clusters formed. In the case $A = 0.5$ the inhomogeneities in the density and the temperature field are much less pronounced. However, even in the very dissipative cases of $A = 0.1$ and $\epsilon = 0$, the omnipresent tidal forces should be able to disrupt the clusters, because of the lack of attractive forces. The decrease of the dissipation (for $\epsilon > 0$) during the cooling should only support this process and the related (viscous) diffusion (heat conductivity) at $T_\infty > 0$ should smooth out density (temperature) gradients.

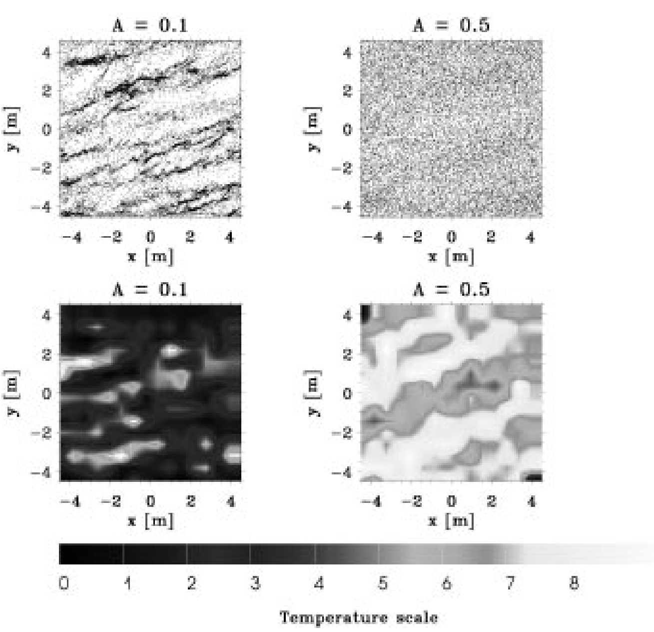


FIG. 2. Comparison of snapshots of the spatial structures as well as of the spatial temperature field for $A = 0.1$ (left), and $A = 0.5$ (right) obtained from numerical experiments.

Next we try to characterize quantitatively the evolution of the density pattern by means of complexity measures. We calculate the Renyi information $H_q$ to describe the spatial inhomogeneity in dependence on time. It is defined by

$$H_q = (1 - q)^{-1}\,\mathrm{ld}\left(\sum_{i=1}^{N} p_i^q\right), \tag{7}$$

which is based on a partition of the space into $N$—boxes and $p_i$ is the probability of particles to be in the $i$th box. The weight $q$ emphasizes clustered structures for $q > 1$, whereas regions with low probability are favored for $0 < q < 1$. $H_q$ converges to the Shannon entropy as $q \to 1$. $H_q$ takes its maximum in the case of equidistributed particles [18]. Recently we have shown that $H_q$ is an appropriate tool to characterize spatiotemporal intermittency [19]. Using a $15 \times 15$ grid partitioning and $q = 5$, we can measure different degrees of clustering for different $A$ in dependence on time (Fig. 3). It should be noted that these do not sensitively depend on $q$.

From Fig. 3 we can conclude (1) there is no clustering in the conservative case $\epsilon = 1$ and almost none for $A = 0.5$. (2) A rapid cluster formation is found for $A < 0.5$. (3) An increase of $H_q$ after a certain time points to a transient nature of all clusters, even in the completely inelastic case. (4) The fluctuations, well observed for $A = 0$ or 0.1, refer to a rapid alternation of freezing and

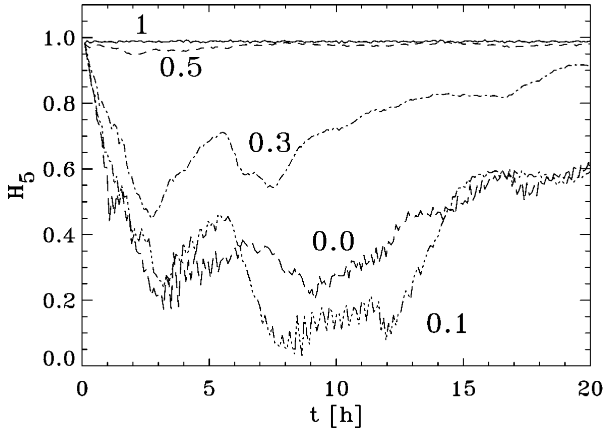


FIG. 3. Renyi entropies $H_q$ are shown for $q = 5$ vs time for the runs varying the dissipation parameter $A = 0; 0.1; 0.3; 0.5; 1$, which measure the disorder in the particle configuration.

melting. Furthermore they are also the reason for the unexpected intersection of the curves $A = 0.0; 0.1$ in the time range $6$ h $< t < 9$ h.

Coming back to the major topic of this Letter we have found that the dynamic behavior of the coefficient of restitution does not prevent granular matter to form clusters. It slows the cooling process down and in course of the time the scales of unstable modes increase. Furthermore, clustering is not stable if tidal forces, but no attractions (interparticle gravity), act between the particles. In this context, recent results that such systems behave as if there were long-range attractions between the granules [3] seem to be questionable and should be seriously revisited. Consequently, collisions alone cannot be responsible for the coagulation of planetesimals into planets after decoupling from the nebular gas, as suggested by Goldhirsch and Zanetti [1]. Only attractive forces as the gravity are able to keep the clusters safe from fragmentation [20,21].

Of course, more detailed work is needed for a deeper understanding of the granular dynamics, as there is, e.g., a better understanding of the collisional behavior of non-spheric particles (fractal surfaces) including the rotational degrees of freedom caused by a tangential friction [13]. Furthermore, not only the dissipation, and thus, the "distance" from equilibrium, is important. In dense systems the Boltzmann and the Enskog theory—both are the base of the constitutive Eqs. (1)–(3)—might not be applicable anymore, because not only binary particle correlations are important. In other words, then the system cannot be described with a single particle distribution function [15]. Therefore, the influence of the particle number density as well as that of a size distribution [12] of the granules on the validity of Eqs. (1)–(3) is one of the interesting points which should be attributed to future investigations.

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