Microscopic Origin of the Logarithmic Time Evolution of Aging Processes in Complex Systems

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There exists compelling experimental evidence in numerous systems for logarithmically slow time evolution, yet its full theoretical understanding remains elusive. We here introduce and study a generic transition process in complex systems, based on nonrenewal, aging waiting times. Each state n of the system follows a local clock initiated at $t = 0$. The random time τ between clock ticks follows the waiting time density $\psi(\tau)$. Transitions between states occur only at local clock ticks and are hence triggered by the local forward waiting time, rather than by $\psi(\tau)$. For power-law forms $\psi(\tau) \simeq \tau^{-1-\alpha}$ $(0 < \alpha < 1)$ we obtain a logarithmic time evolution of the state number $\langle n(t)\rangle \simeq \log(t/t_0)$, while for $\alpha > 2$ the process becomes normal in the sense that $\langle n(t)\rangle \simeq t$. In the intermediate range $1 < \alpha < 2$ we find the power-law growth $\langle n(t)\rangle \simeq t^{\alpha-1}$. Our model provides a universal description for transition dynamics between aging and nonaging states.

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Imagine that you put a thin sheet of paper in a vertical cylinder and let the paper crumple under a heavy piston. If during compression you measure the piston's velocity, you will notice that it decreases over time, well in accordance with your intuition. However, what may appear surprising is that the piston keeps compressing the paper and never seems to come to a full rest. The outcome of such an experiment was reported by Matan *et al.* [1], concluding that the piston's position $z(t)$ at long times t decreases logarithmically, $z(t) \sim a - b\log(t/\sec)$, where a and b are constants. The crumpling of paper is by far not the only example for logarithmically slow dynamics. Experimentally, it is observed in DNA local structure relaxation [2], the time evolution of frictional strength [3], compactification of grains by tapping [4], kinetics of amorphous-amorphous transformations in glasses under high pressure [5], magnetization dynamics in high-T_c superconductors [6], conductance relaxations [7,8], and current relaxation in semiconductor field-effect transistors [9]. Theoretical studies of logarithmic time evolution include decays in colloidal systems [10], aging in simple glasses [11] (see also Supplemental Material [12]), magnetization relaxation in spin glasses [13], evolution of node connectivity in a network with uniform attachment [14], diffusion in a random force landscape (Sinai diffusion) [15], and record statistics [16].

Here we introduce a generic microscopic model displaying logarithmic time evolution which is based on nonrenewal sequential transitions between aging states labeled by n (see Fig. 1). As we show analytically, the qth order moments of the resulting counting process at large times grow as

$$\langle n^q(t)\rangle \sim \left[\frac{\ln(t/t_0)}{\mu}\right]^q \left\{1 + \frac{q}{2}\left[\frac{q\sigma^2}{\mu} - \mu\right]\frac{1}{\ln(t/t_0)}\right\} \quad (1)$$

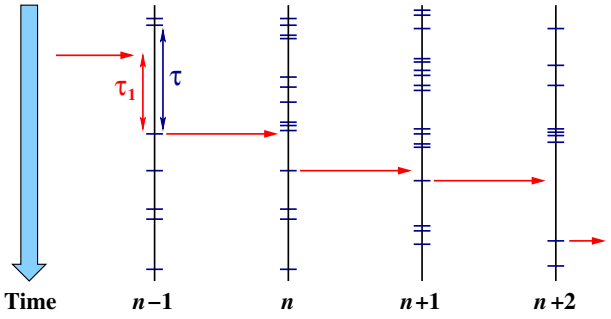

FIG. 1 (color online). Dynamic update of successive states. At each state n, a tick of the local clock allows the transition to the next state, $n + 1$. Local clock ticks are separated by waiting times τ drawn from the distribution $\psi(\tau)$. After transition from state $n - 1$, the system is locked in n until the next clock tick at n, after the forward waiting time $\tau_1 < \tau$. Typically, a transition at a new state arrives during a long waiting time, the statistics of the τ_1 thus slowing down the overall dynamics.

such that, particularly, we find the logarithmically slow counting process $\langle n(t)\rangle \simeq \log(t/t_0)$. The parameters μ and σ depend on the details of the underlying dynamics and are specified below, and t_0 is the time when the counting started after global system initiation at $t = 0$, for instance, by an external perturbation. We also show that under nonaging conditions our model leads to the expected linear growth $\langle n(t)\rangle \simeq t$, and in the intermediate case we observe power-law scaling for $\langle n(t)\rangle$. Our model provides an intuitive mesoscopic approach to the superslow dynamics in aging systems.

We define the dynamics of the system through a series $n(t)$ of consecutive states, each of which is characterized by its own local clock and all being initiated globally at time $t = 0$. The clocks' ticks occur with random time intervals τ, which are drawn from a waiting time density $\psi(\tau)$ (see Fig. 1). If the system arrives at state $n-1$ at a later time t', then it is more likely to encounter a large τ and, therefore, also typically has to wait a correspondingly longer time τ_1 before a transition to state n occurs. For $\psi(\tau) \simeq \tau^{-1-\alpha}$ with $0 < \alpha < 1$, no typical time scale $\langle\tau\rangle = \int_0^\infty \tau\psi(\tau)d\tau$ exists, and we find Eq. (1), whose scaling with the counting initiation time t_0 manifests the nonstationarity of the process [17]. Equation (1) is the central result of this work, but we also obtain $\langle n^q(t)\rangle$ for $\alpha > 1$. Moreover, we find the probability distribution $h_n(t)$ to be in state n at time t given that the counting of transitions (from state 0) began at $t = t_0$.

A simplistic picture for our model is to envision a hitchhiker traveling through a series of towns. In each town, traffic starts in the morning, and friendly drivers (persons willing to pick up our hitchhiker) appear at random intervals τ governed by ψ. The hitchhiker typically arrives to a new town in between two friendly drivers showing up, and the delay time τ_1, i.e., the time the hitchhiker actually has to wait until the next ride, is governed by the forward waiting time density ψ_1 [18]. The probability density ψ_1 is far from trivial: For heavy-tailed $\psi(\tau)$ it displays aging; see below. In this context it is interesting to note that indeed arrival times of English trains, but also response times in human communication patterns, and bursting in queuing models are power-law distributed [19–21].

A more physical picture for our model is defect-mediated crack-type propagation in a solid. Imagine a crack that grows in discrete steps $(\ldots, n-1, n, \ldots)$, the growth being triggered by the arrival of a diffusing defect at the neighboring site of the crack's tip, similar in spirit to Glarum's defect diffusion model [22]. The global initiation in this system occurs when the external stress is applied. Possibly, similar scenarios may apply in the above-mentioned examples of stick-slip dynamics [3] and density relaxation of grains by tapping [4].

We now formulate our process mathematically. To that end, we define the probability density $\rho_n(t)$ for the system to arrive at state n at time t, which fulfills the convolution

$$\begin{aligned}\rho_n(t) &= \int_0^t \rho_{n-1}(t')\psi_1(t-t'|t')dt',\\ \rho_0(t) &= \delta(t-t_0),\end{aligned} \tag{2}$$

where $\psi_1(\tau_1|t')$ is the probability density of the triggering delay time (forward waiting time) τ_1 that the system spends in a new state after having arrived there at time t'. Equation (2) expresses the fact that the probability to arrive at state n in a time interval $[t, t+dt]$ is the probability of having arrived to the state $n-1$ at some earlier time interval $[t', t'+dt']$ with $t' < t$ multiplied by the probability of a triggering event occurring in $[t, t+dt]$, where t' lies anywhere between 0 and t. Now, if $\psi(\tau) \simeq \tau^{-1-\alpha}$ with $0 < \alpha < 1$ ($\alpha > 1$ is discussed below), then the probability density ψ_1 of forward waiting times τ_1 is known from continuous time random walk (CTRW) theory, namely [23–25],

$$\psi_1(\tau_1|t') = \frac{\sin(\pi\alpha)}{\pi}\frac{t'^\alpha}{\tau_1^\alpha(t'+\tau_1)}. \tag{3}$$

This quantity explicitly depends on the arrival time t' and thus mirrors the aging property of the process: While at small t', we observe the scaling $\psi_1 \simeq \tau_1^{-1-\alpha}$ in analogy to the regular waiting time density $\psi(\tau)$, at later t' we have to wait for a longer τ_1 for the next transition event. This intuitively corresponds to the observation of a random walk process governed by the waiting time density $\psi(\tau) \simeq \tau^{-1-\alpha}$ with $0 < \alpha < 1$: When the process evolves (i.e., becomes older), due to the scale-free nature of ψ we see increasingly longer waiting times. The later we arrive at a new state (growing t'), the longer the typical current tick-tick waiting time τ will be, and thus τ_1 grows longer as the overall process develops.

We note that our model is in stark contrast to standard CTRW theory where the waiting time is reset (renewed) after each transition [26,27]; i.e., the renewals are an intrinsic property of the process. Here we update each state *locally* starting at $t = 0$, and each local clock is renewed after a tick. However, the overall process effectively couples all the local clocks, since after a transition to a new state n (i.e., a tick at state $n-1$) the process needs to wait for the next local tick (at n). This bestows the nonrenewal property of the overall process.

Finally, we obtain the probability $h_n(t)$ to find the system in state n at time t. It corresponds to the probability of having arrived at n at $t' < t$ and no transition having occurred since:

$$h_n(t) = \int_0^t \rho_n(t')\int_{t-t'}^\infty \psi_1(\tau_1|t')d\tau_1 dt'. \tag{4}$$

Equations (2)–(4) define the problem we solve here.

To proceed, it is convenient to employ the technique of Mellin transforms [28]. With $G(x) \equiv x\psi_1(x-1|1)\theta(x-1)$, where $\theta(x)$ is the unit step function, Eq. (2) becomes

$$\rho_n(t) = \frac{1}{t}\int_0^\infty \rho_{n-1}(t')G(t/t')dt'. \quad (5)$$

Using the definition of Mellin transforms $f(p) = \int_0^\infty t^{p-1}f(t)dt$, where p is the Mellin variable, along with the Mellin convolution theorem [28] we obtain from Eq. (5) that $\rho_n(p) = G(p-1)\rho_{n-1}(p)$, to which the solution is $\rho_n(p) = [G(p-1)]^n t_0^{p-1}$ [here we used $\rho_0(p) = t_0^{p-1}$]. The Mellin transform of Eq. (4) is $h_n(p) = \rho_n(p+1)[G(p)-1]/p$, and therefore

$$h_n(p) = t_0^p G(p)^n [G(p)-1]/p. \quad (6)$$

This is an exact solution in Mellin space for the sought-after quantity $h_n(t)$ used in the following.

While no simple expression exists for the exact $h_n(t)$, we can obtain all moments of $h_n(t)$ in the limit of long times t. Expanding $G(p)$ for small p to second order, for $0 < \alpha < 1$, we obtain the qth order moments [12]

$$\langle n^q(p)\rangle \sim \frac{\Gamma(q+1)t_0^p}{\mu^q(-p)^{q+1}}\left\{1 + \frac{p}{2}[\mu - q\sigma^2/\mu]\right\}, \quad p \to 0^- \quad (7)$$

in Mellin space, with $\mu = -\Gamma'(\alpha)/\Gamma(\alpha) - \gamma$ and $\sigma^2 = -\pi^2/6 + \partial^2 \ln\Gamma(\alpha)/\partial\alpha^2$. Here, $\Gamma(z)$ is the complete Γ function, and $\gamma = 0.5772\ldots$ denotes Euler's constant. Inverting the Mellin transform, we retrieve Eq. (1) at long t. Thus the leading order behavior of the first two moments follows $\langle n(t)\rangle \sim \ln(t/t_0)/\mu$ and $\langle n^2(t)\rangle \sim \ln^2(t/t_0)/\mu^2$. This shows that the triggering process considered here leads to a nontrivial logarithmic time evolution for heavy-tailed forms of $\psi(\tau)$. The logarithmically slow dynamics contrasts the case $\alpha > 1$ for which $\langle n^q(t)\rangle$ grows as a power law (shown below). In Fig. 2, we compare our analytical result (7) for $\langle n(t)\rangle$ with simulations [29] for the concrete form $\psi(\tau) = \alpha\tau_0^\alpha/(\tau+\tau_0)^{1+\alpha}$. As can be seen, the simulations agree excellently with Eq. (1), except for $\alpha \to 1$. The inset in Fig. 2 shows that the mismatch is due to the fact that t_0 is not sufficiently large (i.e., not much larger than τ_0) and the distribution $\psi_1(\tau_1|t_0)$ thus has not reached its asymptotic form (3).

The q dependence of the dominant term in Eq. (1) corresponds to a δ function for the limiting distribution. This means that the standard deviation versus the mean in our model becomes increasingly small for long times and that the dynamics becomes effectively *deterministic*. Indeed, dividing the variance by the mean we find

$$\frac{\sqrt{\langle n^2(t)\rangle - \langle n(t)\rangle^2}}{\langle n(t)\rangle} \sim \sqrt{\frac{\sigma^2}{\mu \ln(t/t_0)}}, \quad (8)$$

as is nicely corroborated by simulations of this ratio in Fig. 3. Equation (8) contrasts the behavior of the position coordinate in biased subdiffusive CTRW processes where the ratio above tends to a constant [26].

What about the behavior when $\alpha > 1$? In this case, $\psi_1(\tau_1|t')$ has a finite limit independent of t' and is given by $\psi_1(\tau_1) = \int_{\tau_1}^\infty \psi(\tau')d\tau'/\langle\tau\rangle$ [30], where $\langle\tau\rangle = \int_0^\infty \tau\psi(\tau)d\tau$. Assuming the form $\psi(\tau) \sim A/\tau^{\alpha+1}$ for large τ, one obtains $\psi_1(\tau_1) \sim (\alpha-1)A/[\langle\tau\rangle\tau_1^\alpha]$. We find two distinct regimes for the cases $1 < \alpha < 2$ and $\alpha > 2$. For $1 < \alpha < 2$ the system goes through the series of states $n(t)$, as a regular renewal process with power-law waiting times of index $\alpha - 1$. The number of states the system passes in this case thus has the moments [25,31]

$$\langle n^q(t)\rangle \sim \frac{\Gamma(q+1)}{\Gamma(q(\alpha-1)+1)}\left(\frac{\langle\tau\rangle t^{\alpha-1}}{\Gamma(2-\alpha)A}\right)^q \propto t^{(\alpha-1)q}. \quad (9)$$

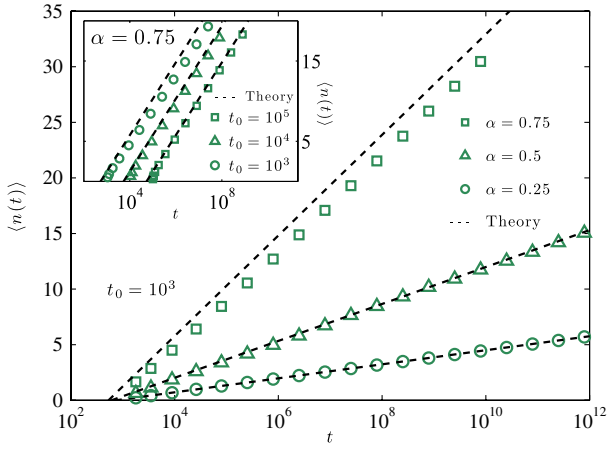

FIG. 2 (color online). Average state number $\langle n(t)\rangle$ versus time t. Symbols represent simulations for various α, as indicated. Dashed lines show the asymptotic behavior, Eq. (1) for $q = 1$. In the simulations we used $t_0 = 10^3$ and $\tau_0 = 1$. Results are ensemble averaged over 10^7 runs, respectively. Inset: Convergence to the theoretical results with t_0 for $\alpha = 0.75$.

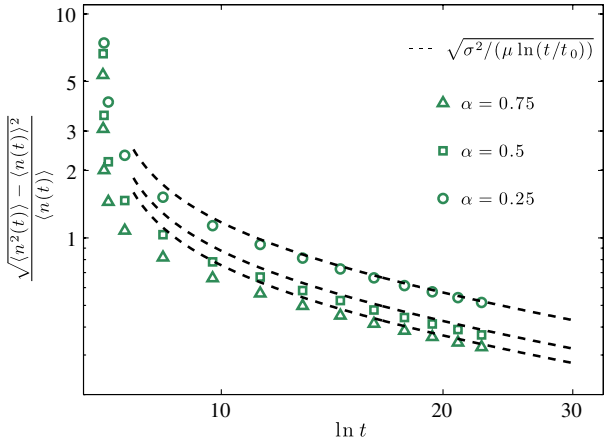

FIG. 3 (color online). Standard deviation versus mean as a function of process time. Simulations (symbols) are compared to the asymptotic results for large t [Eq. (8), dashed lines]. Parameters used: $t_0 = 10^3$, $\tau_0 = 1$, averaged over 7×10^5 simulation runs.

Here we notice that the mean $\langle n(t)\rangle \simeq t^{\alpha-1}$ increases sublinearly rather than logarithmically as in the case $0<\alpha<1$. Moreover, we find that the fluctuations grow as fast as the mean. For $\alpha>2$ we put $\alpha\to 2$ and $\Gamma(2-\alpha)A\to\langle\tau^2\rangle/2$, so that we obtain

$$\langle n^q(t)\rangle \sim \left(\frac{2\langle\tau\rangle t}{\langle\tau^2\rangle}\right)^q \propto t^q. \tag{10}$$

In this case, in particular, the mean grows linearly with time. Interestingly, just as for the case $0<\alpha<1$ (but in contrast to the regime $1<\alpha<2$), the deviations vanish relative to the mean; i.e., the long-time dynamics is effectively deterministic.

We now turn our attention to the full distribution $h_n(t)$ for the case $0<\alpha<1$. To that end, we need to evaluate the inverse Mellin transform of Eq. (6). In the Supplemental Material [12], we derive the approximate form

$$h_n^{(1)}(t) = h_n^{(0)}(t)\left[1+\frac{\sigma^2+\mu^2}{2\mu\sqrt{\sigma^2 n}}y+\frac{\kappa_3 n}{6(\sigma^2 n)^{3/2}}(y^3-3y)\right], \tag{11}$$

where $y=[\ln(t/t_0)-\mu n]/\sqrt{\sigma^2 n}$ and

$$h_n^{(0)}(t) = \frac{\mu}{\sqrt{2\pi\sigma^2 n}}\exp\left(-\frac{[\ln(t/t_0)-\mu n]^2}{2\sigma^2 n}\right). \tag{12}$$

The distribution $h_n(t)$, for fixed (logarithmic) time, is thus a slightly skewed Gaussian in the n domain. In Fig. 4, we compare the result $h_n^{(1)}(t)$ with simulations, demonstrating good agreement for its dominating part.

In particle tracking assays, single trajectories are routinely measured and analyzed [32]. We therefore also consider the time average for a single realization of $n(t)$ defined as $\overline{n(\Delta)}=(t_2-\Delta-t_1)^{-1}\int_{t_1}^{t_2-\Delta}[n(t+\Delta)-n(t)]dt$, where the observation time of the trajectory is from t_1 to t_2 and Δ is the lag time. Here we consider only the heavy-tailed case $0<\alpha<1$. Averaging over many trajectories, the dominant behavior at $t_1, t_2 \gg t_0$ becomes

$$\left\langle\overline{n(\Delta)}\right\rangle \sim \Delta\,\frac{1/\mu}{t_2-t_1}\ln\frac{t_2}{t_1} \tag{13}$$

for $\Delta \ll t_2-t_1$. The linear behavior in Δ contrasts the logarithmic time dependence of $\langle n(t)\rangle$. This discrepancy between ensemble and time average demonstrates that the process considered here is weakly nonergodic [32–34]. Interestingly, while the duration t_2-t_1 and the aging time t_1 factorize from the lag time (Δ) dependence similar to CTRW processes [35], the times t_1 and t_2 enter in terms of the nontrivial combination $(\ln t_2-\ln t_1)/(t_2-t_1)$.

We finally ask whether we can understand the logarithmic time evolution for $\langle n(t)\rangle$. We show in the Supplemental Material [12] that Eq. (5) can, after minor modifications, be interpreted as the probability density for products of independent random variables. The logarithmic time evolution follows from the fact that the product of many random numbers approaches the log-normal distribution. Our work therefore connects to the large number of scientific fields where this distribution appears; see the review [36].

In summary, we developed a generic stochastic framework for systems exhibiting logarithmic time evolution. Our system is initiated globally by some external perturbation (stress, incipient light, etc.), but transitions occur by updates of *local* clocks. Each transition to the following state is thus timed according to the first waiting time. Consequently, the resulting process is "*superaging*" in the sense that at each step a local aging period passes. As a result, we obtain a logarithmic time evolution for power-law forms of the clock-update distribution ψ.

Examples of logarithmically slow dynamics are found in biological, mechanical, and electrical systems. No universal framework has yet been put forward, only certain classes of systems. In Refs. [7,8], the logarithm appears due to a specifically chosen spectral density, whereas in Ref. [6] it stems from assumed relations between macroscopic observables and in Ref. [10] from a complex interplay between hard-core repulsion and short-range attraction. The logarithmic rate of change of network connectivity in Ref. [14] follows by construction, and an effective power-law distribution of potential barriers leads to Sinai diffusion [15]. Finally, record statistics is an extreme value problem where the logarithmic time evolution occurs, since breaking records is (usually) easy in the beginning and becomes increasingly harder as time goes by [16]. In this work, we explore a generic transition process between aging states, in which the logarithmic dynamics is an emergent property. We solved this minimal model exactly and showed results for the temporal distribution

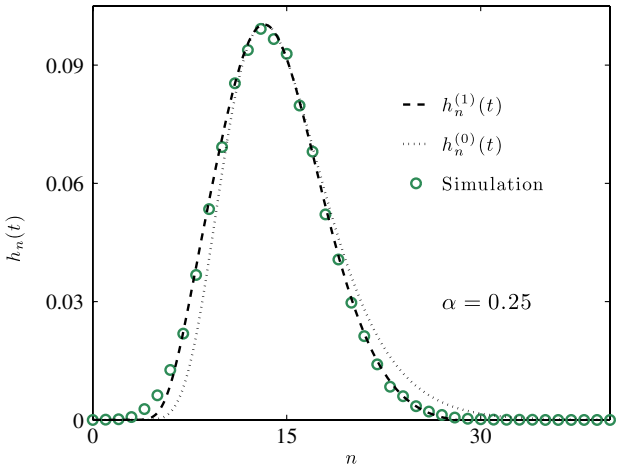

FIG. 4 (color online). Probability distribution to find the system in state n at time t. Lines: Analytical results from Eqs. (11) and (12). Circles: Simulation results, averaged over 7×10^5 runs. Parameters used: $t=7.9\times10^{12}$, $t_0=10^3$, $\tau_0=1$, and $\alpha=0.25$.

to reach a state n as well as the moments $\langle n^q(t)\rangle$. Because of the generic yet simple nature of this model, we are confident that it will be applied in many scientific fields.

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[1] K. Matan, R. B. Williams, T. A. Witten, and S. R. Nagel, Phys. Rev. Lett. **88**, 076101 (2002).
[2] E. B. Brauns, M. L. Madaras, R. S. Coleman, C. J. Murphy, and M. A. Berg, Phys. Rev. Lett. **88**, 158101 (2002).
[3] O. Ben-David, S. M. Rubinstein, and J. Fineberg, Nature (London) **463**, 76 (2010).
[4] P. Richard, M. Nicodemi, R. Delannay, P Ribière, and D. Bideau, Nat. Mater. **4**, 121 (2005).
[5] O. B. Tsiok, V. V Brazhkin, A. G. Lyapin, and L. G. Khvostantsev, Phys. Rev. Lett. **80**, 999 (1998).
[6] A. Gurevich and H. Küpfer, Phys. Rev. B **48**, 6477 (1993).
[7] A. Amir, Y. Oreg, and Y. Imry, Proc. Natl. Acad. Sci. U.S.A. **109**, 1850 (2012).
[8] A. Amir, S. Borini, Y. Oreg, and Y. Imry, Phys. Rev. Lett. **107**, 186407 (2011).
[9] R. Woltjer, A. Hamada, and E. Takeda, IEEE Trans. Electron Devices **40**, 392 (1993).
[10] M. Sperl, Phys. Rev. E **68**, 031405 (2003).
[11] L. Angelani, R. Di Leonardo, G. Parisi, and G. Ruocco, Phys. Rev. Lett. **87**, 055502 (2001).
[12] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.110.208301 for an interpretation of Eq. (5) in terms of products of random variables, derivations of $\langle n^q(p)\rangle$ and $h_n(t)$, and a relation to a simple glass model.
[13] D. Chowdhury and A. Mookerjee, J. Phys. F **14**, 245 (1984).
[14] A.-L. Barabási, R. Albert, and H. Jeong, Physica (Amsterdam) **272A**, 173 (1999).
[15] S. Havlin and D. Ben-Avraham, Adv. Phys. **51**, 187 (2002).
[16] B. Schmittmann and R. K. P. Zia, Am. J. Phys. **67**, 1269 (1999).
[17] Compare to the nonstationary position autocorrelation function in subdiffusive continuous time random walk processes; see S. Burov, R. Metzler, and E. Barkai, Proc. Natl. Acad. Sci. U.S.A. **107**, 13 228 (2010).
[18] Travel speeds in between cities is assumed to be much shorter than forward waiting times.
[19] K. Briggs and C. Beck, Physica (Amsterdam) **378A**, 498 (2007).
[20] J. G. Oliveira and A.-L. Barabási, Nature (London) **437**, 1251 (2005).
[21] A.-L. Barabàsi, Nature (London) **435**, 207 (2005).
[22] S. H. Glarum, J. Chem. Phys. **33**, 639 (1960).
[23] E. B. Dynkin, Izv. Akad. Nauk SSSR, Ser. Math. **19**, 247 (1955); Selected Transl. Math. Stat. Prob. **1**, 171 (1961).
[24] C. Godrèche and J. M. Luck, J. Stat. Phys. **104**, 489 (2001); E. Barkai and Y.-C. Cheng, J. Chem. Phys. **118**, 6167 (2003); E. Barkai, Phys. Rev. Lett. **90**, 104101 (2003).
[25] T. Koren, M. A. Lomholt, A. V. Chechkin, J. Klafter, and R. Metzler, Phys. Rev. Lett. **99**, 160602 (2007).
[26] E. W. Montroll and G. H. Weiss, J. Math. Phys. (N.Y.) **6**, 167 (1965); H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975).
[27] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000); J. Phys. A **37**, R161 (2004).
[28] B. Davies, *Integral Transforms and their Applications* (Springer, New York, 2002).
[29] Our stochastic simulations are based on a forward jumping process on a 1D lattice illustrated in Fig. 1.
[30] J. Klafter and I. M. Sokolov, *First Steps in Random Walks* (Oxford, New York, 2011).
[31] For $\alpha < 1$, the t' dependence in $\psi_1(\tau_1|t')$ hinders us from using Laplace transforms which otherwise is useful for solving convolution problems. For $\alpha > 1$, Eqs. (2)–(4) can be solved by using this transform, yielding Eq. (9).
[32] E. Barkai, Y. Garini, and R. Metzler, Phys. Today **65**, No. 8, 29 (2012).
[33] J.-P. Bouchaud, J. Phys. I (France) **2**, 1705 (1992).
[34] G. Bel and E. Barkai, Phys. Rev. Lett. **94**, 240602 (2005).
[35] J. H. P. Schulz, E. Barkai, and R. Metzler, Phys. Rev. Lett. **110**, 020602 (2013).
[36] E. Limpert, W. A. Stahel, and M. Abbt, BioScience **51**, 341 (2001).