Eur. Phys. J. B (2019) 92: 104
https://doi.org/10.1140/epjb/e2019-90674-0

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Money distribution in agent-based models with position-exchange dynamics: the Pareto paradigm revisited*

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Received 15 November 2018 / Received in final form 11 February 2019
Published online 20 May 2019

Abstract. Wealth and income distributions are known to feature country-specific Pareto exponents for their long power-law tails. To propose a rationale for this, we introduce an agent-based dynamic model and use Monte Carlo simulations to unveil the wealth distributions in closed and open economical systems. The standard money-exchange scenario is supplemented with the position-exchange agent dynamics that vitally affects the Pareto law. Specifically, in closed systems with position-exchange dynamics the power law changes to an exponential shape, while for open systems with traps the Pareto law remains valid.

1 Introduction

For a long time economists, econophysisists, and "inter-disciplinary" scientists have been unveiling the reasons of inequality [1–7] and proposing methods to quantify income, wealth, or money[1] distributions in different societies [8–19]. Pareto was the first to find [1] that wealth distributions in European countries have power-law tails. For $m \geq m_c$ they obey $P_{\rm Par}(m) \propto m^{-(1+\nu)}$, while below a threshold value $m < m_c$ of money one gets $P(m) \propto m^{\alpha} e^{-m/\overline{m}}$. Hereafter, $P(m)$ denotes the number-density of people with money m. The decay of $P_{\rm Par}(m)$ for large m is the Pareto law with exponent ν. The power law for large-income tails of $P(m)$ was confirmed [13], while a close-to-exponential distribution was found at low incomes [20,21].[2] Despite decades [19] of intense research and data analysis, the exact physical mechanisms of wealth distributions are not yet fully understood.

A number of statistical-mechanics models have been proposed to unravel the observed trends, including ideal-gas-like saving-based models for closed and open systems in the absence of position-exchange dynamics [11,13,14,22–33]. Models of these types yield the Pareto law with a certain ν value. However, the analysis of real data revealed variations of ν for different societies, government forms, time evolution of ν [34] and its sub-population dependence [19]. Different Pareto exponents were found for income distributions for the US, the EU, China, Germany, Russia, France, etc. [17,35–38]. Agent-based models without position exchanges of agents face problems explaining these facts that motivated us to propose/examine a model with certain position-exchange dynamics.

2 Money-exchange models

In a one-dimensional model in the absence of position-exchange dynamics the ith agent at time t – the time passed from the start of simulations from the initial state – has an amount of money $m_i(t)$. The total amount of money of all N agents is conserved, $M(t) = \sum_{i=1}^{N} m_i(t) =$ const. In a trading event, a pair of neighboring agents i and j exchange their money, see Figure 1. Also, a saving propensity [11,13] λ is introduced: the ith agent saves a fraction $0 \leq \lambda \leq 1$ of its money and randomly trades the remaining amount. Mathematically, this scheme is $m_i(t+1) = \lambda_i m_i(t) + \varepsilon_{ij}[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)]$ and $m_j(t+1) = \lambda_j m_j(t) + (1-\varepsilon_{ij})[(1-\lambda_i)m_i(t) + (1-\lambda_j)m_j(t)]$, see references [11,14,22–33]. Here, a set of uniformly distributed random numbers $0 \leq \varepsilon_{ij} \leq 1$ is chosen for each exchange of money [33,40]. For a model with

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* Supplementary material in the form of one pdf file available from the Journal web page at https://doi.org/10.1140/epjb/e2019-90674-0

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[1] These three terms are coupled, but clearly not equivalent in terms of their evolution and mathematical description (statistics, distributions, etc.). While salary multiplied by time yields the total amount of money earned by an individual, other sources of income exist as well, generating the total wealth of a person. All income sources have different liquidity and are subject to varying specific inflationary, political, personal and other risks. Our simplistic agent-based model operates solely with the final product: the formal and instantly exchangeable money. We use the term "money" in the model section.

[2] The latter offers a well equilibrated subsystem with the mean value of money in subpopulation acting as temperature in a statistical sense [9,13].

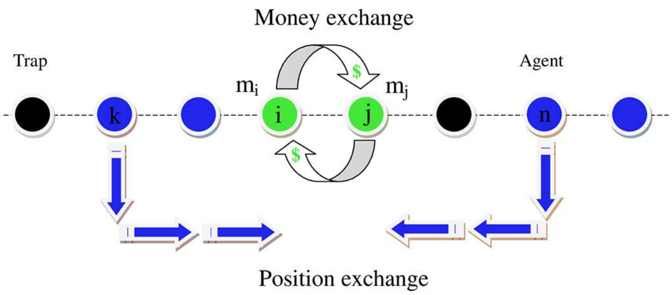

Fig. 1. Schematics of an agent-based economic model on a one-dimensional lattice. Money and position exchange in the presence of traps (the black sites) are shown as arrows for the respective pairs of agents.

constant λ the saving propensity assumes a single value, while for a model with distributed λ we assign a random number for each agent ($0 \leq \lambda_i < 1$, with $1 \leq i \leq N$) in each realization.

Markets obeying the models without any position-exchange dynamics at a fixed λ value are non-interacting at $\lambda = 1$ (the system is "frozen" at this special value of λ), while at $\lambda = 0$ no savings are allowed in the model (trading-only strategy) [11]. For a fixed λ the steady-state distribution $P(m)$ decays exponentially for large m [11]. The most probable amount of money is zero at $\lambda = 0$ (Gibbs distribution), while the average amount $\langle m \rangle = \int_0^\infty mP(m)dm$ is finite at $\lambda \to 1$, see references [11,14,33]. Clearly, other money-exchange scenarios are also possible (multiplicative, greedy exchange, etc. [41]). Various modifications of this model can be proposed, including money exchange with all agents (as in Ref. [24]), money influx in the system, different starting conditions, specific exchange rules, etc. Note that kinetic money-exchange agent-based economic models for growing markets also exist [28].

Recently, we studied the agent-based model without position exchange but with traps by means of computer simulations [33]. These traps on the lattice act as money "sinks": an agent trading with a trap loses the traded money (mimicking unfair trading partners, events of bankruptcy, etc.) [33]. Closed and open systems, with fixed and uniform saving propensities λ, have resulted in a single value of exponent ν [33]. Below, we closely follow the model of reference [33] adding a new important element: a fraction $2f/N$ of traders (random or fixed) can now exchange their positions.

Such moves mimic "migration" of agents in a trading community (geographic, corporate, etc.). In computer simulations, we consider the scenarios of fixed and random number of agent pairs that exchange positions. For instance, at a fixed value $f = 10$ during each money-exchange simulation step 20% of agents are chosen randomly to exchange their positions. For the model with position-exchange dynamics at a variable value of f on each simulation step – in addition to a random number of agent pairs (from 0 to $N/2$) exchanging random amounts of money – a random number of pairs of agents (from 0 to $(f/100) \times N$) are randomly chosen on the lattice to swap their positions. The model is called "fully dynamic" when $f = 50$ and potentially all agents can swap their positions at each step of money exchange. Note here that traps also change their positions in our computational algorithm. Moreover, if traps are present in the system, they are randomly distributed with density ρ. Thus, three or more random-number distributions are used on each step for simulating our kinetic agent-based model with position-exchange dynamics.

The lattice-based system of agents studied below is periodic: the first agent is the same as the last one. Initially, each agent holds a unit amount of money, $m_i(t = 0) = 1$. Money is a positively defined measure, so no debt is allowed in the system, see also reference [39]. For all results below, the simulations are run for $T \geq 10^4$ Monte Carlo steps (the number of steps is equivalent to time t because the time step equals unity), the lattice has $N = 10^2$ sites, and statistical averaging for $P(m)$ is performed over $R = 10^3$ realizations. Each realization here corresponds to a set of λ_i and ε_{ij} values being generated and (if applicable) to a set of traps with density ρ being distributed on the lattice. Note that each site corresponds to a trader in closed systems or a trader or a trap for open systems. The steady state is reached after T steps for all parameters used. We separately simulate closed and open systems, each for the scenario without and with position-exchange dynamics of agents. In the simulations values $\lambda_i \in [0, 1)$ are assigned to each agent and remain constant. Thus, for models with position exchanges certain mixing of agents results in other-than-Pareto distributions of money in the steady-state, as shown below.

3 Closed systems

In Figure S1 (see Supplementary material) we present the results of simulations for the trap-free model without and with position-exchange dynamics featuring a random saving propensity λ. We find rather unexpectedly that $P(m)$ for the model with position-exchange dynamics gives rise to the exponential tail, with $P(m) \propto e^{-m/\overline{m}}$ and $1/\overline{m} \approx 1.29$, as shown in Figure S1b. This exponential decay of $P(m)$ in the large-m regime is our first result. This contrasts the power-law behavior for the model without position-exchange dynamics (at $f = 0$) [11,22–24,31,32], with the Pareto exponent $(1 + \nu) \approx 2.01 \pm 0.02$, see Figure S1a. We checked that simulations for larger systems and more realizations produced the same Pareto exponents for the trap-free model without position-exchange dynamics. We also checked that dramatic changes in the money distribution – from the standard Pareto to the exponential law for the system with position exchange – are independent of the system size and number of realizations in computer simulations, see Figure S2 (Supplementary material).

We find that, as the intensity of position exchange f grows, the Pareto law observed at $f = 0$ gradually turns into an exponential $P(m)$ decay, see Figure 2, as one can expect. The results for both the scenario of constant f value and a variable number of agents exchanging their positions are shown in this plot. The value of f, therefore, controls the transition between the two forms of the steady-state money distribution. Concurrently, as in the

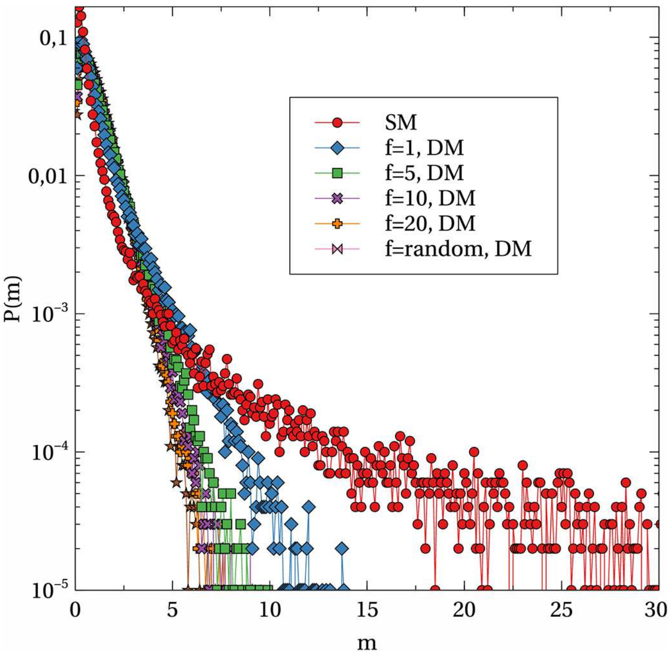

Fig. 2. Distribution $P(m)$ for a closed system for varying intensity f of position exchange of agents, plotted both for constant-f and random-f scenarios.

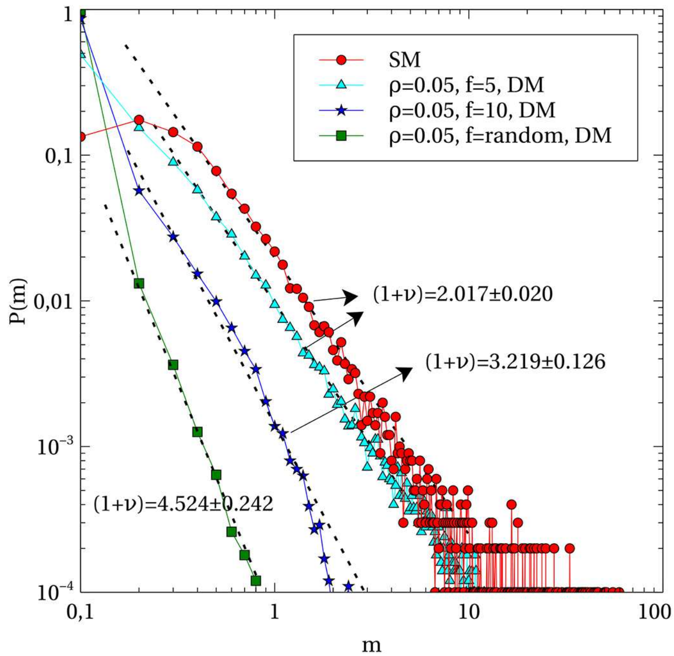

Fig. 3. Power-law $P(m)$ in a closed system without position-exchange (red circles) and an open system with position-exchange dynamics in the presence of traps with density $\rho = 1/20$ plotted for a varying fraction of agents exchanging their positions, f. The Pareto exponents are provided in the graph.

constant-f scenario the values of f increase, the maximum of $P(m)$ shifts slightly towards larger m values. Thus, in the steady state of this model the fractions of very poor and very rich agents is diminished in favor of middle-class agents (as shown in Fig. 2).

4 Open systems

As in reference [33], a non-conservative nature of trading events is implemented in the current model via randomly-distributed traps with occupation fraction ρ, see Figure 1. Traps mimic unsuccessful trading events [33]. Figure S3 (Supplementary material) shows the results of simulations for the models in the absence and in the presence of position-exchange dynamics for trap fraction ρ and varying values of f. The open system with the highest position-exchange fraction of agents f is found to have a larger Pareto exponent than the closed system without position-exchange dynamics, with $(1 + \nu) \approx 2.0$. Thus, for the models with position-exchange dynamics the presence of traps recovers the Paretian decay of $P(m)$, as compared to the exponential decay for trap-free models with position-exchange dynamics (compare Figs. S1 and 2). This observation is our second key result. Note that the value of ν for the system in the absence of position-exchange dynamics (see Fig. S2) is found to be almost insensitive to ρ, see also reference [33].

The distributions $P(m)$ for the model with the position-exchange dynamics and traps are shown in Figure 3 for fixed ρ values and for both constant-f and random-f position-exchange scenarios. The Pareto exponent changes gradually with f from its value for a model in the absence of position-exchange dynamics for a vanishing f, finally to the exponential distribution for the system with position-exchange dynamics at fully random fractions f. The explicit values of ν are given in Figure 3. The trap density ρ tunes the Pareto exponent and, therefore, offers a realistic control parameter to describe the observed variations of ν. This is our third result.

Finally, in Figure S4 (Supplementary material) we show the simulated power-law tails of $P(m)$ for rather small ρ, both for finite f values and for random fractions of agents exchanging their positions. Naturally, for larger trap densities the magnitude of non-normalized $P(m)$ decreases: for larger ρ more money "leaks out" of the system via the traps (after a fixed number of simulation steps). The Pareto exponents are found almost independent of ρ, reaching $(1 + \nu) \approx 4.5$ for fully random fractions f, see Figure S4. This is our fourth key result.

5 Conclusions

We found that a possibility of exchange of positions of the agents in a kinetic economic model has severe implications on the Pareto paradigm for the distribution of money, $P(m)$. We employed Monte Carlo simulations to examine the steady state of this kinetic model, both without and with trapping-agents. Notably, for a closed economic system the Pareto law of $P(m)$ for a system with position-exchange dynamics was demonstrated to turn into an exponential decay as the intensity of position exchange grows, see Figure 2. The physical rationale is as follows: progressive annealing or mixing of the system due to the position-exchange dynamics of agents implies an effective "temperature" in the system [13] that establishes the Boltzmann-Gibbs exponential distribution of money, as expected statistically. This is in contrast to the power-law distribution in a less equilibrated system (in the absence of any position-exchange dynamics). For an open system with position-exchange dynamics, the distribution $P(m)$ revealed again a power law, with the exponents controlled by the intensity of position exchange of the agents, see Figure 3. Lastly, the density of traps in the system was shown not to change considerably the power-law exponent in open systems with position-exchange dynamics, see Figure S4.

Clearly, with this primitive model we do not pretend to explain the realm of extremely rich wealth and income distributions observed for different countries [13,17]. However, we emphasize that the open model in the presence of position-exchange dynamics behaves fundamentally different than the closed model in the absence of position-exchange dynamics, with its paradigmatic Pareto law for $P(m)$. Different mobilities of agents in a global trading system may thus create an impetus for country-specific Pareto exponents, as indeed observed. The traders in their environments possess specific mobilities and trapping partners that possibly affects the forms and exponents of tails of $P(m)$ distributions.

E.A. thanks Potsdam University for hospitality. This work was supported by Istanbul University (Grant BYP-2018-45662) and the Deutsche Forschungsgemeinschaft (DFG Grants ME 1535/6-1 and ME 1535/7-1). R.M. also thanks Foundation for Polish Science within an Alexander von Humboldt Polish Honorary Research Fellowship.

Author contribution statement

E. Aydiner performed computer simulations and analyzed the data; E. Aydiner, A. Cherstvy, and R. Metzler discussed the results, wrote and revised the manuscript.

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