PHYSICAL REVIEW E **100**, 020104(R) (2019)

Rapid Communications

Gumbel central limit theorem for max-min and min-max

Iddo Eliazar,[1,*] Ralf Metzler,[2,†] and Shlomi Reuveni[1,‡]

[1]*School of Chemistry, The Center for Physics and Chemistry of Living Systems, The Raymond and Beverly Sackler Center for Computational Molecular and Materials Science, and The Mark Ratner Institute for Single Molecule Chemistry, Tel Aviv University, Tel Aviv 6997801, Israel*

[2]*University of Potsdam, Institute of Physics and Astronomy, 14476 Potsdam, Germany*

(Received 26 August 2018; revised manuscript received 25 February 2019; published 21 August 2019)

The max-min and min-max of matrices arise prevalently in science and engineering. However, in many real-world situations the computation of the max-min and min-max is challenging as matrices are large and full information about their entries is lacking. Here we take a statistical-physics approach and establish limit laws—akin to the central limit theorem—for the max-min and min-max of large random matrices. The limit laws intertwine random-matrix theory and extreme-value theory, couple the matrix dimensions geometrically, and assert that Gumbel statistics emerge irrespective of the matrix entries' distribution. Due to their generality and universality, as well as their practicality, these results are expected to have a host of applications in the physical sciences and beyond.

DOI: 10.1103/PhysRevE.100.020104

The central limit theorem (CLT), a foundational cornerstone of statistical physics and probability theory, is of prime importance in science and engineering. The CLT and its generalized version assert that the scaled sum of a large number of independent and identically distributed (i.i.d.) random variables is governed, asymptotically, by two limit-law statistics [1,2]: normal and Lévy stable. The CLT considers finite-variance i.i.d. random variables, and yields normal statistics. Departing the finite-variance dominion, the generalized CLT imposes sharp tail conditions on the distribution of the i.i.d. random variables [3], and yields Lévy-stable statistics (that include normal statistics as a boundary case).

Extreme-value theory [4,5] is applied whenever extreme behavior, rather than average behavior, is of relevance; e.g., the prediction of rare events, and the safe design of critical systems such as dams, bridges, and power grids. Extreme-value theory shifts the focus from sums to extrema, i.e., maxima and minima. The Fisher-Tippett-Gnedenko (FTG) theorem is the extreme-value counterpart of the above CLTs. This theorem asserts that the scaled extrema of a large number of i.i.d. random variables are governed, asymptotically, by three limit-law statistics [6,7]: Weibull, Frechet, and Gumbel. As in the case of the generalized CLT, the FTG theorem imposes sharp tail conditions on the distribution of the i.i.d. random variables [3].

The limit-law statistics of the CLTs and the FTG theorem play key roles in physics, e.g., in [8–20] and in [21–27], respectively. Underlying these theorems is a common bedrock: a random-vector setting, with the i.i.d. random variables being the vector entries. Elevating from one-dimensional to two-dimensional arrays, we arrive at a random-matrix setting: matrices whose entries are i.i.d. random variables. Random matrices also play key roles in physics [28,29], and much effort has been directed to the extreme-value analysis of their eigenvalues spectra [30,31]. Here we focus on a different extreme-value analysis of random matrices: their max-min and min-max (see Fig. 1 for the max-min).

The max-min and min-max arise prevalently in science and engineering. Perhaps the best known example is in game theory [32], a field which drew considerable attention from physicists [33–39]. There, a player seeks a strategy that will maximize gain, or minimize loss, in the worst-case scenario. The player has a payoff matrix which specifies the gain or loss for each strategy taken vs each scenario encountered; the player calculates the max-min in the case of gains, and the min-max in the case of losses. However, in real-life situations the payoff matrix is often large and full information about its entries is lacking. In turn, such situations call for a modeling approach employing large random matrices.

The max-min and min-max of large random matrices were investigated in mathematics [40], and in reliability engineering [41–44]. In the pioneering work [40], Chernoff and Teicher established that the scaled max-min and min-max are governed, asymptotically, by the FTG statistics: Weibull, Frechet, and Gumbel. In subsequent works [41–43], Kolowrocki further advanced the topic in the context of (so called) series-parallel and parallel-series systems. In a more

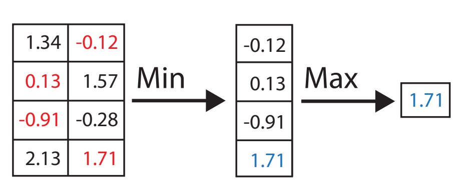

FIG. 1. The max-min of a matrix is obtained by first taking the minimal entry of each row (depicted red), and then taking the maximum of these minimal entries (depicted blue).

[*]eliazar@tauex.tau.ac.il
[†]rmetzler@uni-potsdam.de
[‡]shlomire@tauex.tau.ac.il

recent work [44], Reis and Castro obtained Gumbel limit-law statistics for the max-min via an iterative application of the FTG theorem: first to the minimum of each and every matrix row, and then to the maximum of the rows' minima.

The results in [40–44] are notable and inspiring mathematical theorems. However, from a practical perspective the application of these results is extremely challenging, even on a case by case basis. More importantly, the results in [40–44] do not provide a clear-cut answer to the following focal question: Is there a "central limit theorem" for the max-min and min-max of random matrices?

The CLTs and the FTG theorem stand on two pillars: domain of attraction and scaling scheme. The domain of attraction of the CLT is wide (encompassing all finite-variance distributions), and its scaling scheme is simple; the application of the CLT is thus straightforward, and its use is omnipresent. For the generalized CLT and the FTG theorem matters are more intricate: the domains of attraction are narrow (characterized by the sharp conditions imposed on the distributions' tails [3]), and the scaling schemes are elaborate (they need to be carefully custom-tailored per each admissible distribution [3]). Elevating from a random-vector setting to a random-matrix setting adds a third pillar to the two above: the asymptotic coupling between the matrix dimensions (as these are taken to infinity). In [40–44] the intricacy of all three pillars is prohibitively high. Consequently, there are no available max-min and min-max limit laws with the following features: wide domain of attraction, simple scaling scheme, and simple asymptotic coupling.

Here we present central limit theorem results for the max-min and min-max of large nonsquare random matrices. Circumventing the use of the FTG theorem altogether, the results are based on novel Poisson-process limit laws [45]. The results assert that the scaled max-min and min-max are governed, asymptotically, by Gumbel statistics. The results' domain of attraction is vast, encompassing all distributions with a density. The results' scaling schemes are similar to that of the CLT, and their asymptotic couplings are geometric. The novel results established here are thus highly practical and applicable (see Fig. 2 for the max-min result).

Written for a general physics readership, this Rapid Communication offers a concise brief of the results and their implementation; for a comprehensive exposition, including detailed proofs, see [45]. The brief is organized as follows: we begin with an underlying setting, present Gumbel approximations for the max-min and min-max, and describe the implementation of these approximations; then, we present the Gumbel limit laws (that yield the Gumbel approximations), discuss these limit laws, and conclude with an outlook.

Setting. Consider a random matrix with i.i.d. entries:

$$\mathbf{M} = \begin{pmatrix} X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,n} \end{pmatrix}. \tag{1}$$

Namely, the matrix is of dimensions $m \times n$, with rows labeled $i = 1, \dots, m$, and columns labeled $j = 1, \dots, n$. The matrix entries are i.i.d. copies of a generic real-valued random variable X, with probability density $f(x)$ $(-\infty < x < \infty)$. In what follows we denote by $F(x) = \Pr(X \leqslant x)$ $(-\infty < x < \infty)$ the corresponding distribution function, and by $\bar{F}(x) = \Pr(X > x)$ $(-\infty < x < \infty)$ the corresponding survival function.

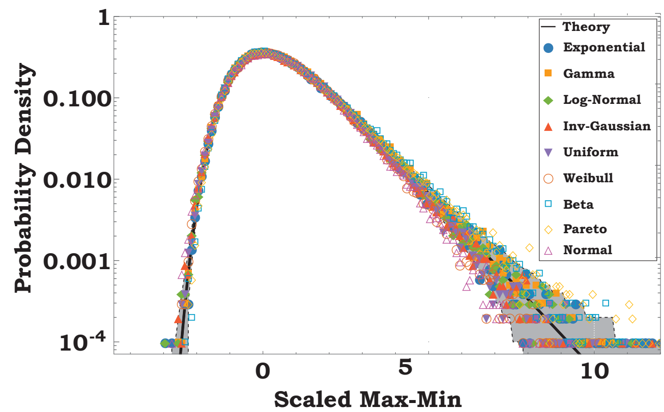

FIG. 2. Gumbel limit-law statistics for the scaled max-min of large random matrices. Universality is demonstrated by data collapse for nine different distributions from which the i.i.d. matrix entries are drawn: the colored symbols depict the simulated data; the solid black line is the probability density of the predicted Gumbel statistics, with its 95% confidence interval shaded in gray. For additional details see discussion below Eq. (8) and the caption of Fig. 3.

We set the focus on the max-min and min-max of the random matrix $\mathbf{M}$. Denoting by $\wedge_i = \min\{X_{i,1}, \dots, X_{i,n}\}$ the minimum over the entries of row i, the max-min is the maximum over the rows' minima:

$$\wedge_{\max} = \max\{\wedge_1, \dots, \wedge_m\}. \tag{2}$$

Similarly, denoting by $\vee_j = \max\{X_{1,j}, \dots, X_{m,j}\}$ the maximum over the entries of column j, the min-max is the minimum over the columns' maxima:

$$\vee_{\min} = \min\{\vee_1, \dots, \vee_n\}. \tag{3}$$

To illustrate the setting, consider the aforementioned game-theory example. If the matrix $\mathbf{M}$ manifests gains, then the rows represent the player's strategies; the columns represent the scenarios the player is facing; $X_{i,j}$ is the player's gain when taking strategy i and encountering scenario j; and $\wedge_{\max}$ is the player's max-min gain. If the matrix $\mathbf{M}$ manifests losses, then the roles of its rows and columns are transposed, $X_{i,j}$ is the player's loss when encountering scenario i and taking strategy j, and $\vee_{\min}$ is the player's min-max loss.

From Eqs. (2) and (3) it follows that the distribution/survival functions of the max-min and min-max are given, respectively, by $\Pr(\wedge_{\max} \leqslant x) = [1 - \bar{F}(x)^n]^m$ and by $\Pr(\vee_{\min} > x) = [1 - F(x)^m]^n$. In the results to be presented here we scale the max-min and min-max appropriately, and establish their convergence (in law) to universal Gumbel statistics. In what follows, Z denotes a "standard" Gumbel random variable, and $G(x)$ denotes the corresponding Gumbel distribution function [7]:

$$\Pr(Z \leqslant x) = G(x) = \exp[-\exp(-x)] \tag{4}$$

$(-\infty < x < \infty)$.

Our results involve an "anchor" x_*, an arbitrary value that can be realized by the generic random variable X. Specifically,

TABLE I. Key statistical features of the Gumbel approximations $Z_{\max}$ of Eq. (5) and $Z_{\min}$ of Eq. (6): mode, median, mean, and standard deviation (SD); in the row for the mean, $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant.

	$Z_{\max}$	$Z_{\min}$
Mode	x_*	x_*
Median	$x_* - \frac{\ln[\ln(2)]}{\alpha}\frac{1}{n}$	$x_* + \frac{\ln[\ln(2)]}{\beta}\frac{1}{m}$
Mean	$x_* + \frac{\gamma}{\alpha}\frac{1}{n}$	$x_* - \frac{\gamma}{\beta}\frac{1}{m}$
SD	$\frac{\pi}{\sqrt{6}\alpha}\frac{1}{n}$	$\frac{\pi}{\sqrt{6}\beta}\frac{1}{m}$

the anchor meets two requirements: (i) $0 < f(x_*) < \infty$ and (ii) $0 < F(x_*) < 1$, which is equivalent to $0 < \bar{F}(x_*) < 1$. For example, with regard to three of the distributions appearing in Fig. 2, the admissible values of the anchor are $-\infty < x_* < \infty$ for the normal, $0 < x_* < \infty$ for the gamma, and $0 < x_* < 1$ for the beta.

Approximations. We present Gumbel approximations for the max-min $\wedge_{\max}$ and the min-max $\vee_{\min}$ of a large random matrix $\mathbf{M}$ with dimensions $m \gg 1$ and $n \gg 1$. The approximations are based on couplings between the matrix dimensions m and n, and the anchor x_*. As we shall show hereinafter, these couplings are always implementable: given two of the triplet $\{m, n, x_*\}$ we can always set the third to satisfy the couplings. Also, in the approximations Z is the standard Gumbel random variable of Eq. (4).

Consider the coupling $m\bar{F}(x_*)^n \simeq 1$; then, the max-min admits the approximation

$$\wedge_{\max} \simeq Z_{\max} := x_* + \frac{1}{n}\frac{1}{\alpha}Z, \tag{5}$$

where $\alpha = f(x_*)/\bar{F}(x_*)$. Similarly, consider the coupling $nF(x_*)^m \simeq 1$; then, the min-max admits the approximation

$$\vee_{\min} \simeq Z_{\min} := x_* - \frac{1}{m}\frac{1}{\beta}Z, \tag{6}$$

where $\beta = f(x_*)/F(x_*)$.

Equations (5) and (6) imply that the deterministic approximation of the max-min $\wedge_{\max}$ and the min-max $\vee_{\min}$ is the anchor x_*; the magnitude of the random fluctuations about x_* is $1/(n\alpha)$ for the max-min, and is $1/(m\beta)$ for the min-max; and the statistics of the random fluctuations about x_* are Gumbel. Key statistical features of the Gumbel approximations $Z_{\max}$ of Eq. (5) and $Z_{\min}$ of Eq. (6) are detailed in Table I: modes, medians, means, and standard deviations. The probability densities of the Gumbel approximations $Z_{\max}$ and $Z_{\min}$ have a unimodal shape: monotone increasing below x_*, and monotone decreasing above x_*.

Implementation. There are two ways of implementing the Gumbel approximations, which we now describe. Both ways exploit the couplings underpinning the approximations.

The first way applies when the matrix dimensions m and n are given; in this case the dimensions determine the anchor x_*. Specifically, for matrix $\mathbf{M}$ with dimensions $m > n \gg 1$ the approximation of Eq. (5) holds with anchor $x_* = \bar{F}^{-1}[(1/m)^{1/n}]$. Similarly, for matrix $\mathbf{M}$ with dimensions $n > m \gg 1$ the approximation of Eq. (6) holds with anchor $x_* = F^{-1}[(1/n)^{1/m}]$.

The second way applies when the anchor x_* is given; in this case the matrix dimensions m and n should be set accordingly. Specifically, for the max-min setting $n \gg 1$ and $m \simeq 1/\bar{F}(x_*)^n$ yields the approximation of Eq. (5). And, for the min-max setting $m \gg 1$ and $n \simeq 1/F(x_*)^m$ yields the approximation of Eq. (6). In this way the magnitudes of the random fluctuations about the anchor x_* are of the order $O(1/n)$ in the approximation of Eq. (5), and of the order $O(1/m)$ in the approximation of Eq. (6).

The first way is a "scientific tool": given a matrix $\mathbf{M}$, it provides us with approximations for the max-min and min-max. The second way is an "engineering tool": given a "target" anchor x_*, it tells us how to design the matrix $\mathbf{M}$ so that x_* will be the deterministic approximation of the max-min and min-max; moreover, we can design the magnitudes of the random fluctuations about x_* to be as small as we wish [45].

Limit laws. The Gumbel approximations of Eqs. (5) and (6) emanate from corresponding Gumbel limit laws which we now present. In the limit laws we fix the anchor x_*, and then grow the matrix dimensions infinitely large: $m, n \to \infty$. Also, in the limit laws $G(x)$ is the standard Gumbel distribution function of Eq. (4).

Grow the matrix dimensions via the coupled limit $\lim_{m,n\to\infty} m\bar{F}(x_*)^n = 1$; then, the max-min limit law is

$$\lim_{m,n\to\infty} \Pr[\alpha n(\wedge_{\max} - x_*) \leqslant x] = G(x) \tag{7}$$

$(-\infty < x < \infty)$, where $\alpha = f(x_*)/\bar{F}(x_*)$ as above. Similarly, grow the matrix dimensions via the coupled limit $\lim_{m,n\to\infty} nF(x_*)^m = 1$; then, the min-max limit law is

$$\lim_{m,n\to\infty} \Pr[\beta m(x_* - \vee_{\min}) \leqslant x] = G(x) \tag{8}$$

$(-\infty < x < \infty)$, where $\beta = f(x_*)/F(x_*)$ as above.

Equations (7) and (8) imply that the scaled max-min $\alpha n(\wedge_{\max} - x_*)$ and the scaled min-max $\beta m(x_* - \vee_{\min})$ converge—in law, as $m, n \to \infty$—to a standard Gumbel random variable Z [recall Eq. (4)]. Hence, the limit laws of Eqs. (7) and (8) yield, respectively, the approximations of Eqs. (5) and (6). The Gumbel limit law of Eq. (7) is tested for nine different distributions from which the i.i.d. matrix entries are drawn (Fig. 3); note that convergence is evident already for moderate values of the dimension n. The data collapse demonstrated in Fig. 2 corresponds to the nine distributions of Fig. 3 with dimension $n = 70$.

The Gumbel limit laws of Eqs. (7) and (8) stem from "bedrock" Poisson-process limit laws. Underlying the max-min $\wedge_{\max}$ is the ensemble of the rows' minima $\{\wedge_1, \ldots, \wedge_m\}$, and underlying the min-max $\vee_{\min}$ is the ensemble of the columns' maxima $\{\vee_1, \ldots, \vee_n\}$. In [45] it is established that appropriately scaled versions of these ensembles converge—in law, as $m, n \to \infty$—to a Poisson process that is characterized by the following exponential intensity function: $\lambda(x) = \exp(-x)$ $(-\infty < x < \infty)$. For the points of this Poisson process one can observe that the maximal point is no larger than a real threshold x if and only if there are no points above this threshold—an event whose probability is $\exp[-\int_x^\infty \lambda(x')dx'] = G(x)$ [46]. Hence, the distribution function of the maximal point is $G(x)$, which is the term that appears on the right-hand sides of Eqs. (7) and (8) [45].

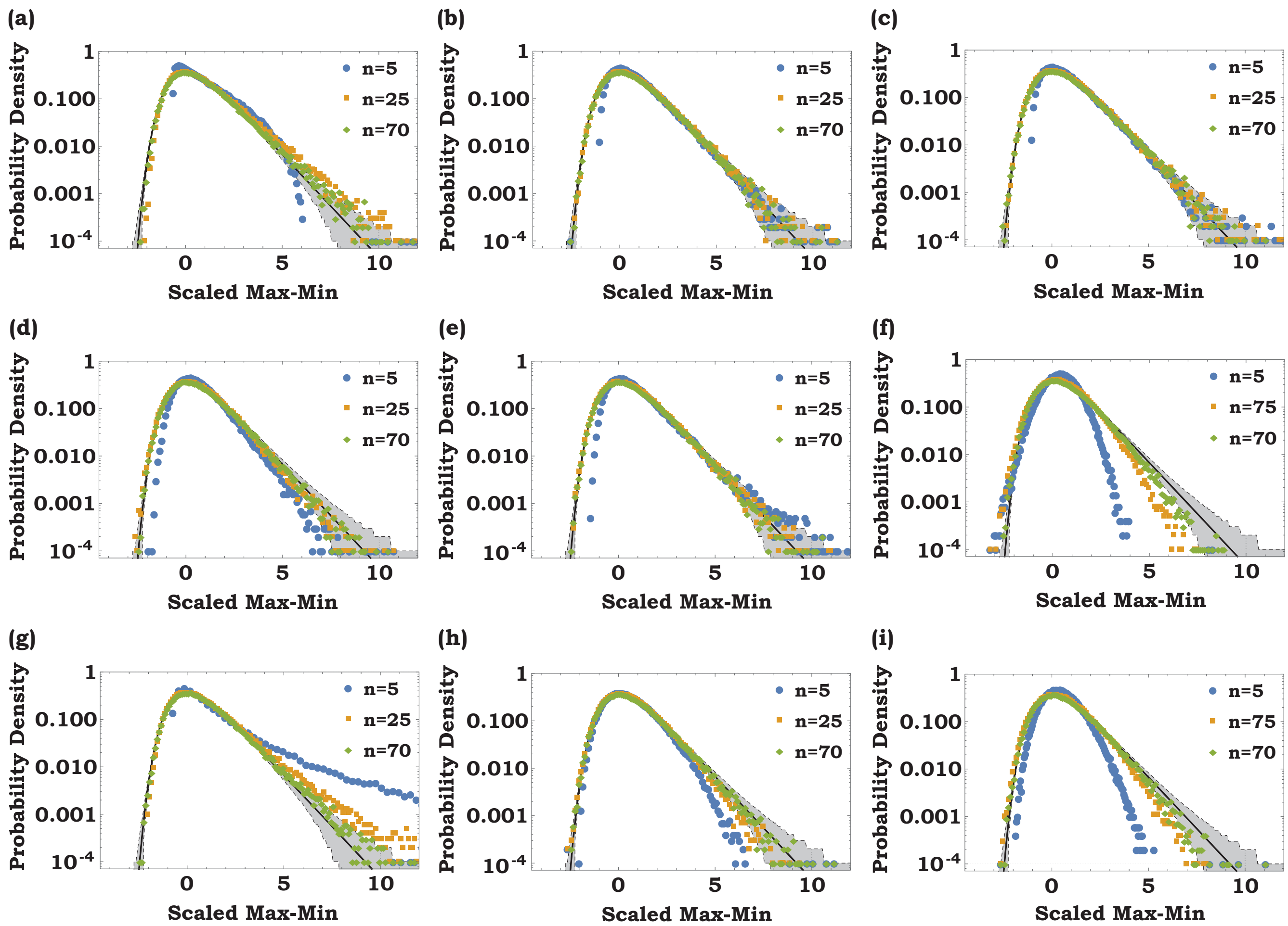

FIG. 3. The Gumbel limit law of Eq. (7) is tested for nine different distributions from which the i.i.d. matrix entries are drawn: (a) beta, (b) exponential, (c) gamma, (d) inverse Gaussian, (e) log-normal, (f) normal, (g) Pareto, (h) uniform, and (i) Weibull. The statistics of the scaled max-min $\alpha n(\wedge_{\max} - x_*)$, with anchor $x_* = \bar{F}^{-1}(0.8)$, were simulated by sampling 10^5 random matrices with the following dimensions: $n = 5, 25, 70$ rows and $m \simeq 1.25^n$ columns. In all cases, the convergence of the simulations (colored symbols) to the probability density of the standard Gumbel law (solid black line, with its 95% confidence interval shaded in gray) is evident.

Discussion. The limit laws of Eqs. (7) and (8) are highly invariant with respect to the i.i.d. entries of the random matrix **M**. Indeed, contrary to the CLT, no moment conditions are imposed on the entries' distribution. And, contrary to the generalized CLT and to the FTG theorem, no tail conditions are imposed on the entries' distribution. The Gumbel limit laws merely require that the entries' distribution has a density. In practice, this smoothness condition is widely satisfied.

The Gumbel limit laws of Eqs. (7) and (8) involve simple scaling schemes. To appreciate their simplicity, we compare these schemes to that of the CLT. Consider A_k to be the average of k i.i.d. random variables with common mean μ and standard deviation σ. The CLT asserts that the scaled average $\sigma^{-1}\sqrt{k}(A_k - \mu)$ converges—in law, as $k \to \infty$—to a standard normal random variable (i.e., with a zero mean and a unit standard deviation). The scaled max-min $\alpha n(\wedge_{\max} - x_*)$ of Eq. (7) and the scaled min-max $\beta m(x_* - \vee_{\min})$ of Eq. (8) are similar, in form, to the scaled average $\sigma^{-1}\sqrt{k}(A_k - \mu)$. Specifically, the anchor x_* is the counterpart of the mean μ; and the scale terms αn and βm are the counterparts of the scale term $\sigma^{-1}\sqrt{k}$. Consequently, the scaling schemes of the limit laws of Eqs. (7) and (8) are as simple and straightforward as that of the CLT.

There are numerously many ways of setting the scaling schemes of the generalized CLT and of the FTG theorem, and each such way corresponds to specific distributions of the underlying i.i.d. random variables. On the other hand, as detailed above, the scaling scheme of the CLT is set in a particular way. This special CLT scaling scheme is universal in the following sense: it yields normal limit-law statistics for all finite-variance distributions.

Addressing limit laws for the max-min and min-max of random matrices [40–44], there are numerous ways of setting the scaling schemes; and there are also numerous ways of asymptotically coupling the matrix dimensions, m and n, when growing them infinitely large ($m, n \to \infty$). Similarly to the CLT, the Gumbel limit laws of Eqs. (7) and (8) employ particular scaling schemes, as well as particular asymptotic couplings. In turn, as for the CLT, these special scaling schemes and asymptotic couplings are universal in the following sense: they yield Gumbel limit-law statistics for all distributions with a density.

The particular asymptotic couplings employed here are geometric, and they are parametrized by the anchor x_*. Specifically, the geometric asymptotic couplings are given by $\lim_{m,n\to\infty} m\bar{F}(x_*)^n = 1$ for the Gumbel limit law of Eq. (7), and $\lim_{m,n\to\infty} nF(x_*)^m = 1$ for the Gumbel limit law of Eq. (8). The couplings' parametrization is a degree of freedom that facilitates tunability. Indeed, the anchor x_*, which is the counterpart of the mean μ in the CLT, can be tuned as we wish within its admissible values.

Outlook. It has long been observed that seemingly identical pieces of matter happen to fail stochastically at different times and under different loads. Consequently, one of the major original drivers for the development of extreme-value theory came from materials science, where statistical predictions for mechanical strength and fracture formation are of prime importance [47,48]. The "weakest link hypothesis" is foundational in materials science [26,27]. This hypothesis suggests that various mechanical systems can be modeled as having a chainlike structure, thus implying that such a system is only as strong as its weakest link. The "weakest link hypothesis" naturally gives rise to the max-min: when statistically similar chainlike systems are compared, either by an evolutionary process or by industrial quality testing, the system with the strongest weakest link prevails.

The min-max also arises naturally from real-world applications. Indeed, consider a backup system in which critical files are stored on multiple separate hard drives. If a file is damaged on one of the drives it could be retrieved from another; however, if all copies of a file are damaged, then the file is lost forever. The loss time of a given file is thus the maximum of its damage times over the different drives. In turn, since all files are critical, system failure occurs at the first loss time of a file. Thus, the system failure time is the min-max of the files' damage times.

Here we adopted the setting of random-matrix theory, considering large matrices with i.i.d. entries. For the max-min and min-max of such matrices we established, respectively, the Gumbel approximations of Eqs. (5) and (6); also, we showed how to apply these approximations as a scientific tool and as an engineering tool. The approximations stem from the limit laws of Eqs. (7) and (8), which assume the role of a "Gumbel central limit theorem" for the max-min and min-max. With their generality and universality, their easy practical implementation, and their many potential applications, e.g., in game theory, in reliability engineering, in materials science, and in the design of backup systems, the results presented herein are expected to serve diverse audiences in the physical sciences and beyond.

Acknowledgments. R.M. acknowledges Deutsche Forschungsgemeinschaft for funding (ME 1535/7-1) and support from the Foundation for Polish Science within an Alexander von Humboldt Polish Honorary Research Scholarship. S.R. gratefully acknowledges support from the Azrieli Foundation and the Sackler Center for Computational Molecular and Materials Science.

[1] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 2008), Vol. 1.

[2] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 2008), Vol. 2.

[3] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation* (Cambridge University Press, Cambridge, UK, 1989), Vol. 27.

[4] E. Castillo, *Extreme Value Theory in Engineering* (Elsevier, New York, 2012).

[5] R. D. Reiss, M. Thomas, and R. D. Reiss, *Statistical Analysis of Extreme Values* (Birkhüser, Basel, 2007), Vol. 2.

[6] B. Gnedenko, Sur La Distribution Limite Du Terme Maximum D'Une Serie Aleatoire, Ann. Math. **44**, 423 (1943) (translated, and reprinted in *Breakthroughs in Statistics I*, edited by S. Kotz and N. L. Johnson (Springer, New York, 1992), pp. 195–225.

[7] E. J. Gumbel, *Statistics of Extremes* (Courier Corp., New York, 2012).

[8] S. Chandrasekhar, Stochastic problems in physics, and astronomy, Rev. Mod. Phys. **15**, 1 (1943).

[9] J. P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: Statistical mechanisms, models, and physical applications, Phys. Rep. **195**, 127 (1990).

[10] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Strange kinetics, Nature (London) **363**, 31 (1993).

[11] M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch, *Lévy flights and related topics in physics* (Springer, New York, 1995).

[12] C. Tsallis, Lévy distributions, Phys. World **10**, 42 (1997).

[13] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. **339**, 1 (2000).

[14] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, UK, 2001).

[15] J. Klafter and I. M. Sokolov, Anomalous diffusion spreads its wings, Phys. World **18**, 29 (2005).

[16] R. Klages, G. Radons, and I. M. Sokolov (eds.), *Anomalous transport: Foundations, and applications* (Wiley-VCH, Weinheim, 2008).

[17] J. Klafter and I. M. Sokolov, *First Steps in Random Walks: From Tools to Applications* (Oxford University Press, New York, 2011).

[18] A. J. Bray, S. N. Majumdar, and G. Schehr, Persistence and first-passage properties in nonequilibrium systems, Adv. Phys. **62**, 225 (2013).

[19] M. L. Ackerman, P. Kumar, M. Neek-Amal, P. M. Thibado, F. M. Peeters, and S. Singh, Anomalous Dynamical Behavior of Freestanding Graphene Membranes, Phys. Rev. Lett. **117**, 126801 (2016).

[20] B. Schäfer, C. Beck, K. Aihara, D. Witthaut, and M. Timme, Non-Gaussian power grid frequency fluctuations characterized by Lévy-stable laws and superstatistics, Nat. Energy **3**, 119 (2018).

[21] J. P. Bouchaud and M. Mézard, Universality classes for extreme-value statistics, J. Phys. A: Math. Gen. **30**, 7997 (1997).

[22] A. Comtet, P. Leboeuf, and S. N. Majumdar, Level Density of a Bose Gas and Extreme Value Statistics, Phys. Rev. Lett. **98**, 070404 (2007).

[23] Y. V. Fyodorov and J. P. Bouchaud, Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential, J. Phys. A: Math. Theor. **41**, 372001 (2008).

[24] A. Perret, A. Comtet, S. N. Majumdar, and G. Schehr, Near-Extreme Statistics of Brownian Motion, Phys. Rev. Lett. **111**, 240601 (2013).

[25] M. Chupeau, O. Bénichou, and R. Voituriez, Cover times of random searches, Nat. Phys. **11**, 844 (2015).

[26] A. L. Sellerio, A. Taloni, and S. Zapperi, Fracture Size Effects in Nanoscale Materials: The Case of Graphene, Phys. Rev. Appl. **4**, 024011 (2015).

[27] A. Taloni, M. Vodret, G. Constantini, and S. Zapperi, Size effects on the fracture of microscale and nanoscale materials, Nat. Rev. Mater. **3**, 211 (2018).

[28] G. Biroli, J. P. Bouchaud, and M. Potters, Extreme value problems in random matrix theory and other disordered systems, J. Stat. Mech.: Theory Exp. (2007) P07019.

[29] J. Bun, J. P. Bouchaud, and M. Potters, Cleaning large correlation matrices: Tools from random matrix theory, Phys. Rep. **666**, 1 (2017).

[30] D. S. Dean and S. N. Majumdar, Large Deviations of Extreme Eigenvalues of Random Matrices, Phys. Rev. Lett. **97**, 160201 (2006).

[31] S. N. Majumdar and M. Vergassola, Large Deviations of the Maximum Eigenvalue for Wishart and Gaussian Random Matrices, Phys. Rev. Lett. **102**, 060601 (2009).

[32] M. Maschler, E. Solan, and S. Zamir, *Game Theory* (Cambridge University Press, Cambridge, UK, 2013).

[33] A. Melbinger, J. Cremer, and E. Frey, Evolutionary Game Theory in Growing Populations, Phys. Rev. Lett. **105**, 178101 (2010).

[34] J. Berg and A. Engel, Matrix Games, Mixed Strategies, and Statistical Mechanics, Phys. Rev. Lett. **81**, 4999 (1998).

[35] J. Eisert, M. Wilkens, and M. Lewenstein, Quantum Games and Quantum Strategies, Phys. Rev. Lett. **83**, 3077 (1999).

[36] D. Challet, M. Marsili, and R. Zecchina, Statistical Mechanics of Systems with Heterogeneous Agents: Minority Games, Phys. Rev. Lett. **84**, 1824 (2000).

[37] D. A. Meyer, Quantum Strategies, Phys. Rev. Lett. **82**, 1052 (1999).

[38] P. Nalcecz-Jawecki and J. Miekisz, Mean-Potential Law in Evolutionary Games, Phys. Rev. Lett. **120**, 028101 (2018).

[39] J. Knebel, M. F. Weber, T. Krüger, and E. Frey, Evolutionary games of condensates in coupled birth-death processes, Nat. Commun. **6**, 6977 (2015).

[40] H. Chernoff and H. Teicher, Limit distributions of the minimax of independent identically distributed random variables, Trans. Am. Math. Soc. **116**, 474 (1965).

[41] K. Kolowrocki, On a class of limit reliability functions of some regular homogeneous series-parallel systems, Reliab. Eng. Syst. Safety **39**, 11 (1993).

[42] K. Kolowrocki, On asymptotic reliability functions of series-parallel and parallel-series systems with identical components, Reliab. Eng. Syst. Safety **41**, 251 (1993).

[43] K. Kolowrocki, Limit reliability functions of some series-parallel and parallel-series systems, Appl. Math. Comput. **62**, 129 (1994).

[44] P. Reis, and L. C. Castro, Limit model for the reliability of a regular and homogeneous series-parallel system, REVSTAT **7**, 227 (2009).

[45] I. Eliazar, R. Metzler, and S. Reuveni, Poisson-process limit laws yield Gumbel max-min and min-max, Phys. Rev. E **100**, 022129 (2019).

[46] J. F. C. Kingman, *Poisson processes* (Clarendon, Oxford, 1992), Vol. 3.

[47] W. Weibull, *A Statistical Theory of Strength of Materials* (Ingeniors Vetenskaps Akademiens, Stockholm, 1939).

[48] W. Weibull, A statistical distribution function of wide applicability, J. Appl. Mech. **18**, 293 (1951).