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Closed-form multi-dimensional solutions and asymptotic behaviors for subdiffusive processes with crossovers: I. Retarding case

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ABSTRACT

Numerous anomalous diffusion processes are characterized by crossovers of the scaling exponent in the mean squared displacement at some correlations time. The bi-fractional diffusion equation containing two time-fractional derivatives is a versatile mathematical tool describing specifically retarded subdiffusive transport, in which the scaling exponents acquires a smaller value, i.e., the diffusion becomes even slower after the crossover. We here derive closed-form multi-dimensional solutions for this integro-differential equation in n spatial dimensions by generalizing the classical Schneider-Wyss solution of the fractional diffusion equation with a single fractional derivative. In the two-dimensional case we develop a limiting approach based on the solution of the space-time fractional diffusion equation. The probabilistic interpretation in higher dimensions is discussed. The asymptotic long- and short-time behaviors are derived. It is shown that the solution of the bi-fractional diffusion equation can be interpreted in terms of the Fox H-transform of the Gaussian distribution.

1. Introduction

Almost a hundred years ago Richardson postulated the characteristic cubic scaling $\langle r^2(t)\rangle \simeq t^3$ of the mean squared displacement (MSD) in the relative diffusion of two pilot balloons and other tracer particles in atmospheric turbulence [1,2]. This Richardson-t^3-law constitutes the first clear reported deviation of the scaling of the MSD from the law of classical Brownian motion characterized by the linear growth in time of the MSD, $\langle r^2(t)\rangle \simeq t$ [3]. By-now the term anomalous diffusion comprises many diffusion phenomena that differ from classical Brownian motion, typically characterized by the non-linear scaling of the MSD in the form [4,5]

$$\langle r^2(t)\rangle \simeq t^{\alpha}, \qquad \alpha \neq 1. \tag{1}$$

When $0 < \alpha < 1$, a diffusive process quantified by the law (1) is termed subdiffusive or dispersive, while for $1 < \alpha < 2$ the process is known as superdiffusion or enhanced diffusion. The limiting cases $\alpha = 1$ and $\alpha = 2$ correspond to classical Brownian motion and ballistic motion, respectively. We note, however, that the observation of the law (1) is not necessarily sufficient to classify a given process as Brownian, as there exist numerous examples in which a linear scaling of (1) coincides with a non-Gaussian shape of the test particles' probability density function (PDF) [6–8].

Anomalous diffusion phenomena quantified by (1) are quite ubiquitous in nature ("anomalous is normal" [9]) and their occurrence is measured over an impressive range of scales, from phospholipids and cholesterols in a lipid bilayer [10] and nanoscale diffusion of membrane hydration water in fluid-phase lipid bilayers [11] to the propagation of cosmic rays in the galaxy [12]. Many more examples can be found in comprehensive reviews [4,5,13–16] and tutorial articles [17–20].

In many circumstances, however, the MSD does not follow the mono-scaling behavior (1), rather a crossover is observed in the sense

$$\langle r^2(t)\rangle \simeq \begin{cases} t^{\alpha_2}, & \text{for } t \to 0^+, \\ t^{\alpha_1}, & \text{for } t \to \infty, \end{cases} \tag{2}$$

where $\alpha_1 \neq \alpha_2$. Examples for such crossover dynamics can be seen in polymeric particle transport in living cells [21], transport of vacuoles in motile amoeba cells [22], active motion of amoeba

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cells themselves [23], viscoelastic flow [24], quasiperiodic interacting systems [25], dewetting-spreading-wetting transitions of run-and-tumble clustered disks in a substrate with randomly placed pinning sites [26], lipid diffusion in bilayer membranes [10,27], or in drug molecule diffusion in a silica nanoslit [28], see also [16,29] and references therein.[1]

The construction of a corresponding mathematical model that describes such crossover behaviors in anomalous diffusion remains in the focus of theoretical modeling of stochastic processes. In fact, starting from the works of Richardson [1], Hurst [35] Mandelbrot [36,37], Scher and Montroll [38], Seshadri and West [39], Klafter and coworkers [40], and Zaslavsky [41] the development of flexible stochastic models to describe anomalous diffusion processes has continued to gain much attention. From a conceptual point of view the formulation of an integro-differential equation with a fractional derivative spanning a range of $0 < \alpha < 2$ of the MSD scaling exponent was the Schneider-Wyss equation and its exact solution in terms of Fox H-functions [42]. Despite the success of the fractional diffusion equation, however, it is confined to a mono-scaling behavior of the form (1) and thus not capable of capturing the crossover behavior (2). To account for such crossovers the next step was to assume that the fractional order in this generalized diffusion equation is not fixed but rather a variable in time. In [43] it was shown that the replacement of the fractional time derivative of fixed order $\alpha \in (0,1)$ with a variable order distributed over the interval (0,1) indeed produces the behavior (2) with a specific choice of the integral kernel. The distributed-order fractional derivative was first supposed by Caputo [44] and successfully employed to model retarding subdiffusion and accelerating superdiffusion [43] as well as accelerating subdiffusion and retarding superdiffusion [45–47], see also a variety of applications in [48–55] and the references therein.

In most of the cases anomalous diffusion has been considered analytically in one-dimensional space, while there is a lack of exact results for the description of anomalous diffusion in higher-dimensional space. In particular, the variable order equation were derived for one-dimensional random walks processes, in contrast to various physical phenomena, such as diffusion in crowded environments [56], diffusion in biological cells [57], protein diffusion in mammalian cell cytoplasm [58], and many other phenomena, which naturally are running off in higher dimensions. Therefore, the aim of the present work is to explore the solution of the multi-dimensional distributed-order diffusion equation and compare it, together with its moments, systematically with the one-dimensional case. Before implementing this goal, it is noteworthy referring to some investigations on the multi-dimensional integrodifferential equations. A series of papers [59–61] examined multi-dimensional propagators of the space-, time- and space-time fractional diffusion and diffusion-wave equation. Closed-form multi-dimensional solutions of the fractional wave equation and fractional diffusion-wave equation and their properties were explored in [62–65]. A fractional elastic model, generalizing the space-time fractional diffusion-wave equation, was solved analytically for certain special cases in [66], alongside with its asymptotic behavior. The multi-dimensional fractional advection-dispersion equation was analyzed in detail in [67], along with establishing some stochastic foundations. The distributed-order fractional diffusion-wave equation was analytically investigated in [68]. The multi-dimensional solution of the time-fractional telegraph equation was derived in terms of the multivariate Mittag-Leffler function and the Fox H-function with two variables in [69], in addition to calculating the multi-dimensional moments, even though their solution exhibits negative values in two and three dimensions, as predicted previously in the ordinary case [70], see also different perspectives in [71].

The paper is organized as follows: In Section II we present analytical solutions to the multi-dimensional distributed-order diffusion equation with two fractional orders, which represents a generalization of the model describing decelerating subdiffusion. Two different approaches for solving the equation are discussed in detail and some special cases of the equation are recovered. The non-negativity of the corresponding solution, analyzed in terms of Bernstein function and subordination approaches, is shown in Section III. The moments of the fundamental solution are calculated and some special cases are analyzed. The various behaviors of the PDF are confirmed by asymptotic analysis of the solution in the short and long time-space limit. In Section IV we provide a summary and expand our future plans.

2. Multi-dimensional propagator

The fractional diffusion equation reads [42,72,73]

$$ {}^{C}{}_{0}\mathcal{D}_t^{\alpha} W^{(n)}(\mathbf{r},t) = \Delta W^{(n)}(\mathbf{r},t), \tag{3}$$

for which we consider the δ initial condition

$$ W^{(n)}(\mathbf{r},0^+) = \delta(\mathbf{r}), \tag{4}$$

where $W^{(n)}(\mathbf{r},t)$ is the multi-dimensional propagator (PDF) in n-dimensional Euclidean space $\mathbb{R}^n$, $\mathbf{r} = (x_1,\ldots,x_n)$ is the position vector, Δ is the Laplacian operator, and ${}^{C}{}_{0}\mathcal{D}_t^{\alpha}$ stands for the Caputo fractional derivative of order α with respect to time t, defined for any generic function $f(t)$ through [74,75]

$$ {}^{C}{}_{0}\mathcal{D}_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial_\zeta f(\zeta)}{(t-\zeta)^\alpha} d\zeta, & \text{for } 0<\alpha<1, \\ \frac{df(t)}{dt}, & \text{for } \alpha = 1. \end{cases} \tag{5}$$

It is well known for the one-dimensional case that the time-fractional diffusion equation can be derived from the continuous-time random walk theory for a random process with long-tailed waiting time probability density $\psi(t) \sim t^{-\alpha-1}$, $0<\alpha<1$, see for example [5]. In an external force field, this equation generalized to the fractional Fokker-Planck equation [76,77]. By calculation of the associated MSD one observes subdiffusion of the form (1) with $0<\alpha<1$.

In analogy to the one-dimensional case we now consider the generalization of Eq. (3) to higher dimensions. Strictly speaking we introduce the distributed-order fractional diffusion equation of natural type [43] in dimensionless form,

$$ \int_0^1 p(\nu)\,{}^{C}{}_{0}\mathcal{D}_t^{\nu} W^{(n)}(\mathbf{r},t) d\nu = \Delta W^{(n)}(\mathbf{r},t), \tag{6}$$

where $p(\nu)$ is a PDF in the sense that $p(\nu) \geq 0$ and $\int_0^1 p(\nu)d\nu = 1$. It is a generalization of the one-dimensional distributed order diffusion equation analyzed in [43,78]. With the specific choice

$$ p(\nu) = p_1\delta(\nu-\alpha_1) + p_2\delta(\nu-\alpha_2), \tag{7}$$

where p_1 and p_2 are positive dimensionless constants satisfying $p_1+p_2=1$ and $0<\alpha_1<\alpha_2\leq 1$, (6) reduces to the bi-fractional (or double-order time-fractional) diffusion equation of the natural type,

$$ p_1\,{}^{C}{}_{0}\mathcal{D}_t^{\alpha_1} W^{(n)}(\mathbf{r},t) + p_2\,{}^{C}{}_{0}\mathcal{D}_t^{\alpha_2} W^{(n)}(\mathbf{r},t) = \Delta W^{(n)}(\mathbf{r},t), \tag{8}$$

which in the one-dimensional setting was introduced in [43] as a model to describe subdiffusive crossover behavior with retardation.

[1] We note that we here do not intend to describe crossovers due to finite size effects of the system, which lead to a plateau of the MSD in ergodic anomalous diffusion systems [30]—see, however, the discussion in [31,32] for transient non-ergodicity in the time-averaged MSD—or a crossover to a confinement-induced power-law of the MSD [33,34]. Our focus is solely on systems with an intrinsic crossover time scale separating dynamical regimes in unconfined systems.

2.1. Generalized Schneider-Wyss solution

We here provide a multi-dimensional closed-form solution for Eq. (8) by generalizing the classical Schneider-Wyss solution [42]. The propagator $W^{(n)}(\mathbf{r},t)$ of Eq. (8) subject to the initial condition (4) can be written in Laplace-Fourier space as

$$\widehat{\widetilde{W}}^{(n)}(|\mathbf{q}|,s)=\frac{p_1s^{\alpha_1-1}+p_2s^{\alpha_2-1}}{p_1s^{\alpha_1}+p_2s^{\alpha_2}+|\mathbf{q}|^2}, \tag{9}$$

where the tilde refers to the Laplace transform $\widetilde{f}(\mathbf{r},s)=\mathscr{L}\{f(\mathbf{r},t);t\}(\mathbf{r},s)=\int_0^\infty f(\mathbf{r},t)\exp(-st)dt$, the hat refers to the Fourier transform $\widehat{f}(\mathbf{q},t)=\mathscr{F}_n\{f^{(n)}(\mathbf{r},t)\}(\mathbf{q},t)=\int_{\mathbb{R}^n}f^{(n)}(\mathbf{r},t)\exp(\imath\mathbf{q}\cdot\mathbf{r})d^n\mathbf{r}$, $s\in\mathbb{C}$ is the Laplace parameter, and $\mathbf{q}=(q_1,\ldots,q_n)\in\mathbb{R}^n$ is the wave vector, $|\mathbf{q}|^2=q_1^2+\ldots+q_n^2$ and $n=1$, 2 or 3. Eq. (9) can be rearranged to take the form [52,79]

$$\widehat{\widetilde{W}}^{(n)}(|\mathbf{q}|,s)=\frac{p_1s^{\alpha_1-1}+p_2s^{\alpha_2-1}}{p_2s^{\alpha_2}+|\mathbf{q}|^2}\left[1+\frac{p_1s^{\alpha_1}}{p_2s^{\alpha_2}+|\mathbf{q}|^2}\right]^{-1}. \tag{10}$$

With the condition $p_1\le p_2$ we can make a formal expansion of the term in the square brackets of Eq. (10) to get

$$\widehat{\widetilde{W}}^{(n)}(|\mathbf{q}|,s)=p_1\widehat{\widetilde{W}}_1^{(n)}(|\mathbf{q}|,s)+p_2\widehat{\widetilde{W}}_2^{(n)}(|\mathbf{q}|,s), \tag{11}$$

where

$$\widehat{\widetilde{W}}_1^{(n)}(|\mathbf{q}|,s)=\sum_{k=0}^{\infty}(-p_1)^k\frac{s^{\alpha_1(k+1)-1}}{(p_2s^{\alpha_2}+|\mathbf{q}|^2)^{k+1}}, \tag{12a}$$

$$\widehat{\widetilde{W}}_2^{(n)}(|\mathbf{q}|,s)=\sum_{k=0}^{\infty}(-p_1)^k\frac{s^{\alpha_1k+\alpha_2-1}}{(p_2s^{\alpha_2}+|\mathbf{q}|^2)^{k+1}}. \tag{12b}$$

The propagator $W^{(n)}(\mathbf{r},t)$ in position-time space can be obtained by inverting the functions $\widehat{\widetilde{W}}_1^{(n)}(|\mathbf{q}|,s)$ and $\widehat{\widetilde{W}}_2^{(n)}(|\mathbf{q}|,s)$ to $(\mathbf{r},t)$ and substituting into $W^{(n)}(\mathbf{r},t)=p_1W_1^{(n)}(\mathbf{r},t)+p_2W_2^{(n)}(\mathbf{r},t)$. Let us start with inverting the Fourier transform in Eq. (12), for which we have

$$\widetilde{W}_1^{(n)}(\mathbf{r},s)=\left(\frac{1}{2\pi}\right)^n\sum_{k=0}^{\infty}(-p_1)^ks^{\alpha_1(k+1)-1}\int_{\mathbb{R}^n}\frac{\exp(-\imath\mathbf{q}\cdot\mathbf{r})}{(p_2s^{\alpha_2}+|\mathbf{q}|^2)^{k+1}}d^n\mathbf{q}, \tag{13a}$$

$$\widetilde{W}_2^{(n)}(\mathbf{r},s)=\left(\frac{1}{2\pi}\right)^n\sum_{k=0}^{\infty}(-p_1)^ks^{\alpha_1k+\alpha_2-1}\int_{\mathbb{R}^n}\frac{\exp(-\imath\mathbf{q}\cdot\mathbf{r})}{(p_2s^{\alpha_2}+|\mathbf{q}|^2)^{k+1}}d^n\mathbf{q}. \tag{13b}$$

Due to the symmetry property $\widehat{f}(\mathbf{q})=\widehat{f}(|\mathbf{q}|)$ we can use the integral [80]

$$\int_{\mathbb{R}^n}\hat{f}(|\mathbf{q}|)\exp(-\imath\mathbf{q}\cdot\mathbf{r})d^n\mathbf{q}=(2\pi)^{n/2}r^{1-n/2}\int_0^\infty J_{(n-2)/2}(qr)q^{n/2}\hat{f}(q)dq, \tag{14}$$

where $q=|\mathbf{q}|$, $r=|\mathbf{r}|$, and $J_\nu(\cdot)$ is the Bessel function of the first kind. Thus Eq. (13) can be reduced to

$$\widetilde{W}_1^{(n)}(r,s)=\frac{r^{1-n/2}}{(2\pi)^{n/2}}\sum_{k=0}^{\infty}(-p_1)^ks^{\alpha_1(k+1)-1}\int_0^\infty\frac{q^{n/2}}{(q^2+p_2s^{\alpha_2})^{k+1}}J_{(n-2)/2}(qr)dq, \tag{15a}$$

$$\widetilde{W}_2^{(n)}(r,s)=\frac{r^{1-n/2}}{(2\pi)^{n/2}}\sum_{k=0}^{\infty}(-p_1)^ks^{\alpha_1k+\alpha_2-1}\int_0^\infty\frac{q^{n/2}}{(q^2+p_2s^{\alpha_2})^{k+1}}J_{(n-2)/2}(qr)dq. \tag{15b}$$

With the help of the integral [81]

$$\int_0^\infty\frac{x^{\nu+1}}{(x^2+a^2)^{\mu+1}}J_\nu(xy)dx=\frac{a^{\nu-\mu}}{2^\mu\Gamma(\mu+1)}y^\mu K_{\nu-\mu}(ay), \tag{16}$$

where $K_\nu(\cdot)$ is the modified Bessel function of the second kind, $\Re\{a\}>0$ and $-1<\Re\{\nu\}<2\Re\{\mu\}+3/2$, Eq. (15) can be simplified to

$$W_1^{(n)}(r,t)=\frac{1}{(2\pi)^{n/2}}\left(\frac{r}{\sqrt{p_2}}\right)^{1-n/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1r}{2\sqrt{p_2}}\right)^k\omega_1(r,t;k), \tag{17a}$$

$$W_2^{(n)}(r,t)=\frac{1}{(2\pi)^{n/2}}\left(\frac{r}{\sqrt{p_2}}\right)^{1-n/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1r}{2\sqrt{p_2}}\right)^k\omega_2(r,t;k), \tag{17b}$$

where $\omega_1(r,t;k)$ and $\omega_2(r,t;k)$ are given in Laplace space as

$$\tilde{\omega}_1(r,s;k)=s^{(\alpha_1-\alpha_2/2)(k+1)+\alpha_2n/4-1}K_{(n-2)/2-k}\left(\sqrt{p_2}rs^{\alpha_2/2}\right), \tag{18a}$$

$$\widetilde{\omega}_2(r,s;k)=s^{(\alpha_1-\alpha_2/2)k+\alpha_2(n+2)/4-1}K_{(n-2)/2-k}\left(\sqrt{p_2}rs^{\alpha_2/2}\right). \tag{18b}$$

To invert the Laplace transform of (18) we first pass from Laplace to Mellin domain, $\tilde{\omega}_{1,2}(r,s;k) \to \check{\omega}_{1,2}(r,z;k)$, where z is the parameter of the Mellin transform defined for any generic function as $\mathscr{M}\{f(x)\}(z) = \check{f}(z) = \int_0^\infty x^{z-1} f(x)dx$, and then convert from the Mellin domain to the physical domain, see [42,82] for details. Utilizing the useful relations [81,83]

$$\mathscr{M}\{K_\nu(s)\}(z) = 2^{z-2}\Gamma\left(\frac{z+\nu}{2}\right)\Gamma\left(\frac{z-\nu}{2}\right), \tag{19a}$$

$$\mathscr{M}\{x^\nu f(ax^p)\}(z) = \frac{1}{p}a^{-(z+\nu)/p}\mathscr{M}\{f(x)\}\left(\frac{z+\nu}{p}\right), \tag{19b}$$

$$\check{f}(z) = \frac{1}{\Gamma(1-z)}\mathscr{M}\{\tilde{f}(s)\}(1-z), \tag{19c}$$

we arrive at

$$\check{\omega}_1(r,z;k) \; = \frac{1}{2\alpha_2}\left(\frac{2}{\sqrt{p_2}r}\right)^{(2\alpha_1/\alpha_2-1)(k+1)+n/2}\left[\left(\frac{2}{\sqrt{p_2}r}\right)^{2/\alpha_2}\right]^{-z}\frac{\Gamma\left(\left[\frac{\alpha_1}{\alpha_2}-1\right](k+1)+\frac{n}{2}-\frac{z}{\alpha_2}\right)\Gamma\left(\frac{\alpha_1}{\alpha_2}(k+1)-\frac{z}{\alpha_2}\right)}{\Gamma(1-z)}, \tag{20a}$$

$$\check{\omega}_2(r,z;k) \; = \frac{1}{2\alpha_2}\left(\frac{2}{\sqrt{p_2}r}\right)^{(2\alpha_1/\alpha_2-1)k+n/2+1}\left[\left(\frac{2}{\sqrt{p_2}r}\right)^{2/\alpha_2}\right]^{-z}\frac{\Gamma\left(\left[\frac{\alpha_1}{\alpha_2}-1\right]k+\frac{n}{2}-\frac{z}{\alpha_2}\right)\Gamma\left(\frac{\alpha_1}{\alpha_2}k+1-\frac{z}{\alpha_2}\right)}{\Gamma(1-z)}. \tag{20b}$$

Now, inverting the Mellin transform of (20), compare (A.4), we obtain

$$\omega_1(r,t;k) = \frac{1}{2\alpha_2}\left(\frac{2}{\sqrt{p_2}r}\right)^{(2\alpha_1/\alpha_2-1)(k+1)+n/2} H_{2,1}^{0,2}\left[\left(\frac{2}{\sqrt{p_2}r}\right)^{2/\alpha_2} t\left|\begin{matrix}\left(1+\left(1-\frac{\alpha_1}{\alpha_2}\right)(k+1)-\frac{n}{2},\frac{1}{\alpha_2}\right),\left(1-\frac{\alpha_1}{\alpha_2}(k+1),\frac{1}{\alpha_2}\right)\\(0,1)\end{matrix}\right.\right], \tag{21}$$

and

$$\omega_2(r,t;k) = \frac{1}{2\alpha_2}\left(\frac{2}{\sqrt{p_2}r}\right)^{(2\alpha_1/\alpha_2-1)k+n/2+1} H_{2,1}^{0,2}\left[\left(\frac{2}{\sqrt{p_2}r}\right)^{2/\alpha_2} t\left|\begin{matrix}\left(1+\left(1-\frac{\alpha_1}{\alpha_2}\right)k-\frac{n}{2},\frac{1}{\alpha_2}\right),\left(-\frac{\alpha_1}{\alpha_2}k,\frac{1}{\alpha_2}\right)\\(0,1)\end{matrix}\right.\right], \tag{22}$$

where $H_{p,q}^{m,n}[x]$ is the Fox H-function [84], refer to Appendix A. By simplifying the above two relations using some properties of the H-function, see pp. 11–13 in [84], we obtain

$$\omega_1(r,t;k) = \frac{1}{2}\left(\frac{2}{\sqrt{p_2}r}\right)^{k+n/2+1} t^{(\alpha_2-\alpha_1)(k+1)}H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)(k+1),\alpha_2)\\\left(\frac{n}{2},1\right),(k+1,1)\end{matrix}\right.\right], \tag{23a}$$

$$\omega_2(r,t;k) = \frac{1}{2}\left(\frac{2}{\sqrt{p_2}r}\right)^{k+n/2+1} t^{(\alpha_2-\alpha_1)k}H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)k,\alpha_2)\\\left(\frac{n}{2},1\right),(k+1,1)\end{matrix}\right.\right]. \tag{23b}$$

Substituting from (23) into (17), we get

$$W_1^{(n)}(r,t) = \frac{t^{\alpha_2-\alpha_1}}{p_2(\pi r^2)^{n/2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)(k+1),\alpha_2)\\\left(\frac{n}{2},1\right),(k+1,1)\end{matrix}\right.\right] \tag{24a}$$

$$W_2^{(n)}(r,t) = \frac{1}{p_2(\pi r^2)^{n/2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)k,\alpha_2)\\\left(\frac{n}{2},1\right),(k+1,1)\end{matrix}\right.\right]. \tag{24b}$$

In view of Eqs. (11) and (24) the multi-dimensional propagator of the bi-fractional diffusion equation of the natural type is given by

$$\begin{aligned}W^{(n)}(r,t) = {} & \frac{1}{(\pi r^2)^{n/2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k \\ & \left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)(k+1),\alpha_2)\\\left(\frac{n}{2},1\right),(k+1,1)\end{matrix}\right.\right]+H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)k,\alpha_2)\\\left(\frac{n}{2},1\right),(k+1,1)\end{matrix}\right.\right]\right),\end{aligned} \tag{25}$$

where $p_1 \le p_2$, $0 < \alpha_1 < \alpha_2 \le 1$ and $n = 1$, 2, or 3. It is worth noting that Eq. (25) can be rewritten in the form

$$\begin{aligned}W^{(n)}(r,t) = {} & \left(\frac{p_2}{4\pi t^{\alpha_2}}\right)^{n/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k \\ & \left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)(k+1),\alpha_2\right)\\(0,1),\left(1-\frac{n}{2}+k,1\right)\end{matrix}\right.\right]+H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\left|\begin{matrix}\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)k,\alpha_2\right)\\(0,1),\left(1-\frac{n}{2}+k,1\right)\end{matrix}\right.\right]\right).\end{aligned} \tag{26}$$

With respect to solutions (25) and (26) we note that the convergence of the infinite series of the H-functions essentially depends on the expansion made in Eq. (10). In general if $p_1/p_2 < 1$ we may argue that the expansion is valid for sufficiently long times. In each particular case the effectiveness of the solution can be verified by numerical simulations employing series representations of the H-function as reported below in Section III.

We also note that, whenever $n=1$ Eq. (26) reduces to the one-dimensional solution of the bi-fractional diffusion equation of natural type [52]. When $p_1=0$, $p_2=1$, and $\alpha_1=\alpha_2=\alpha$, Eqs. (25) and (26) reduce to the classical Schneider-Wyss solution [42]

$$W^{(n)}(r,t)=\frac{1}{\left(\pi r^2\right)^{n/2}}H^{2,0}_{1,2}\left[\frac{r^2}{4t^\alpha}\left|\begin{matrix}(1,\alpha)\\ \left(\frac{n}{2},1\right),(1,1)\end{matrix}\right.\right]=\frac{1}{(4\pi t^\alpha)^{n/2}}H^{2,0}_{1,2}\left[\frac{r^2}{4t^\alpha}\left|\begin{matrix}\left(1-\frac{n\alpha}{2},\alpha\right)\\ (0,1),\left(1-\frac{n}{2},1\right)\end{matrix}\right.\right], \tag{27}$$

namely, the multi-dimensional propagator of the fractional diffusion equation (3).

2.2. Alternative approach

It is known that in the two-dimensional setting ($n=2$) the analysis of the solution $W^{(2)}(r,t)$ of Eq. (27) is complicated by the fact that the corresponding series expansion for the H-function presented in Appendix A diverges [85]. Indeed, disregarding the condition (A.3b) and employing expansion (A.3a) we have that

$$W^{(2)}(r,t)=\frac{1}{4\pi t^\alpha}H^{2,0}_{1,2}\left[\frac{r^2}{4t^\alpha}\Big|\begin{matrix}(1-\alpha,\alpha)\\ (0,1),(0,1)\end{matrix}\right]=\frac{1}{2\pi t^\alpha}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\frac{\Gamma(-\nu)}{\Gamma(1-\alpha-\alpha\nu)}\left(\frac{r^2}{4t^\alpha}\right)^\nu=\infty. \tag{28}$$

In this section we circumvent this problem by replacing the ordinary Laplacian Δ with the fractional Laplacian operator $-(-\Delta)^{\mu/2}$ in our starting Eq. (8). We then solve the corresponding space-time fractional diffusion equation and demonstrate that its solution converges to the solution of the two-dimensional time-fractional diffusion equation when $\mu\to 2$. This trick is used as the series expansion for the H-function converges for $\mu\neq 2$. These properties of the solution to the space-time fractional diffusion equation allow us to use the series expansion with μ close to 2 in numerical calculations for the exclusively time-fractional case. Therefore, let us introduce the fractional Laplacian defined for a generic well-behaved function $f^{(n)}:\mathbb{R}^n\to\mathbb{R}$ as a pseudo-differential operator characterized by its Fourier transform [65,86] as

$$\mathscr{F}_n\left\{(-\Delta)^{\mu/2}f^{(n)}(\mathbf{r})\right\}(\mathbf{q})=|\mathbf{q}|^\mu\mathscr{F}_n\left\{f^{(n)}(\mathbf{r})\right\}(\mathbf{q}), \tag{29}$$

where $1<\mu<2$. Then Eq. (8) with the fractional Laplacian (29) reads

$$p_1{}^C{}_0\mathcal{D}_t^{\alpha_1}W_\mu^{(n)}(\mathbf{r},t)+p_2{}^C{}_0\mathcal{D}_t^{\alpha_2}W_\mu^{(n)}(\mathbf{r},t)=-(-\Delta)^{\mu/2}W_\mu^{(n)}(\mathbf{r},t), \tag{30}$$

where, as we shall see, the propagator of the bi-fractional diffusion equation can be deduced from the solution of (30) by setting $\mu=2$ in the one- and three-dimensional cases or taking the limit

$$W^{(2)}(\mathbf{r},t)=\lim_{\mu\to 2}W_\mu^{(2)}(\mathbf{r},t), \tag{31}$$

in the two-dimensional case. The solution of Eq. (30) in Laplace-Fourier space is given as

$$\widehat{\widetilde{W}}{}_\mu^{(n)}(|\mathbf{q}|,s)=p_1\widehat{\widetilde{W}}{}_{\mu_1}^{(n)}(|\mathbf{q}|,s)+p_2\widehat{\widetilde{W}}{}_{\mu_2}^{(n)}(|\mathbf{q}|,s), \tag{32}$$

where

$$\widehat{\widetilde{W}}{}_{\mu_1}^{(n)}(|\mathbf{q}|,s)=\sum_{k=0}^{\infty}(-p_1)^k\frac{s^{\alpha_1(k+1)-1}}{(p_2s^{\alpha_2}+|\mathbf{q}|^\mu)^{k+1}}, \tag{33a}$$

$$\widehat{\widetilde{W}}{}_{\mu_2}^{(n)}(|\mathbf{q}|,s)=\sum_{k=0}^{\infty}(-p_1)^k\frac{s^{\alpha_1k+\alpha_2-1}}{(p_2s^{\alpha_2}+|\mathbf{q}|^\mu)^{k+1}}. \tag{33b}$$

Inverting the Laplace transform of (33), refer to Eq. (A.16), we obtain (see [66])

$$\widehat{W}_{\mu_1}^{(n)}(|\mathbf{q}|,t)=\frac{t^{\alpha_2-\alpha_1}}{p_2}\sum_{k=0}^{\infty}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^kE^{k+1}_{\alpha_2,(\alpha_2-\alpha_1)(k+1)+1}\left(-\frac{t^{\alpha_2}}{p_2}|\mathbf{q}|^\mu\right), \tag{34a}$$

$$\widehat{W}_{\mu_2}^{(n)}(|\mathbf{q}|,t)=\frac{1}{p_2}\sum_{k=0}^{\infty}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^kE^{k+1}_{\alpha_2,(\alpha_2-\alpha_1)k+1}\left(-\frac{t^{\alpha_2}}{p_2}|\mathbf{q}|^\mu\right), \tag{34b}$$

where $E^\gamma_{\alpha,\beta}(\cdot)$ is the Prabhakar generalization of the Mittag-Leffler function [75,87–89]. Making use of Eq. (14) the inversion of the Fourier transform in expression (34) is given by

$$W_{\mu_1}^{(n)}(r,t)=\frac{r^{1-n/2}t^{\alpha_2-\alpha_1}}{p_2(2\pi)^{n/2}}\sum_{k=0}^{\infty}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\int_0^\infty q^{n/2}J_{(n-2)/2}(qr)E^{k+1}_{\alpha_2,(\alpha_2-\alpha_1)(k+1)+1}\left(-\frac{t^{\alpha_2}}{p_2}q^\mu\right)dq, \tag{35a}$$

$$W_{\mu_2}^{(n)}(r,t)=\frac{r^{1-n/2}}{p_2(2\pi)^{n/2}}\sum_{k=0}^{\infty}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\int_0^\infty q^{n/2}J_{(n-2)/2}(qr)E^{k+1}_{\alpha_2,(\alpha_2-\alpha_1)k+1}\left(-\frac{t^{\alpha_2}}{p_2}q^\mu\right)dq. \tag{35b}$$

To evaluate the integrals of Eq. (35) we use the Hankel transform of the Prabhakar generalization of the Mittag-Leffler function (A.17),

$$W_{\mu_1}^{(n)}(r,t)=\frac{t^{\alpha_2-\alpha_1}}{p_2}\left(\frac{p_2}{2^\mu\pi^{\mu/2}t^{\alpha_2}}\right)^{n/\mu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^kH^{2,1}_{2,3}\left[\frac{p_2r^\mu}{2^\mu t^{\alpha_2}}\left|\begin{matrix}\left(1-\frac{n}{\mu},1\right);\left(1-\frac{n\alpha_2}{\mu}+(\alpha_2-\alpha_1)(k+1),\alpha_2\right)\\ \left(0,\frac{\mu}{2}\right),\left(1+k-\frac{n}{\mu},1\right);\left(1-\frac{n}{2},\frac{\mu}{2}\right)\end{matrix}\right.\right], \tag{36a}$$

$$W^{(n)}_{\mu_2}(r,t)=\frac{1}{p_2}\left(\frac{p_2}{2^\mu\pi^{\mu/2}t^{\alpha_2}}\right)^{n/\mu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k H^{2,1}_{2,3}\left[\frac{p_2r^\mu}{2^\mu t^{\alpha_2}}\middle|\begin{matrix}\left(1-\frac{n}{\mu},1\right);\left(1-\frac{n\alpha_2}{\mu}+(\alpha_2-\alpha_1)k,\alpha_2\right)\\ \left(0,\frac{\mu}{2}\right),\left(1+k-\frac{n}{\mu},1\right);\left(1-\frac{n}{2},\frac{\mu}{2}\right)\end{matrix}\right], \tag{36b}$$

and the multi-dimensional propagator of Eq. (30) is then determined from

$$W^{(n)}_{\mu}(r,t)=p_1W^{(n)}_{\mu_1}(r,t)+p_2W^{(n)}_{\mu_2}(r,t). \tag{37}$$

We note that when $p_1=0$, $p_2=1$ and $\alpha_1=\alpha_2=\alpha$ we get the multi-dimensional solution of the space-time fractional diffusion equation,

$$W^{(n)}_{\mu}(r,t)=\left(\frac{1}{2^\mu\pi^{\mu/2}t^{\alpha}}\right)^{n/\mu}H^{2,1}_{2,3}\left[\frac{r^\mu}{2^\mu t^{\alpha}}\middle|\begin{matrix}\left(1-\frac{n}{\mu},1\right);\left(1-\frac{n\alpha}{\mu},\alpha\right)\\ \left(0,\frac{\mu}{2}\right),\left(1-\frac{n}{\mu},1\right);\left(1-\frac{n}{2},\frac{\mu}{2}\right)\end{matrix}\right], \tag{38}$$

which is equivalent to the Mellin-Barnes representation derived in [65] (refer to relation (30) of Ref. [65]). In contrast to the two-dimensional Schneider-Wyss solution, the series expansion of the H-function in Eq. (38) in the two-dimensional case does not diverge when the space-fractality is incorporated. By use of expansion (A.3) we get

$$W^{(2)}_{\mu}(r,t)=\left(\frac{1}{2^\mu\pi^{\mu/2}t^{\alpha}}\right)^{n/\mu}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\left\{\frac{2\Gamma\left(\frac{2}{\mu}(1+\nu)\right)\Gamma\left(1-\frac{2}{\mu}(1+\nu)\right)}{\mu\Gamma\left(1-\frac{2\alpha}{\mu}(1+\nu)\right)\Gamma(1+\nu)}\left(\frac{r^\mu}{2^\mu t^\alpha}\right)^{2\nu/\mu}+\frac{\Gamma(1+\nu)\Gamma\left(1-\frac{\mu}{2}(1+\nu)\right)}{\Gamma(1-\alpha(1+\nu))\Gamma\left(\frac{\mu}{2}(1+\nu)\right)}\left(\frac{r^\mu}{2^\mu t^\alpha}\right)^{1-2/\mu+\nu}\right\}. \tag{39}$$

Moreover, it can be similarly shown that the two-dimensional limit of expressions (36) to (37) does not diverge in contrast to the two-dimensional case of Eq. (26).

If we set $\alpha=1$ in Eq. (38) we obtain the propagator of the space-fractional diffusion equation,

$$\begin{aligned}W^{(n)}_{\mu}(r,t)&=\left(\tfrac{1}{2^\mu\pi^{\mu/2}t}\right)^{n/\mu}H^{2,1}_{2,3}\left[\tfrac{r^\mu}{2^\mu t}\middle|\begin{matrix}\left(1-\frac{n}{\mu},1\right);\left(1-\frac{n}{\mu},1\right)\\ \left(0,\frac{\mu}{2}\right),\left(1-\frac{n}{\mu},1\right);\left(1-\frac{n}{2},\frac{\mu}{2}\right)\end{matrix}\right]\\ &=\left(\tfrac{1}{2^\mu\pi^{\mu/2}t}\right)^{n/\mu}H^{1,1}_{1,2}\left[\tfrac{r^\mu}{2^\mu t}\middle|\begin{matrix}\left(1-\frac{n}{\mu},1\right)\\ \left(0,\frac{\mu}{2}\right);\left(1-\frac{n}{2},\frac{\mu}{2}\right)\end{matrix}\right]\\ &=\tfrac{2}{\mu}\left(\tfrac{1}{2^\mu\pi^{\mu/2}t}\right)^{n/\mu}H^{1,1}_{1,2}\left[\tfrac{r^2}{4t^{2/\mu}}\middle|\begin{matrix}\left(1-\frac{n}{\mu},\frac{2}{\mu}\right)\\ (0,1);\left(1-\frac{n}{2},1\right)\end{matrix}\right].\end{aligned} \tag{40}$$

Using the relation between the H-function and the generalized Wright function, see Eq. (1.140) in [84], we get (*cf.* relation (36) in [65])

$$W^{(n)}_{\mu}(r,t)=\frac{2}{\mu}\left(\frac{1}{2^\mu\pi^{\mu/2}t}\right)^{n/\mu}{}_1\Psi_1\left[-\frac{r^2}{4t^{2/\mu}}\middle|\begin{matrix}\left(\frac{n}{\mu},\frac{2}{\mu}\right)\\ \left(\frac{n}{2},1\right)\end{matrix}\right], \tag{41}$$

where ${}_p\Psi_q[z|{}^{(a_p,A_p)}_{(b_q,B_q)}]$ is the generalized Wright function. We refer the reader to [90,91], where the one-dimensional space-time fractional diffusion equation was investigated.

3. Properties of the solution

We now proceed to analyze the properties of our general solution. In particular, we investigate the non-negativity, the moments, and the asymptotic behavior of the PDF.

3.1. Non-negativity

The non-negativity of solutions (25) and (26) can be verified along two different routes.

3.1.1. Bernstein function approach

This first approach is based on finding the solutions in the Laplace domain and then check whether they are completely monotone functions or not [92,93]. Inverting the Fourier transform in Eq. (9), we get [94]:

$$\widetilde{W}^{(1)}(x,\lambda)=\frac{1}{2\lambda}\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}}\,\exp\left(-|x|\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}}\right), \tag{42}$$

$$\widetilde{W}^{(2)}(r,\lambda)=\frac{1}{2\pi\lambda}(p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2})K_0\left(r\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}}\right), \tag{43}$$

$$\widetilde{W}^{(3)}(r,\lambda)=\frac{1}{4\pi r\lambda}(p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2})\exp\left(-r\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}}\right), \tag{44}$$

where $\lambda=\Re\{s\}>0$.

Since we have that λ^α is a complete Bernstein function (CBF) for $\lambda>0$ and $\alpha\in(0,1]$ then λ^{α_1} and λ^{α_2} are CBFs. Since the linear combination of two complete Bernstein functions is also a complete Bernstein function, $p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}\in$ CBF. Furthermore, since $\sqrt{\lambda}\in$ CBF and the composition of two complete Bernstein functions is a complete Bernstein function, we see that $\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}} \in$ CBF. From the fact that the function $\varphi(\lambda)/\lambda$ is a completely monotone function if $\varphi(\lambda) \in$ CBF we have that $(p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2})/\lambda$ and $\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}}/\lambda$ are completely monotone functions. Conversely, since $\exp(-a\lambda)$ and $K_0(a\lambda)$ are completely monotone functions for $a, \lambda > 0$, and since the completely monotone function of a complete Bernstein function is completely monotone function, we see that $\exp(-r\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}})$ and $K_0(r\sqrt{p_1\lambda^{\alpha_1}+p_2\lambda^{\alpha_2}})$ are completely monotone functions.

From this discussion we conclude that $\widetilde{W}^{(1)}(x,\lambda)$, $\widetilde{W}^{(2)}(r,\lambda)$, and $\widetilde{W}^{(3)}(r,\lambda)$ are completely monotone functions as products of two completely monotone functions, see Eqs. (42) to (44), thus proving the non-negativity of the solutions of the bi-fractional diffusion equation of the natural type in higher dimensions.

3.1.2. Subordination approach and H-transform

The second approach for proving the non-negativity of solutions in higher dimensions is based on the subordination principle [43,52,95,96]. The subordination approach is a mathematical tool transforming a given distribution at a physical timescale u to another distribution at an operational time t. It depends on the integral transformation [85,97,98]

$$W^{(n)}(r,t) = \int_0^\infty N(u,t)G^{(n)}(r,u)du, \tag{45}$$

where $G^{(n)}(r,t)$ is the solution of the ordinary diffusion equation, namely,

$$G^{(n)}(r,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{r^2}{4t}\right), \tag{46}$$

and $N(u,t)$ is a PDF. Utilizing the identity $\zeta^{-1} = \int_0^\infty \exp(-u\zeta)du$ [99], Eq. (9) can be recast into the form (45) such that

$$\tilde{N}(u,s) = \left(p_1 s^{\alpha_1-1} + p_2 s^{\alpha_2-1}\right)\exp\left(-[p_1 s^{\alpha_1} + p_2 s^{\alpha_2}]u\right). \tag{47}$$

In [43] the non-negativity of the solution of the one-dimensional space was verified by showing that the kernel (47) is a completely monotone function. In [52] it was shown that $N(u,t)$ can be expressed in the convolution form

$$N(u,t) = \int_0^t [\Theta_1(u,\tau)\Phi_2(u,t-\tau) + \Theta_2(u,\tau)\Phi_1(u,t-\tau)]d\tau, \tag{48a}$$

where

$$\Theta_i(u,t) = \frac{p_i t}{\alpha_i (p_i u)^{1+1/\alpha_i}} \ell_{\alpha_i}\left(\frac{t}{(p_i u)^{1/\alpha_i}}\right), \quad \Phi_i(u,t) = \frac{1}{(p_i u)^{1/\alpha_i}} \ell_{\alpha_i}\left(\frac{t}{(p_i u)^{1/\alpha_i}}\right). \tag{48b}$$

Here $\ell_{\alpha_i}(\cdot)$ is the one-sided Lévy stable density, and $i = 1, 2$.

In what follows, we give an interpretation for the integral transform (45). Using the expansion $\exp(-x) = \sum_{k=0}^\infty (-x)^k/k!$, Eq. (47) can be rewritten as

$$N(u,t) = \sum_{k=0}^\infty \frac{(-p_1 u)^k}{k!}[p_1 N_1(u,t) + p_2 N_2(u,t)], \tag{49}$$

where $N_1(u,t)$ and $N_2(u,t)$ are given in the Laplace domain by

$$\tilde{N}_1(u,s) = s^{\alpha_1(k+1)-1}\exp\left(-p_2 u s^{\alpha_2}\right), \tag{50a}$$

$$\tilde{N}_2(u,s) = s^{\alpha_1 k + \alpha_2 - 1}\exp\left(-p_2 u s^{\alpha_2}\right). \tag{50b}$$

Using the useful relation (A.11) we can invert the Laplace transform in Eq. (50) and obtain the following form

$$N_1(u,t) = t^{-\alpha_1(k+1)} H_{1,1}^{1,0}\left[\frac{p_2 u}{t^{\alpha_2}}\middle| \begin{matrix}(1-\alpha_1(k+1),\alpha_2)\\(0,1)\end{matrix}\right], \tag{51a}$$

$$N_2(u,t) = t^{-\alpha_1 k - \alpha_2} H_{1,1}^{1,0}\left[\frac{p_2 u}{t^{\alpha_2}}\middle| \begin{matrix}(1-\alpha_1 k-\alpha_2,\alpha_2)\\(0,1)\end{matrix}\right]. \tag{51b}$$

Now substituting (51) into (49) and rearranging the arguments, we arrive at

$$N(u,t) = \frac{p_2}{t^{\alpha_2}}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k \left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1} H_{1,1}^{1,0}\left[\frac{p_2 u}{t^{\alpha_2}}\middle| \begin{matrix}(1+(\alpha_2-\alpha_1)k-\alpha_1,\alpha_2)\\(k,1)\end{matrix}\right] + H_{1,1}^{1,0}\left[\frac{p_2 u}{t^{\alpha_2}}\middle| \begin{matrix}(1+(\alpha_2-\alpha_1)k-\alpha_2,\alpha_2)\\(k,1)\end{matrix}\right]\right\}. \tag{52}$$

We now use the H-transform of a generic well-behaved function $f(t)$ as [100]

$$(\mathscr{H}f)(x) = \int_0^\infty H_{p,q}^{m,n}\left[xt\middle| \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] f(t)dt. \tag{53}$$

Then substituting from (52) into (45) and utilizing the definition (53) we have

$$W^{(n)}(r,t) = \frac{p_2}{t^{\alpha_2}}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k \left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}(\mathscr{H}_1 G^{(n)})(r,t) + (\mathscr{H}_2 G^{(n)})(r,t)\right\}, \tag{54}$$

where $G^{(n)}(r,t)$ is the Gaussian distribution (46), whose H-transforms are given by

$$\left(\mathscr{H}_1 G^{(n)}\right)(r,t)=\int_0^\infty H^{1,0}_{1,1}\left[\frac{p_2 u}{t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)k-\alpha_1,\alpha_2)\\(k,1)\end{matrix}\right.\right]G^{(n)}(r,u)du, \tag{55a}$$

$$\left(\mathscr{H}_2 G^{(n)}\right)(r,t)=\int_0^\infty H^{1,0}_{1,1}\left[\frac{p_2 u}{t^{\alpha_2}}\left|\begin{matrix}(1+(\alpha_2-\alpha_1)k-\alpha_2,\alpha_2)\\(k,1)\end{matrix}\right.\right]G^{(n)}(r,u)du. \tag{55b}$$

Therefore, the subordination trick, or alternatively the integral transform (45) for the bi-fractional diffusion equation of natural type can be viewed as an infinite series of H-transforms of the Gaussian distribution.

It is interesting to show that the H-transforms (54) and (55) are exactly equivalent to the generalized Schneider-Wyss solution (26). Indeed if we use the Gaussian (A.12) together with the integral relation (A.13) we can rewrite the H-transforms of the Gaussian (55) as

$$\left(\mathscr{H}_1 G^{(n)}\right)(r,t)=\frac{t^{\alpha_2}}{p_2}\left(\frac{p_2}{4\pi t^{\alpha_2}}\right)^{n/2}H^{2,0}_{1,2}\left[\frac{p_2 r^2}{4t^{\alpha_2}}\left|\begin{matrix}\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)(k+1),\alpha_2\right)\\(0,1),\left(1-\frac{n}{2}+k,1\right)\end{matrix}\right.\right], \tag{56a}$$

$$\left(\mathscr{H}_2 G^{(n)}\right)(r,t)=\frac{t^{\alpha_2}}{p_2}\left(\frac{p_2}{4\pi t^{\alpha_2}}\right)^{n/2}H^{2,0}_{1,2}\left[\frac{p_2 r^2}{4t^{\alpha_2}}\left|\begin{matrix}\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)k,\alpha_2\right)\\(0,1),\left(1-\frac{n}{2}+k,1\right)\end{matrix}\right.\right]. \tag{56b}$$

Combining (54) with (56) it is easy to check the equivalence between the H-transforms (54) and (55) and the generalized Schneider-Wyss solution (26).

As a special case, if we set $p_1=0$, $p_2=1$, and $\alpha_1=\alpha_2=\alpha$ in Eq. (54) the H-transform reduces to

$$W^{(n)}(r,t)=\frac{1}{t^\alpha}\left(\mathscr{H}G^{(n)}\right)(r,t)=\frac{1}{t^\alpha}\int_0^\infty H^{1,0}_{1,1}\left[\frac{u}{t^\alpha}\left|\begin{matrix}(1-\alpha,\alpha)\\(0,1)\end{matrix}\right.\right]G^{(n)}(r,u)du \tag{57}$$

$$=\frac{1}{t^\alpha}\int_0^\infty M_\alpha\left(\frac{u}{t^\alpha}\right)G^{(n)}(r,u)du, \tag{58}$$

where $M_\alpha(\cdot)$ is the M-Wright function. Using the relation between the M-Wright function and the one-sided Lévy stable density, i.e., $\ell_\alpha(t)=\frac{\alpha}{t^{\alpha+1}}M_\alpha\left(\frac{1}{t^\alpha}\right)$, the transform (57) reduces to the inverse Lévy transform suggested in [85,98],

$$W^{(n)}(r,t)=\frac{t}{\alpha}\int_0^\infty\frac{1}{u^{1+1/\alpha}}\ell_\alpha\left(\frac{t}{u^{1/\alpha}}\right)G^{(n)}(r,u)du. \tag{59}$$

3.2. Moments

We now derive the moments of the PDF $W^{(n)}(r,t)$ in n dimensions. The non-zero, even moments are defined through [80]

$$\begin{aligned}\mathrm{M}(2m_1,\dots,2m_n)&=\int_{\mathbb{R}^n}x_1^{2m_1}\dots x_n^{2m_n}W^{(n)}(\mathbf{r},t)d^n\mathbf{r}=\int_{\mathbb{R}^n}e_1^{2m_1}\cdots e_n^{2m_n}W^{(n)}(\mathbf{r},t)r^{2m}d^n\mathbf{r}\\&=\int_{\mathbb{R}^n}e_1^{2m_1}\cdots e_n^{2m_n}W^{(n)}(\mathbf{r},t)r^{2m}r^{n-1}drd^{n-1}\mathbf{e},\end{aligned} \tag{60}$$

where m_i are positive integers for all $i=1,2,\dots,n$, $m=\sum_{i=1}^n m_i$, and $d^{n-1}\mathbf{e}$ is the angular integral with the radial unit vector $\mathbf{e}=(e_1,\dots,e_n)$ and $e_i=x_i/r$. Therefore,

$$\begin{aligned}\mathrm{M}(2m_1,\dots,2m_n)&=\left(\int_{\mathbb{R}^{n-1}}e_1^{2m_1}\cdots e_n^{2m_n}d^{n-1}\mathbf{e}\right)\left(\int_0^\infty r^{2m+n-1}W^{(n)}(r,t)dr\right)\\&=\Omega_n(m_1,\dots,m_n,n)(\mathscr{M}W^{(n)}(r,t))(2m+n,t),\end{aligned} \tag{61}$$

where

$$\Omega_n(m_1,\dots,m_n,n)=\int_{\mathbb{R}^{n-1}}e_1^{2m_1}\cdots e_n^{2m_n}d^{n-1}\mathbf{e}=\frac{2}{\Gamma\left(m+\frac{n}{2}\right)}\prod_{i=1}^n\Gamma\left(m_i+\frac{1}{2}\right). \tag{62}$$

The Mellin transform of solution (26) with respect to r, using Eq. (A.4), yields

$$\begin{aligned}(\mathscr{M}W^{(n)}(r,t))(z,t)&=\frac{1}{2}\left(\frac{p_2}{4\pi t^{\alpha_2}}\right)^{n/2}\sum_{k=0}^\infty\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\\&\times\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\left(\frac{p_2}{4t^{\alpha_2}}\right)^{-z/2}\frac{\Gamma\left(\frac{z}{2}\right)\Gamma\left(1-\frac{n}{2}+k+\frac{z}{2}\right)}{\Gamma\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)(k+1)+\frac{\alpha_2 z}{2}\right)}+\left(\frac{p_2}{4t^{\alpha_2}}\right)^{-z/2}\frac{\Gamma\left(\frac{z}{2}\right)\Gamma\left(1-\frac{n}{2}+k+\frac{z}{2}\right)}{\Gamma\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)k+\frac{\alpha_2 z}{2}\right)}\right\}.\end{aligned} \tag{63}$$

Now, upon inserting Eq. (63) into (61) we obtain

$$\begin{aligned}\mathrm{M}(2m_1,\dots,2m_n)&=\frac{\Omega_n\Gamma\left(m+\frac{n}{2}\right)\Gamma(m+1)}{2\pi^{n/2}}\left(\frac{4t^{\alpha_2}}{p_2}\right)^m\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\sum_{k=0}^\infty\frac{(m+1)_k\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k}{k!\Gamma(1+\alpha_2 m+(\alpha_2-\alpha_1)(k+1))}\right.\\&\left.+\sum_{k=0}^\infty\frac{(m+1)_k\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k}{k!\Gamma(1+\alpha_2 m+(\alpha_2-\alpha_1)k)}\right\}.\end{aligned} \tag{64}$$

Recalling the definition of the generalized Mittag-Leffler function and utilizing the recurrence relation (A.18), we write the closed-form moments of the bi-fractional diffusion equation in n dimensions as

$$\mathrm{M}(2m_1,\dots,2m_n)=C_{m,n}t^{\alpha_2 m}E^{m}_{\alpha_2-\alpha_1,1+\alpha_2 m}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right), \tag{65}$$

where

$$C_{m,n}=\frac{2^{2m}}{\pi^{n/2}p_2^m}\Gamma(m+1)\prod_{i=1}^{n}\Gamma\left(m_i+\frac{1}{2}\right). \tag{66}$$

Next we calculate the qth order moment from the solution (26), given by

$$\langle r^q(t)\rangle=\int_{\mathbb{R}^n}|\mathbf{r}|^q W^{(n)}(\mathbf{r},t)d^n\mathbf{r}. \tag{67}$$

Due to the radial symmetry, $W^{(n)}(\mathbf{r},t)=W^{(n)}(|\mathbf{r}|,t)=W^{(n)}(r,t)$, we have that

$$\begin{aligned}\langle r^q(t)\rangle &=\int_{\mathbb{R}^n}r^qW^{(n)}(r,t)d^n\mathbf{r}=\int_{\mathbb{R}^n}r^qW^{(n)}(r,t)r^{n-1}drd^{n-1}\mathbf{e}\\ &=\left(\int_{\mathbb{R}^{n-1}}d^{n-1}\mathbf{e}\right)\left(\int_0^\infty r^{q+n-1}W^{(n)}(r,t)dr\right),\end{aligned} \tag{68}$$

where

$$\int_{\mathbb{R}^{n-1}}d^{n-1}\mathbf{e}=\Omega_n(0,\dots,0,n)=\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}, \tag{69}$$

and $\Omega_n(m_1,\dots,m_n,n)$ is defined in Eq. (62). Therefore,

$$\langle r^q(t)\rangle=\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty r^{q+n-1}W^{(n)}(r,t)dr. \tag{70}$$

From result (B.4) in Appendix B, we obtain for the qth order moment

$$\langle r^q(t)\rangle=\frac{2^q\Gamma\left(\frac{q+n}{2}\right)\Gamma\left(1+\frac{q}{2}\right)}{\Gamma\left(\frac{n}{2}\right)p_2^{q/2}}t^{\alpha_2 q/2}E^{q/2}_{\alpha_2-\alpha_1,\alpha_2 q/2+1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right). \tag{71}$$

For the one-dimensional case, $n=1$, one recovers the result obtained in [52],

$$<|x|^q(t)>=\frac{\Gamma(q+1)}{p_2^{q/2}}t^{\alpha_2 q/2}E^{q/2}_{\alpha_2-\alpha_1,\alpha_2 q/2+1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right). \tag{72}$$

From here we conclude that the process governed by the multi-dimensional bi-fractional diffusion equation of natural type belongs to a multifractal process of the form

$$\langle r^q(t)\rangle\simeq t^{\mu(q,t)}, \tag{73}$$

where $\mu(q,t)$ is a time-dependent exponent [52].

For $q=2$ we find the second moment

$$\langle r^2(t)\rangle=\frac{2n}{p_2}t^{\alpha_2}E_{\alpha_2-\alpha_1,\alpha_2+1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right), \tag{74}$$

which for the one-dimensional case reduces to the MSD

$$\langle x^2(t)\rangle=\frac{2}{p_2}t^{\alpha_2}E_{\alpha_2-\alpha_1,\alpha_2+1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right). \tag{75}$$

3.3. Asymptotic behaviors of the PDF

We finally turn to the PDF. While the exact form of the PDF for given time and position needs to be analyzed numerically, we obtain explicit expressions in the limits of short time-long distance and long time-short distance and evaluate the asymptotic expressions graphically.

3.3.1. Short-time and long distance asymptotics

From the calculations carried out in Appendix C.1 in the limit $\frac{p_2r^2}{4t^{\alpha_2}}\gg 1$, i.e., short-time and/or long-distance behavior, we find the asymptotic shape of the PDF (25) in the form

$$W^{(n)}(r,t)\simeq A_1r^{-(1-\alpha_2)n/(2-\alpha_2)}t^{-\alpha_2 n/[2(2-\alpha_2)]}\left[1+A_2r^{-2(\alpha_2-\alpha_1)/(2-\alpha_2)}t^{2(\alpha_2-\alpha_1)/(2-\alpha_2)}\right]\mathrm{Exp}_1(r,t)\mathrm{Exp}_2(r,t), \tag{76}$$

where

$$\begin{aligned}A_1&=\frac{1}{\pi^{n/2}\sqrt{2-\alpha_2}}\alpha_2^{[\alpha_2(n+1)-2]/[2(2-\alpha_2)]}\left(\frac{p_2}{4}\right)^{n/[2(2-\alpha_2)]},\\ A_2&=p_1p_2^{(\alpha_1-1)/(2-\alpha_2)}2^{2(\alpha_2-\alpha_1)/(2-\alpha_2)}\alpha_2^{2(\alpha_1-\alpha_2)/(2-\alpha_2)},\end{aligned}$$

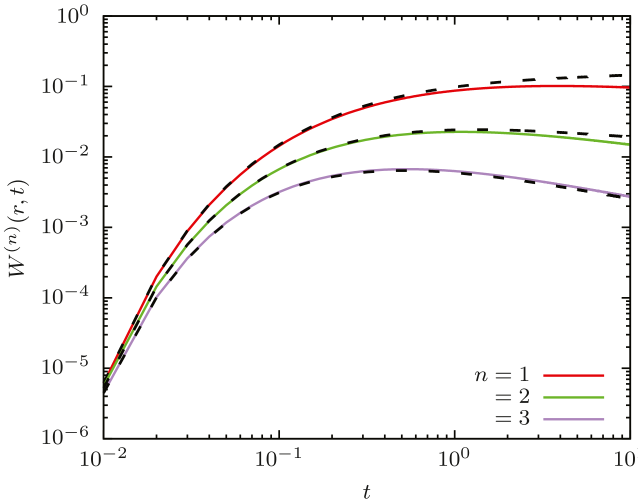

Fig. 1. Short-time asymptotic behavior (76) (black dashed line) and n-dimensional propagator, Eqs. (36) to (37), as function of time for the fixed position $r=2$. For $n=1$ and 3 we set $\mu=2$, and for $n=2$ we take $\mu=1.9995$. The model parameters are chosen as $p_1=0.1$, $p_2=0.9$, $\alpha_1=0.3$ and $\alpha_2=0.7$.

$$\mathrm{Exp}_2(r,t)=\exp\left(-2^{2(\alpha_2-\alpha_1-1)/(2-\alpha_2)}p_1p_2^{(\alpha_1-1)/(2-\alpha_2)}\alpha_2^{(2\alpha_1-\alpha_2)/(2-\alpha_2)}r^{2(\alpha_1-\alpha_2+1)/(2-\alpha_2)}t^{(\alpha_2-2\alpha_1)/(2-\alpha_2)}\right), \tag{77}$$

and

$$\mathrm{Exp}_1(r,t)=\exp\left(-(2-\alpha_2)\alpha_2^{\alpha_2/(2-\alpha_2)}\left(\frac{p_2r^2}{4t^{\alpha_2}}\right)^{1/(2-\alpha_2)}\right). \tag{78}$$

For very short times the asymptotic behavior (76) can be approximated to

$$W^{(n)}(r,t)\sim A_1r^{(\alpha_2-1)n/(2-\alpha_2)}t^{-\alpha_2n/[2(2-\alpha_2)]}\mathrm{Exp}_1(r,t)\mathrm{Exp}_2(r,t), \tag{79}$$

while for $p_1=0$, $p_2=1$, and $\alpha_2=\alpha$, Eq. (76) reduces to the classical asymptotic behavior (see Eqs. (2.25) and (2.26) in Ref [42].),

$$W^{(n)}(r,t)\sim Ar^{(\alpha-1)n/(2-\alpha)}t^{-\alpha n/[2(2-\alpha)]}\exp\left(-(2-\alpha)\alpha^{\alpha/(2-\alpha)}\left(\frac{r^2}{4t^{\alpha}}\right)^{1/(2-\alpha)}\right), \tag{80}$$

where $A=2^{-n/(2-\alpha)}\pi^{-n/2}(2-\alpha)^{-1/2}\alpha^{[\alpha(n+1)-2]/[2(2-\alpha)]}$. The graphical representation of the asymptotic behavior (76) is compared in Fig. 1 with the full propagator in Eqs. (36) and (37). We observe very good agreement between the multi-dimensional propagator and its short-time asymptotic behavior given the fixed distance $r=2$. Note that approximation (76) fits the propagator well even for relatively long times up to 10^1, at which the inequality $\frac{p_2r^2}{4t^{\alpha_2}}\gg 1$ is already violated.

3.3.2. Long-time and short-distance behavior

Recalling the asymptotic behavior of the H-function near zero according to Eq. (A.5) we now derive the asymptotic behavior of the propagator (8).

1D solution: From the calculations in Appendix C.2.1 for $\frac{p_2x^2}{4t^{\alpha_2}}\ll 1$ we find the asymptotic of the PDF (26) (or (C.4)) in terms of the three parameter Mittag-Leffler function (A.14),

$$W^{(1)}(x,t)\sim W^{(1)}(x=0,t)=\frac{p_1}{2\sqrt{p_2}}t^{\alpha_2/2-\alpha_1}E^{1/2}_{\alpha_2-\alpha_1,1+\alpha_2/2-\alpha_1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)+\frac{\sqrt{p_2}}{2}t^{-\alpha_2/2}E^{1/2}_{\alpha_2-\alpha_1,1-\alpha_2/2}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right). \tag{81}$$

Moreover, with the use of the asymptotic behavior (A.15) of the three parameter Mittag-Leffler function the solution (81) near the origin reads

$$W^{(1)}(x,t)\sim W^{(1)}(x=0,t)\sim\frac{\sqrt{p_1}}{2}\frac{t^{-\alpha_1/2}}{\Gamma\left(1-\frac{\alpha_1}{2}\right)}. \tag{82}$$

In Fig. 2 we represent the one-dimensional solution (26) and compare it with the asymptotic behaviors (81) and (82). Note that the behavior of the curves is in full agreement with the inequality indicating the short-distance regime: with increasing distance x the asymptotic solution exhibits good agreement at longer times. The solution (82) for the long time asymptotics of Eq. (81) converges to the solution at $x=0$ at longer times, as it should be.

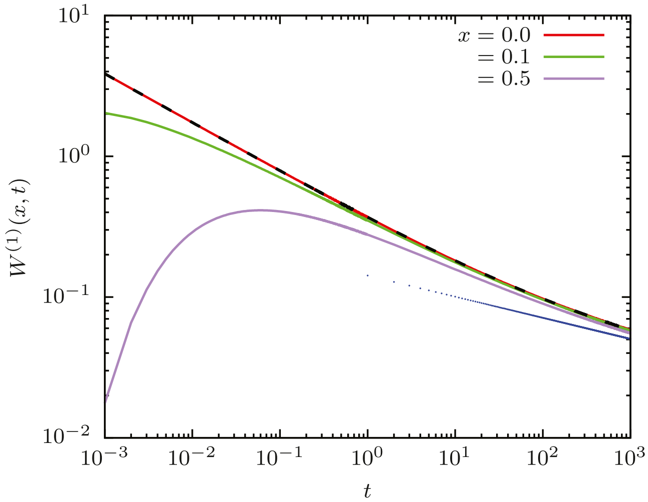

Fig. 2. Temporal evolution of the asymptotic behavior (81) (black dashed line) compared with the one-dimensional propagator (C.4) for different values of the position variable. The blue dotted line represents the long-time behavior near the origin, (82). The model parameters are chosen as $p_1 = 0.1$, $p_2 = 0.9$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.7$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

2D solution: Since the corresponding expansion (28) for the two-dimensional (2D) solution of the bi-fractional diffusion equation diverges, it is difficult to obtain an asymptotic formula near the origin based on the propagator (26). We follow an alternate approach to derive the asymptotic behavior based on the two-dimensional solution in the Laplace space,

$$\widetilde{W}^{(2)}(r,s) = \frac{1}{2\pi s}(p_1 s^{\alpha_1} + p_2 s^{\alpha_2})K_0\left(r\sqrt{p_1 s^{\alpha_1} + p_2 s^{\alpha_2}}\right). \tag{83}$$

With $K_0(z) \sim -\gamma - \ln(z/2)$ for $z \to 0$, where γ is the Euler-Mascheroni constant [101], near the origin $r = 0$ and for intermediate and long times, Eq. (83) behaves as

$$\widetilde{W}^{(2)}(r,s) \sim -\frac{\gamma p_1}{2\pi}s^{\alpha_1-1} - \frac{(\gamma+0.5)p_2}{2\pi}s^{\alpha_2-1} - \frac{\alpha_1}{4\pi}(p_1 s^{\alpha_1-1} + p_2 s^{\alpha_2-1})\ln\left(\left[\frac{\sqrt{p_1}r}{2}\right]^{2/\alpha_1} s\right). \tag{84}$$

Inverting the Laplace transform of (84) we then obtain the asymptotic behavior of the two-dimensional solution near the origin,

$$W^{(2)}(r,t) \sim \frac{p_1 t^{-\alpha_1}}{2\pi\Gamma(1-\alpha_1)}\left\{-\gamma - \frac{\alpha_1\psi_0(1-\alpha_1)}{2} + \ln\left(\frac{2t^{\alpha_1/2}}{\sqrt{p_1}r}\right)\right\}$$
$$+\frac{p_2 t^{-\alpha_2}}{2\pi\Gamma(1-\alpha_2)}\left\{-(\gamma+0.5) - \frac{\alpha_1\psi_0(1-\alpha_2)}{2} + \ln\left(\frac{2t^{\alpha_1/2}}{\sqrt{p_1}r}\right)\right\}, r \to 0, \tag{85}$$

where $\psi_0(\cdot)$ is the polygamma function. Finally, in the long-time limit (near the origin $r = 0$), Eq. (85) reduces to

$$W^{(2)}(r,t) \sim \frac{p_1 t^{-\alpha_1}}{2\pi\Gamma(1-\alpha_1)}\ln\left(\frac{2t^{\alpha_1/2}}{\sqrt{p_1}r}\right), \quad r \to 0, \ t \to \infty. \tag{86}$$

In view of expression (85) we deduce the asymptotic behavior of the two-dimensional solution of the fractional diffusion equation near the origin by setting $p_1 = 1$, $p_2 = 0$, and $\alpha_1 = \alpha$,

$$W^{(2)}(r,t) \sim \frac{t^{-\alpha}}{2\pi\Gamma(1-\alpha)}\left\{-\gamma - \frac{\alpha\psi_0(1-\alpha)}{2} + \ln\left(\frac{2t^{\alpha/2}}{r}\right)\right\}, \quad r \to 0, \tag{87}$$

which in the long time limit reduces to

$$W^{(2)}(r,t) \sim \frac{t^{-\alpha}}{2\pi\Gamma(1-\alpha)}\ln\left(\frac{2t^{\alpha/2}}{r}\right), \quad r \to 0, \ t \to \infty. \tag{88}$$

Although formula (88) was previously derived by Saichev [85,102], to our best knowledge formula (87) was not presented before in literature. The comparison between the two-dimensional propagator in Eqs. (36) to (37) and its asymptotics (85) to (86) is presented in Fig. 3. Note that we show three dashed lines in contrast to Fig. 2 since the asymptotic behavior depends on the distance r. However, the qualitative behavior remains the same.

3D solution: From the calculations in Appendix C.2.2 for $\frac{p_2 r^2}{4t^{\alpha_2}} \ll 1$ we find the asymptotic behavior of the PDF in terms of the three parameter Mittag-Leffler function (A.14) in the form

$$W^{(3)}(r,t) \sim \frac{1}{4\pi}\left\{\frac{1}{r}\left[\frac{p_1 t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + \frac{p_2 t^{-\alpha_2}}{\Gamma(1-\alpha_2)}\right] + \sqrt{p_2}\left[\frac{p_1 t^{-\alpha_1-\frac{\alpha_2}{2}}}{\Gamma\left(1-\alpha_1-\frac{\alpha_2}{2}\right)} + \frac{p_2 t^{-3\alpha_2/2}}{\Gamma\left(1-\frac{3\alpha_2}{2}\right)}\right]\right.$$
$$\left. -\sqrt{p_2}\left[p_1 t^{-\alpha_1-\alpha_2/2}E^{-1/2}_{\alpha_2-\alpha_1,1-\alpha_2/2-\alpha_1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right) + p_2 t^{-3\alpha_2/2}E^{-1/2}_{\alpha_2-\alpha_1,1-3\alpha_2/2}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)\right]\right\}, \quad r \to 0. \tag{89}$$

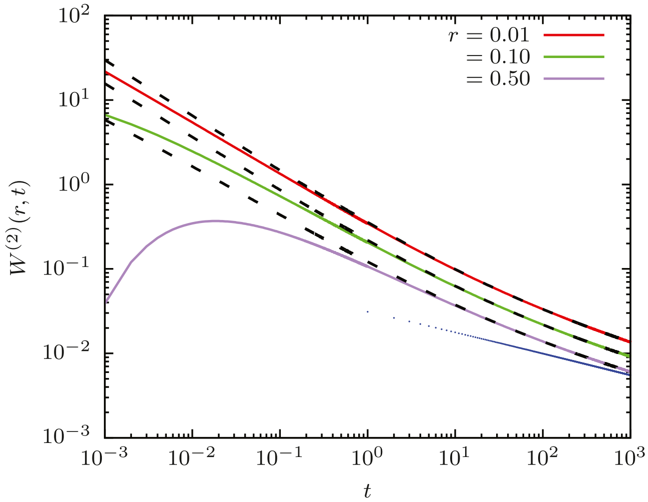

Fig. 3. Asymptotic behavior (85) (black dashed line) compared with the two-dimensional propagator, Eqs. (36) to (37) with $n = 2$ and $\mu = 1.9999$, for different values of the spatial variable. The long-time behavior (86) at $r = 0.5$ is shown, as well (blue dotted line). The model parameters are chosen as $p_1 = 0.1$, $p_2 = 0.9$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.7$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

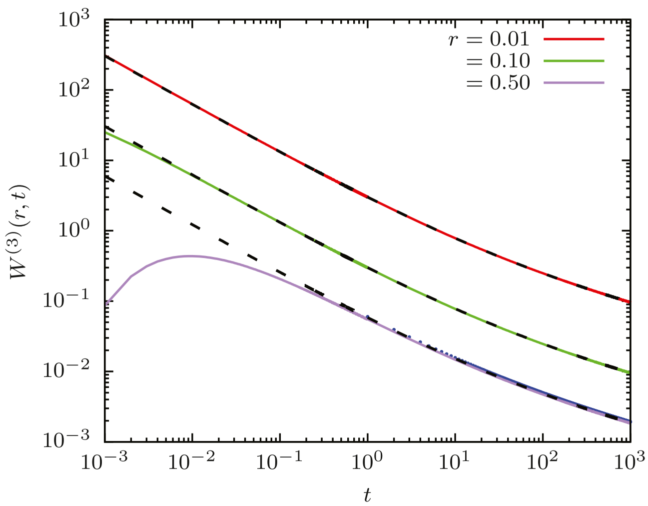

Fig. 4. Asymptotic behavior (89) (black dashed line) as function of time compared with the three-dimensional propagator (C.6) for different values of the spatial variable r. The long-time behavior (91) at $r = 0.5$ is also shown (blue dotted line). The model parameters are chosen as $p_1 = 0.1$, $p_2 = 0.9$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.7$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

At long times the asymptotic behavior (89) near the origin $r = 0$ reduces to

$$W^{(3)}(r,t) \sim \frac{1}{4\pi}\left\{\frac{1}{r}\left[\frac{p_1 t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + \frac{p_2 t^{-\alpha_2}}{\Gamma(1-\alpha_2)}\right] + \sqrt{p_2}\left[\frac{p_1 t^{-\alpha_1-\alpha_2/2}}{\Gamma\left(1-\alpha_1-\frac{\alpha_2}{2}\right)} + \frac{p_2 t^{-3\alpha_2/2}}{\Gamma\left(1-\frac{3\alpha_2}{2}\right)}\right]\right.$$
$$\left. -\sqrt{p_1}\left[\frac{p_1 t^{-3\alpha_1/2}}{\Gamma\left(1-\frac{3\alpha_1}{2}\right)} + \frac{p_2 t^{-\alpha_2-\frac{\alpha_1}{2}}}{\Gamma\left(1-\alpha_2-\frac{\alpha_1}{2}\right)}\right]\right\}, \quad r \to 0, \ t \to \infty. \tag{90}$$

Alternatively, the long-time behavior of the three-dimensional solution near the origin reads

$$W^{(3)}(r,t) \sim \frac{1}{4\pi r}\left[\frac{p_1 t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + \frac{p_2 t^{-\alpha_2}}{\Gamma(1-\alpha_2)}\right], \quad r \to 0, \ t \to \infty, \tag{91}$$

where (A.15) was used. This result generalizes the result for the mono-fractional diffusion equation (for $p_1 = 1$, $p_2 = 0$, $\alpha_1 = \alpha$), see relation (33) in [102]. We finally verify our analysis by providing a comparison between the three-dimensional propagator (C.6) and the asymptotic solutions (89) and (91) in Fig. 4.

4. Conclusion

We analyzed the multi-dimensional bi-fractional time-fractional diffusion equation of natural type describing a retarding crossover between two subdiffusive states, as a generalization of the one-dimensional case. We derived a closed-form multi-dimensional solution for this equation in terms of infinite series of Fox H-functions by generalizing the classical Schneider-Wyss solution for the mono-fractional case. The divergent series of the two-dimensional propagator was approximated by accommodating space-fractality of order $1 < \mu < 2$, for which the behavior is non-divergent, and then the two-dimensional propagator was obtained from this in the limit $\mu \to 2$. The solution was interpreted as the H-transform of the Gaussian distribution. Certain restriction are hereby imposed on the model coefficients, $p_1 \le p_2$, in order to guarantee the convergence of the series in the long-time limit. The non-negativity of the solution, in different dimensions, was shown by employing two different arguments, the Bernstein function approach and the subordination approach. The moments of the fundamental solution were obtained in terms of the Prabhakar three-parameter Mittag-Leffler function and the limiting cases were analyzed, providing a rich behavior of the dynamics in the considered multi-dimensional model. Finally, we derived the asymptotic formulas for the propagator in different dimensions, in the short-time long-space domain, and in the long-time short-space domain. Graphical comparisons between the multidimensional propagator and its asymptotic formulas show good agreements.

As shown previously, the bi-fractional diffusion equation of modified type [46] shows a different behavior and thus provides an alternative behavior for the description of crossover dynamics. Its higher-dimensional solution is the topic of our future investigation. Another important step is the consideration of confinement in bi-fractional processes in higher dimensions.

Declaration of Competing Interest

The authors declare that there is no conflict of interest.

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Appendix A. A primer on some special functions

In this appendix we present some special functions that are used throughout the paper. The Fox H-function is defined in terms of the Mellin-Barnes integral [84]

$$H^{m,n}_{p,q}\left[x\middle|\begin{array}{l}(a_1,A_1),\cdots,(a_p,A_p)\\(b_1,B_1),\cdots,(b_q,B_q)\end{array}\right]=\frac{1}{2\pi\imath}\int_\Omega\Theta(s)x^s\,ds, \tag{A.1}$$

where m, n, p, and q are integers satisfying $0 \le n \le p$, $1 \le m \le q$, $a_i,\ b_j \in \mathbb{C}$, $A_i, B_j \in \mathbb{R}_+$, $i = 1,\dots,p$, $j = 1,\dots,q$, and the function $\Theta(s)$ is given by

$$\Theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\ B_js)\prod_{j=1}^{n}\Gamma(1-a_j+\ A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\ B_js)\prod_{j=n+1}^{p}\Gamma(a_j-\ A_js)}, \tag{A.2}$$

where $\Gamma(\cdot)$ is the Gamma function. The contour Ω in the right side of Eq. (A.1) separates the poles of $\Gamma(b_j + B_js)$, $j = 1,\dots,m$ from the poles of $\Gamma(1 - a_i - A_is)$, $i = 1,\dots,n$. If the poles of $\prod_{j=1}^{m}\Gamma(b_j-\ B_js)$ are simple, the following series expansion holds

$$H^{m,n}_{p,q}\left[x\middle|\begin{array}{l}(a_p,A_p)\\(b_q,B_q)\end{array}\right]=\sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu x^{\frac{b_h+\nu}{B_h}}}{\nu!B_h}\frac{\prod_{j=1,j\neq h}^{m}\Gamma\left(b_j-\ B_j\frac{b_h+\nu}{B_h}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\ A_j\frac{b_h+\nu}{B_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\ B_j\frac{b_h+\nu}{B_h}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-\ A_j\frac{b_h+\nu}{B_h}\right)}, \tag{A.3a}$$

provided that

$$B_h(b_j+k)\neq B_j(b_h+\nu)\quad\text{for}\quad j\neq h,\quad j,h=1,2,\cdots,m,\quad k,\nu=0,1,2,\cdots, \tag{A.3b}$$

$$A_j(b_h+\nu)\neq B_h(a_j-k-1),\quad\text{for}\quad j,h=1,2,\cdots,n,\quad k,\nu=0,1,2,\cdots. \tag{A.3c}$$

The Mellin transform of the H-function is defined by

$$\int_0^\infty x^{z-1}H^{m,n}_{p,q}\left[ax\middle|\begin{array}{l}(a_p,A_p)\\(b_q,B_q)\end{array}\right]dx=a^{-z}\Theta(-z), \tag{A.4}$$

where $\Theta(z)$ is given by (A.2).

The asymptotic behavior of the H-function near zero is given by [84,103]

$$H^{m,n}_{p,q}\left[z\middle|\begin{array}{l}(a_p,A_p)\\(b_q,B_q)\end{array}\right]\sim|z|^c,\quad|z|\to0, \tag{A.5}$$

where $c = \min_{1\le j\le m}\{\Re(b_j)/B_j\}$, and provided that $\rho > 0$, $|\arg z| < \pi\rho/2$, and where ρ is given by

$$\rho=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j. \tag{A.6}$$

Using the asymptotic behavior (A.5) and the series expansion (A.3) of the H-function we find

$$H^{2,0}_{1,2}\left[z\middle|\begin{array}{l}(a_1,A_1)\\(0,1),\left(\frac12+k,1\right)\end{array}\right]\sim\frac{\Gamma\left(\frac12+k\right)}{\Gamma(a_1)},\quad|z|\to0, \tag{A.7}$$

$$H_{1,2}^{2,0}\left[z\left|\begin{array}{l}(a_1,A_1)\\(0,1),\left(-\frac{1}{2}+k,1\right)\end{array}\right.\right]\sim\begin{cases}\frac{\Gamma\left(\frac{1}{2}\right)z^{-\frac{1}{2}}}{\Gamma\left(a_1+\frac{A_1}{2}\right)}, & \text{if } k=0,\ |z|\to 0;\\ \frac{\Gamma\left(-\frac{1}{2}+k\right)}{\Gamma(a_1)}, & \text{if } k\geq 1,\ |z|\to 0,\end{cases} \tag{A.8}$$

The asymptotic behavior of the H-function for large values reads [103]

$$H_{p,q}^{m,0}\left[z\left|\begin{array}{l}(a_p,A_p)\\(b_q,B_q)\end{array}\right.\right]\sim\ \kappa z^{\gamma/\delta}\exp\left[-\delta(\varepsilon z)^{1/\delta}\right],\quad |z|\to\infty, \tag{A.9}$$

where

$$\delta=\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j,\qquad \gamma=\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j+\frac{p-q+1}{2},\qquad \varepsilon=\prod_{j=1}^{p}A_j^{A_j}\prod_{j=1}^{q}B_j^{-B_j},$$

$$\kappa=(2\pi)^{\frac{q-p-1}{2}}\ \varepsilon^{\gamma/\delta}\ \delta^{-1/2}\prod_{j=1}^{p}A_j^{\frac{1}{2}-a_j}\ \prod_{j=1}^{q}B_j^{b_j-\frac{1}{2}}. \tag{A.10}$$

The following inverse Laplace transform relation is used throughout the paper,

$$\mathscr{L}^{-1}\left\{s^{-\rho}\exp\left(-as^{\sigma}\right)\right\}=t^{\rho-1}H_{1,1}^{1,0}\left[\frac{a}{t^{\sigma}}\left|\begin{array}{l}(\rho,\sigma)\\(0,1)\end{array}\right.\right],\quad a,\sigma>0, \tag{A.11}$$

The Gaussian distribution (46) can be expressed in terms of the H-function through

$$G^{(n)}(r,u)=\frac{1}{\left(\pi r^2\right)^{n/2}}H_{1,0}^{0,1}\left[\frac{4u}{r^2}\left|\begin{array}{l}(1-n/2,1)\\ \text{———}\end{array}\right.\right], \tag{A.12}$$

The integral of the product of two H-functions is given by

$$\int_0^{\infty}H_{p,q}^{m,n}\left[zx^{\sigma}\left|\begin{array}{l}(a_p,A_p)\\(b_q,B_q)\end{array}\right.\right]H_{p_1,q_1}^{m_1,n_1}\left[\eta x\left|\begin{array}{l}(d_{p_1},D_{p_1})\\(e_{q_1},E_{q_1})\end{array}\right.\right]dx \tag{A.13}$$

$$=\frac{1}{\eta}H_{p+q_1,q+p_1}^{m+n_1,n+m_1}\left[\frac{z}{\eta^{\sigma}}\left|\begin{array}{l}(a_n,A_n),(1-e_{q_1}-E_{q_1},\sigma E_{q_1}),(a_{n+1},A_{n+1}),\cdots,(a_p,A_p)\\(b_m,B_m),(1-d_{p_1}-D_{p_1},\sigma D_{p_1}),(b_{m+1},B_{m+1}),\cdots,(b_q,B_q)\end{array}\right.\right]. \tag{A.13}$$

The Prabhakar generalization of the Mittag-Leffler function (PML) is defined in terms of the series [87–89]

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad \alpha,\beta,\gamma,z\in\mathbb{C},\Re\{\alpha\}>0, \tag{A.14}$$

where $(\gamma)_n$is the ascending Pochhammer symbol defined by $(\gamma)_0=1$, $(\gamma)_n=\gamma(\gamma+1)\cdots(\gamma+n-1)=\Gamma(\gamma+n)/\Gamma(\gamma)$. The PML function $E_{\alpha,\beta}^{\gamma}(-\lambda t^{\alpha})$ is a completely monotone function for $t\geq 0$, and λ is positive constant, $0<\alpha,\beta\leq 1$, and $0<\gamma\leq\beta/\alpha$. The PML function has the asymptotic representation

$$E_{\alpha,\beta}^{\gamma}\left(-\lambda t^{\alpha}\right)\sim\begin{cases}\frac{1}{\Gamma(\beta)}-\frac{\lambda\gamma t^{\alpha}}{\Gamma(\alpha+\beta)}, & \text{for } t\to 0^+;\\ \frac{\left(\lambda t^{\alpha}\right)^{-\gamma}}{\Gamma(\beta-\alpha\gamma)}, & \text{for } t\to\infty,\end{cases} \tag{A.15}$$

The Laplace transform of the PML function is given by

$$\mathscr{L}\left\{t^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(-\lambda t^{\alpha}\right);s\right\}=\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+\lambda)^{\gamma}},\quad \Re\{\beta\}>0, \tag{A.16}$$

and its Hankel transform reads

$$\int_0^{\infty}x^{\rho-1}J_{\nu}(ax)E_{\alpha,\beta}^{\gamma}(-bx^{\sigma})dx=\frac{2^{\rho-1}}{a^{\rho}\Gamma(\gamma)}H_{2,3}^{2,1}\left[\frac{1}{b}\left(\frac{a}{2}\right)^{\sigma}\left|\begin{array}{l}(1,1)\ ;(\beta,\alpha)\\ \left(\frac{\rho+\nu}{2},\frac{\sigma}{2}\right)(\gamma,1)\ ;\left(\frac{\rho-\nu}{2},\frac{\sigma}{2}\right)\end{array}\right.\right], \tag{A.17}$$

The latter can be readily derived from the relation between the generalized Mittag-Leffler function and the H-function, and the Hankel transform of the H-function [84].

For the three parameter Mittag-Leffler function the following recurrence relation holds true [104]

$$E_{\alpha,\beta}^{\gamma+1}(-t^{\alpha})+t^{\alpha}E_{\alpha,\beta+\alpha}^{\gamma+1}(-t^{\alpha})=E_{\alpha,\beta}^{\gamma}(-t^{\alpha}). \tag{A.18}$$

Appendix B. Calculation of moments

In the calculation of the moments an integral of the form

$$I(t)=\int_0^{\infty}r^{q+n-1}W^{(n)}(r,t)dr, \tag{B.1}$$

occurs, where $W^{(n)}(r,t)$ is given by (26). Therefore, we have

$$I(t)=\left(\frac{p_2}{4\pi t^{\alpha_2}}\right)^{\frac{n}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k$$

$$\times\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\int_0^\infty r^{q+n-1}H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\middle|\begin{array}{l}\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)(k+1),\alpha_2\right)\\(0,1),\left(1-\frac{n}{2}+k,1\right)\end{array}\right]dr\right.$$
$$\left.+\int_0^\infty r^{q+n-1}H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\middle|\begin{array}{l}\left(1-\frac{n\alpha_2}{2}+(\alpha_2-\alpha_1)k,\alpha_2\right)\\(0,1),\left(1-\frac{n}{2}+k,1\right)\end{array}\right]dr\right\}. \tag{B.2}$$

From formula (A.4) for the Mellin transform of the H-function we find

$$I(t)=\left(\frac{p_2}{4\pi t^{\alpha_2}}\right)^{\frac{n}{2}}\frac{\Gamma\left(\frac{q+n}{2}\right)}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k$$
$$\times\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\left(\frac{4t^{\alpha_2}}{p_2}\right)^{\frac{q+n}{2}}\frac{\Gamma(1+q/2+k)}{\Gamma(1+\alpha_2q/2+\alpha_2-\alpha_1+(\alpha_2-\alpha_1)k)}+\left(\frac{4t^{\alpha_2}}{p_2}\right)^{\frac{q+n}{2}}\frac{\Gamma(1+q/2+k)}{\Gamma(1+\alpha_2q/2+(\alpha_2-\alpha_1)k)}\right\}$$
$$=\left(\frac{p_2}{4\pi t^{\alpha_2}}\right)^{\frac{n}{2}}\frac{\Gamma\left(\frac{q+n}{2}\right)\Gamma(1+q/2)}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k$$
$$\times\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\frac{(1+q/2)_k\left(\frac{4t^{\alpha_2}}{p_2}\right)^{\frac{q+n}{2}}}{\Gamma(1+\alpha_2q/2+\alpha_2-\alpha_1+(\alpha_2-\alpha_1)k)}+\frac{(1+q/2)_k\left(\frac{4t^{\alpha_2}}{p_2}\right)^{\frac{q+n}{2}}}{\Gamma(1+\alpha_2q/2+(\alpha_2-\alpha_1)k)}\right\}. \tag{B.3}$$

From the definition of the three parameter Mittag-Leffler function (A.14) we then have

$$I(t)=\frac{2^q\Gamma\left(\frac{q+n}{2}\right)\Gamma\left(1+\frac{q}{2}\right)}{2\pi^{\frac{n}{2}}p_2^{q/2}}t^{\alpha_2q/2}\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}E_{\alpha_2-\alpha_1,\alpha_2-\alpha_1+\alpha_2q/2+1}^{1+q/2}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)+E_{\alpha_2-\alpha_1,\alpha_2q/2+1}^{1+q/2}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)\right\}$$
$$=\frac{2^q\Gamma\left(\frac{q+n}{2}\right)\Gamma\left(1+\frac{q}{2}\right)}{2\pi^{\frac{n}{2}}p_2^{q/2}}t^{\alpha_2q/2}E_{\alpha_2-\alpha_1,\alpha_2q/2+1}^{q/2}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right), \tag{B.4}$$

where we have used relation (A.18).

Appendix C. Deriving the asymptotic forms of the PDF

C1. The limit $p_2r^2/(4t^{\alpha_2})\gg 1$

Let us firstly rewrite the multi-dimensional propagator (25) as

$$W^{(n)}(r,t)=\frac{1}{\left(\pi r^2\right)^{\frac{n}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}\right)^k\left\{\left(\frac{p_2r^2}{4}\right)^{\left(1-\frac{\alpha_1}{\alpha_2}\right)k}f_1\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)+\left(\frac{p_2r^2}{4}\right)^{\left(1-\frac{\alpha_1}{\alpha_2}\right)(k+1)}f_2\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)\right\}, \tag{C.1}$$

where

$$f_1\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)=H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\middle|\begin{array}{l}(1,\alpha_2)\\\left(\frac{n}{2}+\left(\frac{\alpha_1}{\alpha_2}-1\right)k,1\right),\left(1+\frac{\alpha_1}{\alpha_2}k,1\right)\end{array}\right], \tag{C.2a}$$

$$f_2\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)=H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\middle|\begin{array}{l}(1,\alpha_2)\\\left(\frac{n}{2}+\left(\frac{\alpha_1}{\alpha_2}-1\right)(k+1),1\right),\left(1+\frac{\alpha_1}{\alpha_2}(k+1),1\right)\end{array}\right]. \tag{C.2b}$$

Then, by using Eqs. (A.9) and (A.10) the functions $f_1\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)$ and $f_2\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)$ have the following asymptotic behavior for $\frac{p_2r^2}{4t^{\alpha_2}}\gg 1$

$$f_1\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)\sim\frac{1}{\sqrt{2-\alpha_2}}\alpha_2^{\frac{\alpha_2(n+1)-2}{2(2-\alpha_2)}}\left(\frac{p_2r^2}{4t^{\alpha_2}}\right)^{\frac{n}{2(2-\alpha_2)}}\left[\alpha_2^{\frac{2\alpha_1-\alpha_2}{2-\alpha_2}}\left(\frac{p_2r^2}{4t^{\alpha_2}}\right)^{\frac{2\alpha_1-\alpha_2}{\alpha_2(2-\alpha_2)}}\right]^k\mathrm{Exp}_1(r,t), \tag{C.3a}$$

$$f_2\left(\frac{p_2r^2}{4t^{\alpha_2}};k\right)\sim\frac{1}{\sqrt{2-\alpha_2}}\alpha_2^{\frac{\alpha_2(n+1)+4(\alpha_1-\alpha_2)-2}{2(2-\alpha_2)}}\left(\frac{p_2r^2}{4t^{\alpha_2}}\right)^{\frac{n\alpha_2+4(\alpha_1-\alpha_2)}{2\alpha_2(2-\alpha_2)}}\left[\alpha_2^{\frac{2\alpha_1-\alpha_2}{2-\alpha_2}}\left(\frac{p_2r^2}{4t^{\alpha_2}}\right)^{\frac{2\alpha_1-\alpha_2}{\alpha_2(2-\alpha_2)}}\right]^k\mathrm{Exp}_1(r,t), \tag{C.3b}$$

where $\mathrm{Exp}_1(r,t)$ is given by Eq. (78). In view of the asymptotic behavior (C.3) and Eq. (C.1) we get our result (76).

C2. The limit $p_2r^2/(4t^{\alpha_2})\ll 1$

C2.1. 1D case

The one-dimensional solution is given by setting $n=1$ in (26) [52],

$$W^{(1)}(x,t)=\sqrt{\frac{p_2}{4\pi t^{\alpha_2}}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}H_{1,2}^{2,0}\left[\frac{p_2x^2}{4t^{\alpha_2}}\middle|\begin{array}{l}\left(1-\frac{\alpha_2}{2}+(\alpha_2-\alpha_1)(k+1),\alpha_2\right)\\(0,1),\left(\frac{1}{2}+k,1\right)\end{array}\right]\right.$$

$$+H_{1,2}^{2,0}\left[\frac{p_2x^2}{4t^{\alpha_2}}\middle|\begin{array}{l}\left(1-\frac{\alpha_2}{2}+(\alpha_2-\alpha_1)k,\alpha_2\right)\\(0,1),\left(\frac{1}{2}+k,1\right)\end{array}\right]\Bigg\}. \tag{C.4}$$

In view of the asymptotic behavior (A.7), therefore, near the origin $x=0$ and at intermediate and long times the one-dimensional solution (C.4) behaves asymptotically as

$$W^{(1)}(x,t)\sim\sqrt{\frac{p_2}{4\pi t^{\alpha_2}}}\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left[\left(1+\frac{\alpha_2}{2}-\alpha_1\right)+(\alpha_2-\alpha_1)k\right]}\right.$$
$$\left.+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left[\left(1-\frac{\alpha_2}{2}\right)+(\alpha_2-\alpha_1)k\right]}\right\}$$

which can be summed, using the definition (A.14) of the generalized Mittag-Leffler function, to yield

$$W^{(1)}(x,t)\ \sim\ \frac{p_1}{2\sqrt{p_2}}t^{\frac{\alpha_2}{2}-\alpha_1}E^{\frac{1}{2}}_{\alpha_2-\alpha_1,1+\frac{\alpha_2}{2}-\alpha_1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)+\sqrt{\frac{p_2}{4t^{\alpha_2}}}E^{\frac{1}{2}}_{\alpha_2-\alpha_1,1-\frac{\alpha_2}{2}}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right). \tag{C.5}$$

C2.2. 3D case

The three-dimensional solution of the bi-fractional diffusion equation is given by setting $n=3$ in (26), i.e.,

$$W^{(3)}(r,t)=\left(\sqrt{\frac{p_2}{4\pi t^{\alpha_2}}}\right)^3\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\left\{\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\middle|\begin{array}{l}\left(1-\frac{3\alpha_2}{2}+(\alpha_2-\alpha_1)(k+1),\alpha_2\right)\\(0,1),\left(-\frac{1}{2}+k,1\right)\end{array}\right]\right.$$
$$\left.+H_{1,2}^{2,0}\left[\frac{p_2r^2}{4t^{\alpha_2}}\middle|\begin{array}{l}\left(1-\frac{3\alpha_2}{2}+(\alpha_2-\alpha_1)k,\alpha_2\right)\\(0,1),\left(-\frac{1}{2}+k,1\right)\end{array}\right]\right\}. \tag{C.6}$$

We note that the asymptotic behavior of the H-function in Eq. (C.6) near the origin depends on the value of k, see Eq. (A.5). Therefore the solution (C.6) near the origin $r=0$ and for intermediate and long times behaves as

$$W^{(3)}(r,t)\sim\frac{p_1\sqrt{p_2}}{8\pi^{\frac{3}{2}}}t^{-\frac{\alpha_2}{2}-\alpha_1}\left\{\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1-\alpha_1)}\left(\frac{4t^{\alpha_2}}{p_2r^2}\right)^{\frac{1}{2}}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma\left[\left(1-\frac{\alpha_2}{2}-\alpha_1\right)+(\alpha_2-\alpha_1)k\right]}\right\}$$
$$+\frac{p_2\sqrt{p_2}}{8\pi^{\frac{3}{2}}}t^{-\frac{3\alpha_2}{2}}\left\{\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1-\alpha_2)}\left(\frac{4t^{\alpha_2}}{p_2r^2}\right)^{\frac{1}{2}}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)^k\frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma\left[\left(1-\frac{3\alpha_2}{2}\right)+(\alpha_2-\alpha_1)k\right]}\right\}, \tag{C.7}$$

where the asymptotic behavior (A.8) has been used. Moreover, utilizing the relation $\Gamma(k-1/2)=\Gamma(-1/2)(-1/2)_k$ and definition (A.14) we obtain

$$W^{(3)}(r,t)\ \sim\ \frac{1}{4\pi}\left\{\frac{1}{r}\left[\frac{p_1t^{-\alpha_1}}{\Gamma(1-\alpha_1)}+\frac{p_2t^{-\alpha_2}}{\Gamma(1-\alpha_2)}\right]+\sqrt{p_2}\left[\frac{p_1t^{-\alpha_1-\frac{\alpha_2}{2}}}{\Gamma\left(1-\alpha_1-\frac{\alpha_2}{2}\right)}+\frac{p_2t^{-\frac{3\alpha_2}{2}}}{\Gamma\left(1-\frac{3\alpha_2}{2}\right)}\right]\right.$$
$$\left.-\sqrt{p_2}\left[p_1t^{-\alpha_1-\frac{\alpha_2}{2}}E^{-\frac{1}{2}}_{\alpha_2-\alpha_1,1-\frac{\alpha_2}{2}-\alpha_1}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)+p_2t^{-\frac{3\alpha_2}{2}}E^{-\frac{1}{2}}_{\alpha_2-\alpha_1,1-\frac{3\alpha_2}{2}}\left(-\frac{p_1}{p_2}t^{\alpha_2-\alpha_1}\right)\right]\right\},\quad r\to 0. \tag{C.8}$$

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